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Topological Field Theories, con.

Thm 1 0-1 dimensional topological field theories with values in $\mathcal{C}$

$\longleftrightarrow$ dualizable objects in $\mathcal{C}$

$\mathcal{C}$ = ~~∞ category~~ ∞ 1-category
symmetric monoidal

0+1 manifolds form a symmetric monoidal ∞ 1-category.

Simplicial sets
(combinatorial model for the algebraic topology of spaces)

$X = \{X_n,\ n=0,1,2 \dots\}$.

X_n = set of n-simplices in X.

$\partial^i : X_n \to X_{n-1},\ i=0 \to n$ face maps

$s^i : X_n \to X_n,\ i=0,\dots,n-2$ degeneracy maps

Δ = category of finite ordered sets

$[n] = \{0 < \dots < n\}$.

Simplicial set = functor $\Delta \to$ Sets

Simplicial sets don't have the homotopy extension property

$$\begin{array}{l} K \\ \cap \\ L \end{array} \longrightarrow X$$

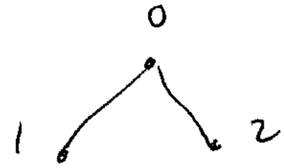
<u>Def</u> X is a <u>Kan complex</u> if each horn can be filled.

$$\begin{tikzcd} V_{k,n} \arrow[r] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

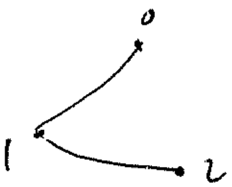
Δ^n = standard n-simplex

$V_{k,n}$ = union of all codim 1 faces except the one opposite the k^{th} vertex

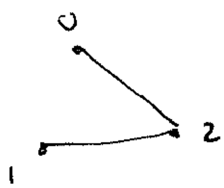
$V_{0,2}$



$V_{1,2}$



$V_{2,2}$

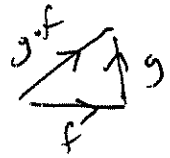


$\mathcal{C}$ = a cat.

$\Rightarrow$ simplicial set $N\mathcal{C}$

0-simplices = objects

1-simplices = $\bullet\!\longrightarrow\!\bullet$

2-simplices = 

n-simplices = { n-tuples of composable arrows }

Is NC a Kan complex?

$V_{1,2}$
$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] & \\ \bullet \arrow[rr, "g"'] & & \bullet \end{tikzcd} \hookrightarrow \begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "g \circ f"] & \\ \bullet \arrow[rr, "g"'] & & \bullet \end{tikzcd}$$

$V_{0,2}$
$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "h"] & \\ \bullet & & \bullet \end{tikzcd}$$

want to solve $h = g \circ f$
$$\begin{tikzcd} & \bullet \arrow[dl, "f"'] \arrow[dr, "h"] & \\ \bullet \arrow[rr, "g"'] & & \bullet \end{tikzcd}$$

(can't always...)

Similarly for $V_{2,2}$.

<u>Def</u> (Boardman-Vogt)

A simplicial set S is a <u>weak Kan complex</u>, or <u>quasi-category</u>, or <u>∞ 1-category</u>, if every mirror horn has a filler.

$V_{k,n}$ is <u>inner</u> if $0 < k < n$.

Every Kan complex is an ∞ 1-category.

Spaces $\longleftrightarrow$ groupoids

∞ 1-category $\longleftrightarrow$ category

X	Space (Kan cx)	∞ 1-cat
	π_0 – set	π_0 – set
	π_1 – group	π_1 – set
	π_n – abelian grp, $n \geq 2$	π_2 – grp
		π_n – abelian grp, $n \geq 3$
	$\pi_{\leq 1}$ fundamental groupoid	$\pi_{\leq 1}$ fundamental category
	π_n – module of $\pi_{\leq 1}$	$\pi_{\leq 2}$ fundamental 2-groupoid

$\pi_{\leq 1} X$ objects $= X_0$ (0-simplices)
maps $=$ 1-simplices$/\sim$
composition law $=$ 2-simplices$/\sim$

Obstruction theory

$$\begin{tikzcd}
K^{(n-1)} \arrow[r] \arrow[d, hook] & X \\
K^{(n)} \arrow[ur, dashed] \arrow[d, hook] & \\
K^{(n+1)} \arrow[uur, dashed] &
\end{tikzcd}$$

obstruction in $H^n(K, \pi_{n-1} X)$ ← local system on X

choices in $H^n(K, \pi_n X)$.

