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Self-duality, K-theory, and orientifolds

(with Jacques Distler, Greg Moore)

High Energy
perturbative string
2 dim

Low Energy
field theory in 10 dim

Gravitational Ramond-Ramond charge
computed for orientifolds in
perturbative string by
Morales-Scrucca-Serone

Data:

X = smooth 10-manifold (compact), spin structure, (metric)

$\sigma : X \to X$ involution : $\sigma^2 = id_X$

Spacetime = quotient of X by σ.

Dichotomy: $\sigma \begin{Bmatrix} \text{preserves} \\ \text{reverses} \end{Bmatrix}$ orientation of X. Type $\begin{Bmatrix} B \\ A \end{Bmatrix}$.

① σ preserves orientation

$$\begin{tikzcd} \mathcal{B}_{Spin}(X) \arrow[r, "\tilde\sigma"] \arrow[d] & \mathcal{B}_{Spin}(X) \arrow[d, "2:1"] \\ \mathcal{B}_{SO}(X) \arrow[r, "\sigma"] \arrow[d] & \mathcal{B}_{SO}(X) \arrow[d] \\ X \arrow[r, "\sigma"] & X \end{tikzcd}$$

~~Let~~ $\tilde{\sigma}$ has order $\left\{\begin{matrix}2\\4\end{matrix}\right\}$

(2) σ reverses orientation

$$\begin{array}{ccc}
\mathcal{B}_{Pin^-}(X) & \longrightarrow & \mathcal{B}_{Pin^-}(X) \\
{\scriptstyle 2:1}\downarrow & & \downarrow \\
\mathcal{B}_0(X) & \xrightarrow{\sigma} & \mathcal{B}_0(X) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X
\end{array}$$

$\tilde{\sigma}$ has order $\left\{\begin{matrix}2\\4\end{matrix}\right\}$

Let $\mathcal{X} = X/\sigma$ V-manifold, orbifold or smooth Deligne-Mumford stack
i.e., $X/\!/\sigma$ = notation for stack quotient

<u>Def</u> A <u>string background</u> is the following data

1) $\mathcal{X}$ = smooth DM stack of dim 10

2) $r = 0, 1$ (Type B, A)

3) Double cover $\mathcal{X}_w \xrightarrow{2:1} \mathcal{X}$

$w \in H^1(\mathcal{X}, \mathbb{Z}/2\mathbb{Z})$

4) Twisted Pin^- structure on $\mathcal{X}$

<u>Constraint:</u> $w_1(\mathcal{X}) = rw$, $(w_1^2 + w_2)(\mathcal{X}) = \left\{\begin{matrix}0\\w^2\end{matrix}\right.$

<u>Remark</u> Think $\mathfrak{X} = X/\Gamma$ (Γ discrete)

$$H^\bullet(\mathfrak{X}) = H^\bullet_\Gamma(X)$$

5) metric, dilaton, <u>B-field</u>

<u>Standard cases</u>

<u>Type II</u> $\mathfrak{X}$ = smooth 10-manifold, spin

$w = 0$

<u>Type I</u> $\mathfrak{X} = X^{10}_{\text{smooth}}/\text{trivial} = X^{10} \times pt/\!/\mathbb{Z}/2\mathbb{Z}$ ← stack quotient

w = generator of $H^1(pt/\mathbb{Z}/2\mathbb{Z}; \mathbb{Z}/2\mathbb{Z})$

$r = 0$

The worldsheet of the string is

- Σ a pin$^-$ compact 2-manifold
- $\phi : \Sigma \to \mathfrak{X}$

[if $\mathfrak{X} = X/\Gamma$ then a Γ-equivariant
$$\begin{array}{ccc} \tilde{\Sigma} & \to & X \\ \downarrow & & \downarrow \\ \Sigma & \to & \mathfrak{X} \end{array}$$
]

such that $w_1(\Sigma) = \phi^* w$.

