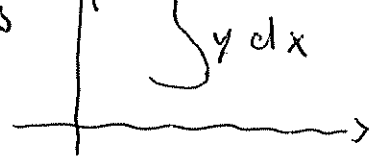
R. Dijkgraaf — part III

matrix models, top strings $\Big\}$ $\longrightarrow$ spectral curves $F(x,y)=0$

$\int y\,dx$ $\quad dx\wedge dy$



$\rightsquigarrow$ wave fct.

q.m. – system

$[x,y]=\hbar \quad \Longrightarrow \quad y = -\hbar\frac{\partial}{\partial x}$

$F(x,y) \rightsquigarrow$ diff op. $\hat F(x,\partial)$, $\hat F\psi = 0$

spectral curve $\quad A(x)$ matrix

$$\det(y\,\mathbb{1} - A(x)) = F(x,y) = 0$$

$\Longrightarrow \quad D_A = \partial_x - A(x)$ $\quad$ rank-n holom. connection $[D,\bar D]=0$

4d top $N=4$ YM on M^4 Hk, $u(N)$ gauge group

$$Z = \sum_{p,n} d(n,p)\, e^{2\pi i(\vec p\cdot\vec v + \cancel{n\cdot\tau}(n-\frac{c}{24})\tau)}$$

Euler # $M^{instan}_{p,n} \in \mathbb{Z}$

$c_1 = p \in H^2$, $ch_2 = n$

$C = N\cdot\chi(M^4)$

then Z Jacobi form $Sl(2;\mathbb{Z})$

$\tau \rightarrow \frac{a\tau+b}{c\tau+d}$

take for $M^4 = \widetilde{\text{ALE}}$ space

$= \widetilde{\mathbb{C}^2/\Gamma}$ $\qquad \Gamma \subset SU(2)$

$\Gamma = \mathbb{Z}_K$

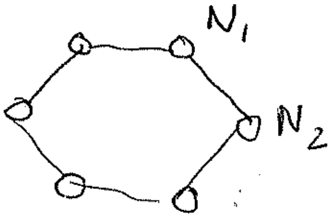
S^3/Γ

pick flat connection

$\rho \in \mathrm{Hom}(\Gamma, U(N))$

$\rho = \oplus N_i \, \rho_i$ irrep of Γ

$\mathbb{Z}_6$



MacKay correspondence

$\Gamma \longleftrightarrow \hat{g}$

$\mathbb{Z}_K \qquad SU(k)$

ρ dim $N \qquad \hat{\rho}$

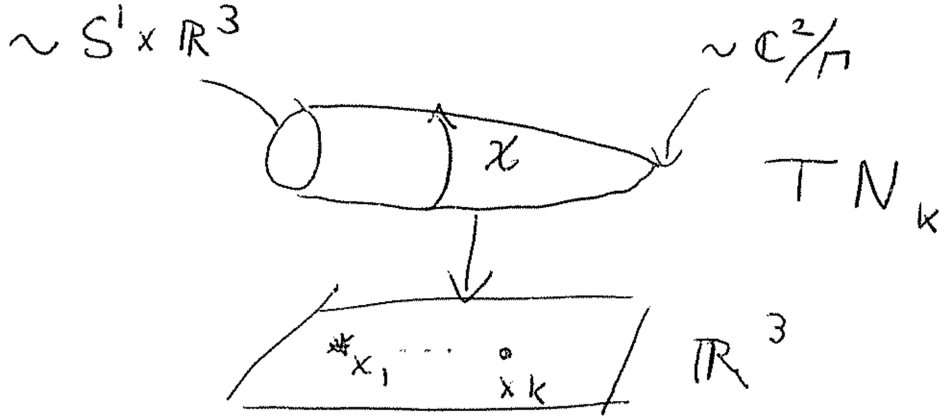
<u>claim</u>:

$$\mathcal{Z}\left(\widetilde{\mathbb{C}^2/\Gamma}, \tau, v\right)$$

$$= \mathrm{Tr}_{\hat{\rho}}\left(e^{2\pi i L_0} e^{2\pi i v J_0}\right)$$

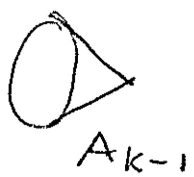
$=$ character of rep $\hat{\rho}$ of $\hat{g}_N$

$\Gamma = \mathbb{Z}_k \qquad M^4 \qquad A_{k-1}\text{-sing} \subset TN_k$ (3)