The <u>same obstruction theory</u> works for ∞ 1-cat.

local system

$$x \in X \rightsquigarrow M_x \quad \text{abelian group}$$

$$x \to y \rightsquigarrow M_x \to M_y, \quad \text{which compose}$$

$$\begin{tikzcd} & z \\ x \arrow[ur] \arrow[r] & y \arrow[u] \end{tikzcd} \rightsquigarrow \begin{tikzcd} M_x \arrow[rr] \arrow[dr] & & M_z \\ & M_y \arrow[ur] & \end{tikzcd}$$

local 2-system for a quasi-category

$$e \in X_1 \longrightarrow M_e, \text{ an abelian group.}$$

$$\begin{tikzcd} & 2 & \\ 0 \arrow[ur, "h"] \arrow[rr, "f"'] & & 1 \arrow[ul, "g"'] \end{tikzcd} \qquad \begin{array}{l} M_f \to M_h. \\ M_g \to M_h \end{array}$$

+ a constraint associated to every 3-simplex.

<u>obstruction theory works.</u>

$\Pi_{\le 1} \Delta^1 \quad \bullet \to \bullet \qquad \Delta^1 \not\simeq *$

Symmetric monoidal ∞ 1-category

$$\otimes : \underline{X} \times \underline{X} \to \underline{X}$$

commutative + associative up to ∞ homotopy

X is symmetric monoidal ∞ 1-category $\Rightarrow \Pi_{\leq 1} X$ is a symmetric monoidal category.

An object V in X is dualizable if V is dualizable in $\Pi_{\leq 1} X$.

V is dualizable in $\mathcal{C}$ if $\exists\, W$

$$V \otimes W \to \mathbb{1}$$

$$\mathbb{1} \to V \otimes W.$$

$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend right, "id"'] & W \otimes V \otimes W \arrow[r] & W \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r] \arrow[rr, bend right, "id"'] & V \otimes W \otimes V \arrow[r] & V \end{tikzcd}$$

Bord_{01} $\quad$ ∞ 1-cat. of 0 + 1-manifolds.

ob: oriented 0-manifolds.

symmetric monoidal, using $\underline{\amalg}$

<u>Def</u> A $0+1$ dim TFT with values in $\mathcal{C}$ ← symmetric monoidal ∞ 1-cat

is a map of symmetric monoidal ∞-1-categories

$$\mathrm{Bord}_{01} \to \mathcal{C}.$$

If V is dualizable then the dual W is characterized.

The space of dualizable objects

$$= \underbrace{\{(V,W) + \text{duality data}\}}_{\text{space}}$$

map $(V,W) \to (V',W')$

is $V \to V'$
$W \to W'$
respecting duality data

forces $V \to V'$ to be an equivalence

space of dualizable objects in $\mathcal{C}$

object dualizable V.

maps equivalence

--- a Kan complex

In $\mathrm{Bord}_{0,1}$ $\quad +$

TFT $\quad \mathrm{Bord}_{01} \xrightarrow{Z} \mathcal{C}$

$Z(+)$ = dualizable object.

TFT with values in $\mathcal{C}$ $\longrightarrow$ space of dualizable objects

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Ω $\quad$ symmetric monoidal ∞ 1-category

map$(\Omega, \mathcal{C})$ = space of dual objects in $\mathcal{C}$

Ω has cells only in low dimension

$\Rightarrow H^*(\Omega; M) = 0 \quad * > 2$

Need to show:

$\Omega \to \mathrm{Bord}_{01}$

is an equivalence.

1) $\pi_{\leq 2}$ is iso

2) $H^*(\_\,; M)$ is iso, for M any local 2-system

$\Leftrightarrow H^*(\mathrm{Bord}_{01}; M) = 0$ for M any local 2-system.