Self-duality, K-theory, orientifolds

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High Energy	Low Energy
perturbative string 2 dim	field theory in 10 dim

- Gravitational Ramond-Ramond charge

computed in perturb string on orientifolds by Morales-Scrucca-Serone

Data: X smooth 10 mfld (cpt) spin (metric)

$\sigma : X \to X$ involution $\sigma^2 = id_X$

spacetime = quotient of X by σ

Dichotomy: $\sigma \left\{ \begin{matrix} \text{preserves} \\ \text{reverses} \end{matrix} \right\}$ orientation of X

Type $\left(\begin{matrix} A \\ B \end{matrix} \right)$

1. σ preserves orientation

$$\begin{tikzcd} B_{SO}(X) \arrow[r, "\sigma"] \arrow[d, "2:"] & B_{SO}(X) \arrow[d] \\ X \arrow[r, "\sigma"] & X \end{tikzcd}$$

assume it lifts to spin-bdl

$$\begin{tikzcd} B_{spin}(X) \arrow[r, "\tilde{\sigma}"] \arrow[d, "2:1"] & B_{spin}(X) \arrow[d, "2:1"] \\ B_{SO}(X) \arrow[r, "\sigma"] \arrow[d] & B_{SO}(X) \arrow[d] \\ X \arrow[r] & X \end{tikzcd}$$

$\tilde{\sigma}$ order $\left\{ \begin{matrix} 2 \\ 4 \end{matrix} \right)$ today: order 2

2. σ reverses orientation

$$\begin{tikzcd} B_{Pin^-}(X) \arrow[r, "\tilde{\sigma}"] \arrow[d, "2:1"] & B_{Pin^-}(X) \arrow[d] \\ B_O(X) \arrow[r, "\sigma"] \arrow[d] & B_O(X) \arrow[d] \\ X \arrow[r] & X \end{tikzcd}$$

$\mathfrak{X} = X/\!\!/\sigma$ V-manifold
orbifold
smooth Deligne-Mumford stack

<u>Def</u>: A <u>string backgrd</u> is the following data:

1) $\mathfrak{X}$ smooth DM stack of dim 10
2) $r = 0, 1$ for Type B, A
3) Double cover $\mathfrak{X}_w \xrightarrow{2:1} X$
$w \in H^1(\mathfrak{X}; \mathbb{Z}/2\mathbb{Z})$
4) Twisted pin^--structure on $\mathfrak{X}$.
5) metric, dilaton, <u>B-field</u>

<u>constraint</u>

$$w_1(\mathfrak{X}) = r \cdot w$$

$$(w_1^2 + w_2)(\mathfrak{X}) = \begin{cases} 0 \\ w^2 \end{cases}$$

<u>Remark</u>: Think $\mathfrak{X} = X/\Gamma$ (Γ discrete)

$$H^\bullet(\mathfrak{X}) = H^\bullet_\Gamma(X)$$

<u>standard cases</u> Type II: $\mathfrak{X}$ smooth 10-mfld
$w = 0 \rightsquigarrow$ spin

Type I: $\mathfrak{X} = \underset{\text{smooth}}{X^{10}}/\text{trivial}$
$= X \times pt/\!\!/\mathbb{Z}/2\mathbb{Z}$

$r = 0$ $w =$ generator of $H^1(pt/\!\!/\mathbb{Z}_2; \mathbb{Z}_2)$

the string is

- worldsheet Σ : pin$^-$ 2-mfld (cpct)
- $\Phi : \Sigma \to \mathcal{X}$

[If $\mathcal{X} = X/\Gamma$ this is

$$\begin{tikzcd} \tilde{\Sigma} \arrow[r, "\phi"] \arrow[d, "\Gamma"'] & X \arrow[d, "F"] \\ \Sigma \arrow[r, "\phi"] & \mathcal{X} \end{tikzcd}$$

satisfying $\boxed{w_1(\Sigma) = \phi^* w}$

$$\begin{tikzcd} \text{Orientation double cover} \arrow[r, "\tilde{\phi}"] \arrow[d] & \tilde{\mathcal{X}} \arrow[d, "2:1"', "W"] \\ \Sigma \arrow[r, "\phi"] & \mathcal{X} \end{tikzcd}$$