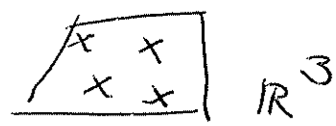
metric $\qquad ds^2 = H(d\vec{x})^2 + \frac{1}{H}(d\chi + \omega)^2$

$$H = 1 + \sum_{i=1}^{k} \frac{1}{|\vec{x} - \vec{x}_i|}$$



$\xleftarrow{R \to \infty}$

$\xrightarrow{R \to 0}$



D-branes

$M^4 \times S^1 \qquad$ put N D4-branes on it

↘ lift to M-theory

$M^4 \times T^2$

N M5-branes

modulus of T^2 is τ

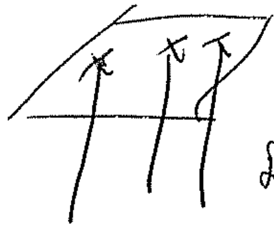
↙

$B \times T^2$

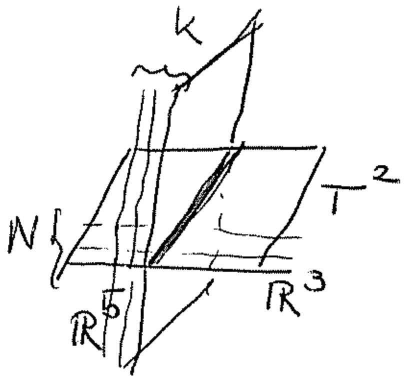
if you take TN_k

N D4-brane $\nearrow$ $TN_k \times T^2$ $\searrow$ $(\mathbb{R}^3 \times T^2)$

$TN_k \times S^1$ N M5-brane N D4-brane

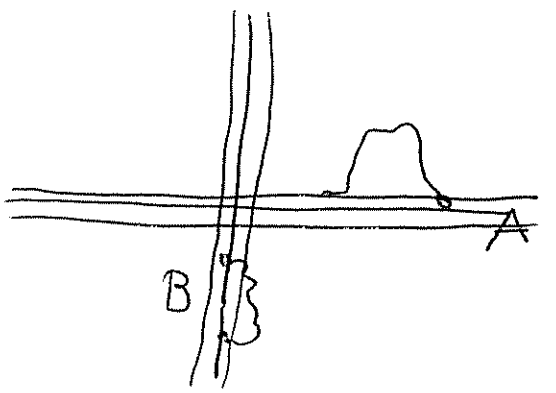


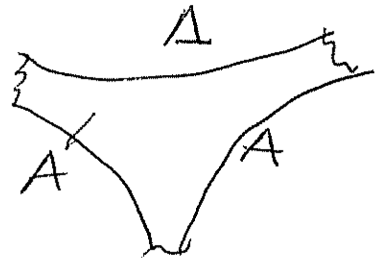
$U(N)$ gauge theory on TN_k $\cong$



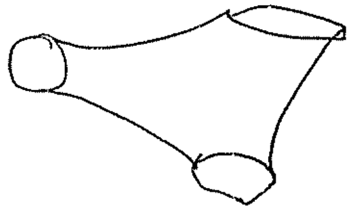
N D4-brane

I branes





non-com. algebra



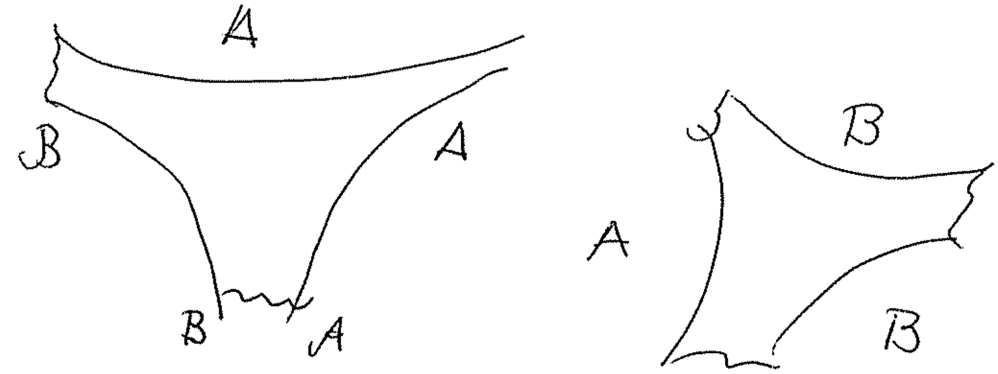
comm alg.