orientation cover
$$\begin{array}{ccc} \tilde{\Sigma} & \longrightarrow & \mathfrak{X}_w \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \mathfrak{X} \end{array}$$

Low energy theory

Field theory on $\mathcal{X}$

Assume $\mathcal{X} = X // \sigma$

X^{10} = compact spin manifold i.e. "spin"

$\sigma : X \to X$ involution

Ramond-Ramond field : a gauge field

- charge quantization — charges live in $K(\mathcal{X}) = K_{\mathbb{Z}/2}(X)$.
- self-dual field

Digression on self-duality

Maxwell field on X^{10}

$F \in \Omega^2(X)$, field strength

$\cancel{dF = 0}$ $dF = j_{mag}$

$\cancel{d*F = 0}$ $d*F = j_{elec}$

E-M duality : $j_{mag} \longleftrightarrow j_{elec}$

$F \longleftrightarrow *F$

Note: $dj = 0$

$j_{mag} = j_3$

$j_{elec} = j_9$

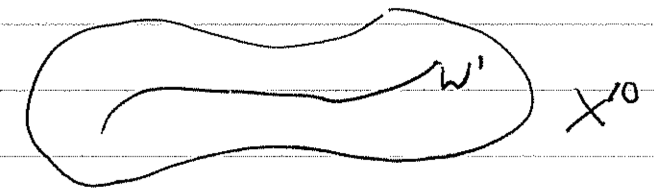
Self-dual version

$\mathbb{F} = (F_2, F_8)$

$j = (j_3, j_9)$ $d\mathbb{A} = \mathbb{F}$

$\mathbb{A} = (A_1, A_7)$

Pairing of electric + magnetic



$$\int_{W^1} A_1 = \int_X A_1 \wedge j_9$$

self-dual: $\frac{1}{2}\int_X \mathbb{A} \wedge \mathbb{j}$

Pairing of electric + magnetic on M^{12}

$$Q = (Q_3, Q_9) \in H^3(M;\mathbb{Z}) \times H^9(M;\mathbb{Z})$$

$$(Q, Q') = \int_M Q \wedge Q' = \langle Q \cup Q'; [M] \rangle$$

$$= \langle Q_3 \cup Q_9' + Q_9 \cup Q_3', [M] \rangle$$

New data:

① $\Theta : H^3 \times H^9 \longrightarrow H^3 \times H^9$

$$(Q_3, Q_9) \longmapsto (Q_3, -Q_9)$$

$b(\Theta Q, Q')$ is symmetric.

② $q : H^3 \times H^9 \longrightarrow \mathbb{Z}$.

$$(Q_3, Q_9) \longmapsto \langle Q_3 \cup Q_9, [M] \rangle$$

<u>Property</u>

$$q(Q+Q') - q(Q') - q(Q) + q(0) = b(\theta Q, Q')$$

<u>Remark</u> $b : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$n, n' \to n n'$

$q : \mathbb{Z} \to \mathbb{Z}$

$n \longmapsto \frac{1}{2}(n^2+n)$

Ramond-Ramond story

X^{10} compact, $\sigma : X \to X$, equivariantly spin

RR charge group: $KR^0(X)$.

<u>K-theory</u> $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ cplx v.b

<u>KR-theory</u> $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ $\begin{matrix} \sigma^* E \to \overline{E} \\ \downarrow \\ X \end{matrix}$

<u>KO-theory</u> $\begin{matrix} F \\ \downarrow \\ X \end{matrix}$ real v.b.

<u>Self-dual data</u>: ① $\theta : KR^0(M^{12}) \to KR^8(M^{12})$

$E \longmapsto u^4 \overline{E}$

$u \in K^2(pt)$ is the Bott element; u^4 lifts to $KO^8(pt)$

(2) $q : KR^0(M) \longrightarrow \mathbb{Z}$.