Low Energy theory

Field theory on $\mathcal{X}$

Assume : $\mathcal{X} = X /\!/ \sigma$

equi spin $\begin{cases} X^{10} & \text{cpct spin mfld} \\ \sigma : X \to X & \text{involutions} \end{cases}$

Ramond – Ramond field: gauge field
- charge quantization
- self-dual field

charges live in $K(\mathcal{X}) = K_{\mathbb{Z}_2}(X)$

Digression:

Maxwell Field on X^{10}

$F \in \Omega^2(X)$ field strength

$dF = 0$ $\qquad\qquad$ or $\qquad dF = j_{mag}$

$d*F = 0$ $\qquad\qquad\qquad\quad d*F = j_{mag}$

EM duality

$$j_{mag} \longleftrightarrow j_{elec}$$

$$F \longleftrightarrow *F$$

Note $\quad dj = 0$

$j_{mag} = j_3$

$j_{elec} = j_9$

$\mathbb{F} = (F_2 \; F_8)$

$dA = F \qquad \mathbb{j} = (j_3, j_9)$

$\mathbb{A} = (A_1, A_7)$

pairing of electric + magnetic

$$\int_{W'} A_1 = \int_X A_1 \wedge j_q$$

self-dual $\quad \frac{1}{2}\int_X A \wedge j$

Pairing of el. and magnetic. M^k

$$\mathbb{Q} = (Q_3, Q_9) \in H^3(M;\mathbb{Z}) \times H^9(M;\mathbb{Z})$$

$$b(\mathbb{Q} \cup \mathbb{Q}') = \langle \mathbb{Q}\mathbb{Q}', [M] \rangle$$
$$= \langle Q_3 \cup Q_9' + Q_9 \cup Q_3' \rangle$$

New data

$$\Theta : H^3 \times H^9 \to H^3 \times H^9$$

$b(\Theta \mathbb{Q} \cancel{*} \mathbb{Q}')$ symmetric

(2) $\quad g : H^3 \times H^9 \to \mathbb{Z}$

$$Q_3 \times Q_1 \mapsto \langle Q_3 \cup Q_9 [M] \rangle$$

$$g : \mathbb{Z} \to \mathbb{Z}$$

$$n \to \frac{1}{2}(n^2+1)$$

Ramond-Ramond story

X^{10} cpct, $\sigma : X \to X$, equivariantly spin

RR : charge group : $KR^0(X)$

K-theory : $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ cplx. vect

KR-theory : $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ $\sigma : E \to \overline{E}$, $\begin{array}{c} \\ \downarrow \\ X \end{array}$

KO-theory $\begin{array}{c} F \\ \downarrow \\ X \end{array}$ vol, b

~~X^{10}~~ cpct self-dual data. $\ominus$

(1) $\phi : KR^0(M) \to KR^8(M)$

$u \in K^2(pt)$ Bott elt.

$E \to u^4 \overline{E}$

u^4 lifts to $KO^8(pt)$

(2) $q : KR^0(M) \to \mathbb{Z}$

To compute $q(E)$:

(i) Form $\Theta(E)\cdot E = u^4 \bar{E} E$ ...