algebras A, B

$\Longrightarrow$ gauge groups

$U(N)$, $U(K)$

in A – B strings is A – B bi-module



D0 – D8 system

$\Rightarrow$ chiral fermions on T^2

$$\int d^2z \; \psi^{+}_{i,a} \, \bar{\partial} \, \psi^{i,a} \qquad \begin{array}{l} i = 1, \dots N \\ a = 1, \dots, k \end{array}$$

Shur – Weyl duality $\quad U(N) \quad S_k$

$\mathcal{F} = (\mathbb{C}^N)^{\otimes k}$ decompose $U(N) \times S_k$

$$= \bigoplus \underset{\substack{\text{irrep} \\ U(N)}}{V} \otimes \underset{\substack{\text{irrep} \\ S_k}}{W}$$

$N \cdot k$ free fermions $\quad U(N \cdot k)_1$

$$SU(N)_k \times SU(k)_N \times U(1)_{Nk} \subset U(Nk),$$

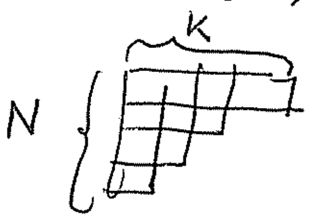
conformal embedding

$\mathcal{F}$ free fermion module

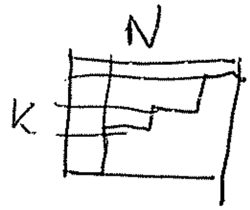
$$\mathcal{F}^{\otimes Nk} = \bigoplus V_\rho \otimes W_{\rho^T}$$

irrep of $SU(N)_k$

irrep of $SU(k)_N$

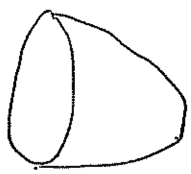


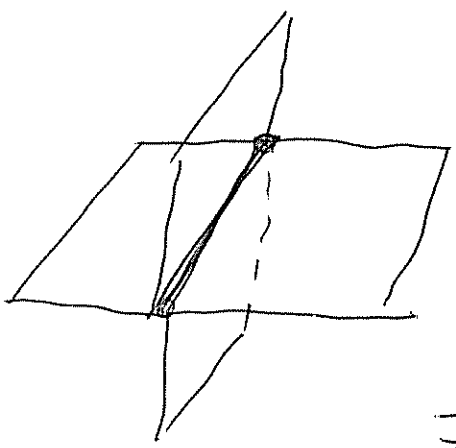
ρ



ρ^T

Nakajima $\quad \mathbb{C}^2/\mathbb{Z}_k \quad U(N) \to SU(k)_N \times SU(N)_k$





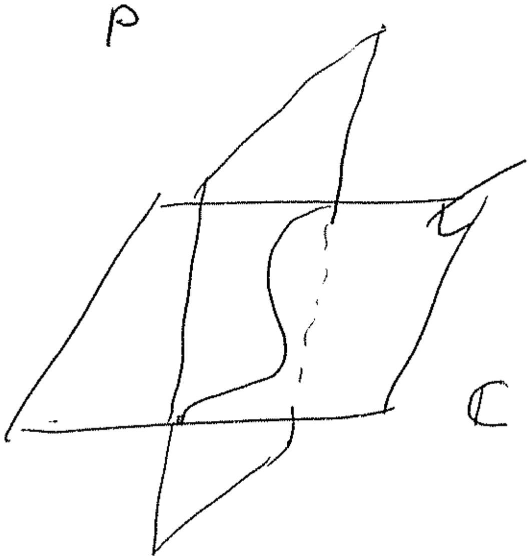
free fermions

$N = k = 1$

$$Z = \det \bar{\partial}_{T^2} = \frac{\sum e^{\pi i p^2 \tau + 2\pi i p v}}{\eta(\tau)}$$

$$\eta = \prod_{n>0} (1 - e^{2\pi i n t}) = \det{}^{1/2} \Delta$$

$$Z = \sum_{p} e^{F_0 + F_1}$$



$$\frac{dx \wedge dy}{\lambda}$$

$\mathbb{C}^2$

$\Rightarrow$ free fermions on C

$N = 2$ SW solutions

$$Z = \det \bar{\partial}_C$$

Chain of dualities.