<u>To compute $q(E)$</u>

(i) Form $\Theta(E)\cdot E = u^4 \bar{E}\cdot E$

(ii) Lift to $KO^8_{\mathbb{Z}/2\mathbb{Z}}(M)$

(iii) Integrate: let $\pi^M : M \to pt$

$$\pi^M_* : KO^8_{\mathbb{Z}/2\mathbb{Z}}(M) \longrightarrow KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(pt)$$

(iv) Use $\iota : KO^{-4}(pt) \xrightarrow{\cong} \mathbb{Z}$

$$KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(pt) \longrightarrow KO^0_{\mathbb{Z}/2\mathbb{Z}}(pt) = RO(\mathbb{Z}/2\mathbb{Z})$$

$$\shortparallel$$

$$\mathbb{Z}[\underset{\substack{\uparrow \\ \text{sign rep.}}}{\varepsilon}]/1-\varepsilon^2$$

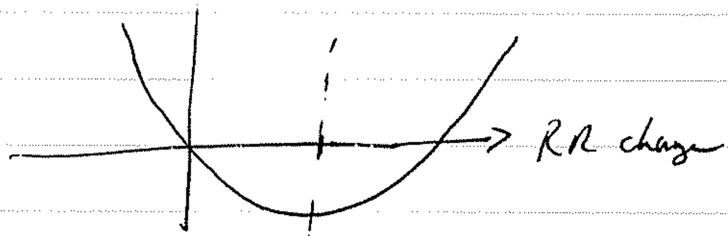
(v) take coef of ε to get $q(E)$.

<u>Remark</u>: Special case (type I) σ trivial

$$KR^0(M) = KO^0(M)$$

$$q(\underset{\substack{\uparrow \\ \text{real bundle}}}{E}) = \pi^M_*(\lambda^2 E)$$

Recall that a quadratic function has a graph which is a parabola.



$\mu \in KR^0(X)$ is the center of symmetry.

The R-R field will, in the end, be an ~~trivialization~~ "isomorphism"

$F: \mu \longrightarrow j$ ← the current of D-branes

$$\left[\begin{array}{l} j: \mathrm{Pic}(\Sigma) \to \mathrm{Lines} \\ \quad L \longmapsto \mathrm{Det}\, \not{D}_\Sigma(L) \\ \text{or } L \longmapsto \mathrm{Det}\, \bar\partial(L) \end{array}\right]$$

<u>Remark</u>: $KR^0(M)$ is a module over $RO(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}[\varepsilon]/_{1-\varepsilon^2}$

$$S = \{(1-\varepsilon)^n\}_{n \geq 1}$$

localize $RO(\mathbb{Z}/2\mathbb{Z})$: S^{-1}

(i.e., "set $\varepsilon = -1$"), because

$$1+\varepsilon = 1+\varepsilon \cdot \frac{1-\varepsilon}{1-\varepsilon} = \frac{1-\varepsilon^2}{1-\varepsilon} = 0$$

<u>Thm</u> X^{10} compact, spin

$\sigma: X \to X$ spin involution

$i: F \hookrightarrow X$ fixed point set

$\nu \to F$ is the normal bundle

<u>Then</u> $\mu = \frac{1}{2} i_* \left[\psi_{1/2}\left(\frac{u^4}{\mathrm{Euler}(\nu)} \right) \Xi(F) \right]$

in $S^{-1} KR^0(X)$.

where $\psi_{1/2}$ is the inverse of Adams operation ψ_2,

and $\Xi(F)$ is a KO-theory Wu class.

$$" \frac{1}{2} \int_X A \cdot g " = g(F) = \int_F A \frac{\sqrt{L(R^F/4)}}{\sqrt{L(R^\nu/4)}}$$

$$L(R/4) = \frac{x}{\tanh \frac{x}{2}} \qquad L(R/4) = \frac{x/4}{\tanh x/4}$$

$$x \longmapsto \pi_*^{F} \psi_2(x)$$
$$= \pi_*^{F}(x \cdot \Xi(F))$$