(ii) Lift to $KO^8_{\mathbb{Z}/2\mathbb{Z}}(M)$

(iii) Integrate

$$\pi^M : M \to pt$$
$$\pi^M_* : KO^8_{\mathbb{Z}/2\mathbb{Z}}(M) \to KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(pt)$$

(iv) Use $\iota : KO^{-4}(pt) \xrightarrow{\simeq} \mathbb{Z}$

$$KO^{-4}_{\mathbb{Z}/2\mathbb{Z}}(pt) \to KO^0_{\mathbb{Z}/2\mathbb{Z}}(pt)$$
$$= RO(\mathbb{Z}/2\mathbb{Z})$$
$$= \mathbb{Z}[\varepsilon]/1-\varepsilon^2$$

sign-rep

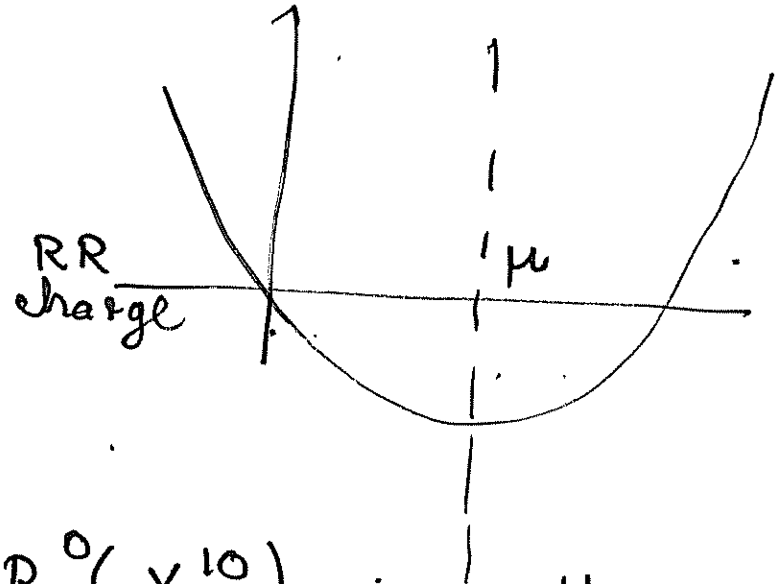
(v) Take coeff of ε.

<u>Remark</u>: Special case (Type I) or trivial

$$KR^0(M) = KO^0(M)$$
$$q(E) = \pi^M_*(\lambda^2 E)$$

real bdl

Recall that a quadratic function has a graph which is a parabola



$\mu \in KR^0(X^{10})$ is the center of symmetry

The RR field will in the end be an isomorphism $F : \mu \longrightarrow \check{j}$ ← current of D-branes

$q : \mathrm{Pic}(\Sigma) \to \text{Lines}$

$L \mapsto \mathrm{Det}\,\bar{\partial}^L$

scaling → acts as q^{ind}

<u>Remark</u> $KR^{\circ}(M)$ is a involution over $RO(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}[\varepsilon]/1-\varepsilon^2$

$$S = \{(1-\varepsilon)^n\}_{n\geq 1}$$

Localize S^{-1} [setting $\varepsilon = -1$]

$$1+\varepsilon = 1+\varepsilon \frac{1-\varepsilon}{1-\varepsilon} = \frac{1-\varepsilon^2}{1-\varepsilon} = 0$$

<u>Thm</u>: X^{10} cpct, spin

$\sigma : X \to X$ spin - involution

$i : F \hookrightarrow X$ fixed pt set

$\nu \longrightarrow F$ normal bdl

<u>Then</u>

$$\mu = \frac{1}{2}\, i_* \left[\psi_{1/2}\left(\frac{u^4}{\mathrm{Euler}(\nu)} \right) \Xi(F) \right]$$

in $S^{-1}KR^{0}(X)$

where $\psi_{1/2}$ is the inverse of Adams ψ_2, and $\Xi(F)$ is a KO-theory Wu-class

$$\frac{1}{2} \int_X A \cdot \gamma = g(F)$$

$$= \int_F A \frac{\sqrt{L(R^F/4)}}{\sqrt{L(R^\nu/4)}}$$

$$L(R/2) \quad \frac{x/2}{\tanh x/2} \qquad L(R/4) = \frac{x/4}{\tanh(x/4)}$$

Adams operation.

$$x \longmapsto \pi_*^F \psi_2(x)$$

$$= \pi_*^F (x \cdot \Xi(F))$$