strong coupling const $\quad B = \dfrac{dx \wedge dy}{\lambda}$

$\mathcal{A} = \mathbb{C}[x,y]$ B-field $\rightsquigarrow$ $[x,y] = \lambda$

$f(x,y)$

$y = -\lambda \frac{\partial}{\partial x}$

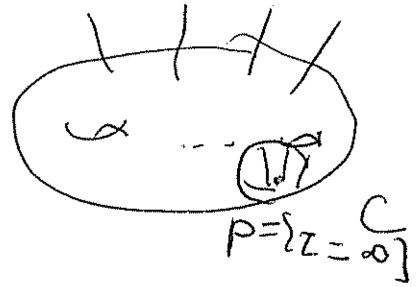
$\mathcal{A} = \langle x, \partial_x]$

sheaf of diff operators

$D = \sum p_i(x)\, \partial^i$

$\rightsquigarrow$ moduli for $\mathcal{A}$

free fermions + tau-fcts



$\det \bar\partial_C = Z$

$$\psi(z) = \sum_{n \in \mathbb{Z}+1/2} \psi_n \, z^{-n-1/2} (dz)^{1/2}$$

$\psi^+(z)$

$$\{\psi_n, \psi^*_m\} = \delta_{n+m,0}$$

$|0\rangle$

$$\psi_n |0\rangle = \psi^*_n |0\rangle = 0 \quad n > 0$$

$$F = \prod \psi_{-n} \prod \psi^*_{-m} |0\rangle$$

$\psi(z) \in K^{1/2}$

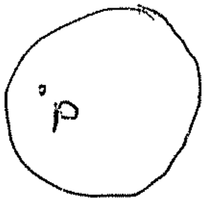
$f \in K^{1/2}$

$$\psi[f] = \oint f(z)\, \psi(z)$$

$f(z) = z^{n+\frac{1}{2}}$ $\qquad \psi[z^n] = \psi_n$

$|C\rangle$ $\qquad \psi[f]\,|C\rangle = 0$

$\Longleftrightarrow \quad f \in H^0(C-p)$



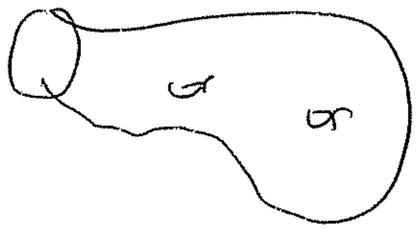
$\mathbb{P}^1$

disc

regular fcts $z^n \quad n>0$

$\rightsquigarrow \quad |0\rangle$ vacuum

ψ



$\det \bar{\partial}_C = Z = \langle 0|C\rangle$

$|0\rangle, |C\rangle \in \mathcal{F}$

$|t\rangle = \exp \sum t_n \alpha_{-n} |0\rangle$

$\alpha_{-n} = \sum \psi_k^+ \psi_{-k-n}$

$\det \bar{\partial}_t = \langle t|C\rangle$

related to integrable hierarchy

$\langle t | W \rangle = Z(t)$

τ - fct of the KP - hierarchy for any $|W\rangle \in \mathcal{F}$

if $|W\rangle = |C\rangle \to$ geometric solution (KricheVer)

$\Rightarrow \vartheta$ - fcts

if we have a $\mathcal{D}$ - module

$$\underset{\text{curve}}{y^2 = u(x)} \rightsquigarrow \left(\partial^2 - u(x)\right)\psi = 0$$

module $\mathcal{M} = \{ D\psi \,;\ D$ is a $\mathcal{D} = \langle x, \partial \rangle \}$

$$\left.\begin{array}{l} \psi,\ x\psi,\ x^2\psi, \ldots \\ \partial\psi,\ x\partial\psi,\ \ldots \\ \partial^2 \cancel{\psi = u(x)}\ \psi \end{array}\right\} \text{module } \mathcal{M}$$

if we have a $\mathcal{D}$ – module M

$$\Psi[f] \,|\, M \rangle = 0 \qquad f \in M$$

old matrix models

$$P = \sum p_i(x)\, \partial^i$$
$$Q = \sum q_i(x)\, \partial^i \qquad \text{on } \mathbb{C}[x]$$

$$[P, Q] = 1$$

$\langle x, \partial \rangle$ sheaf of diff-op

representation $\langle x, \partial \rangle \longrightarrow \langle P, Q \rangle$

$$Z = \langle t \,|\, M \rangle$$