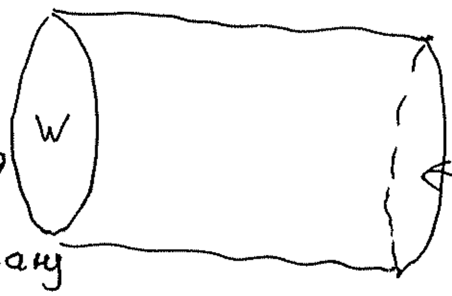
S. Gukov : „D-branes + representations" (1)

work in progress w/ E. Witten

* understand representations of $G_{\mathbb{R}} \subset G_{\mathbb{C}}$ in terms of geometry (D-branes)

Toy model: top. twist of $\mathcal{N}=4$ SYM
motivation
G cpct real form of $G_{\mathbb{C}}$
(GL-twist)

take GL-twist of $\mathcal{N}=4$ SYM on
$M_4 = W \times I$, $I = [0;1]$



boundary cond

B' brane:
(Higgs-field $\phi = 0$
susy bdy cond.)

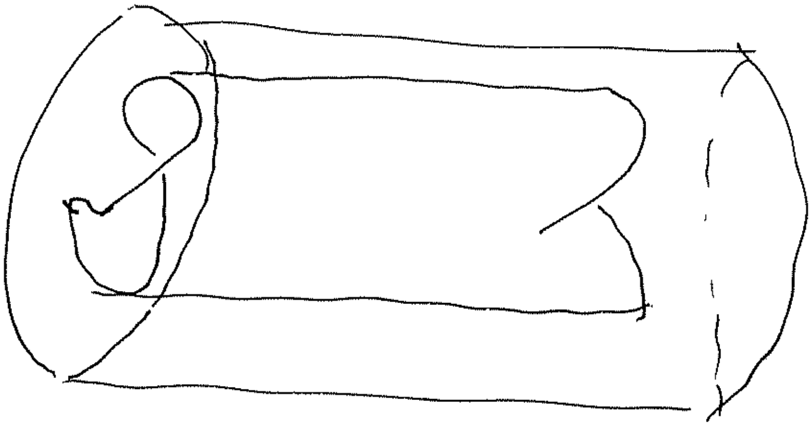
$B_{c.c.}$:
$F_A^+ \big|_W = 0$

boundary cond. preserve topol. susy

theory $\underset{\text{(collapse interval)}}{\simeq}$ 3d TQFT on W
Chern-Simons gauge theory w/ g-grp. G

interesting Questions: how does S-duality act on Chern-Simons theory?

non-local operators in 4d theory supported on two-dim surface $D \subset M_4$
e.g. $D = \gamma \times I$

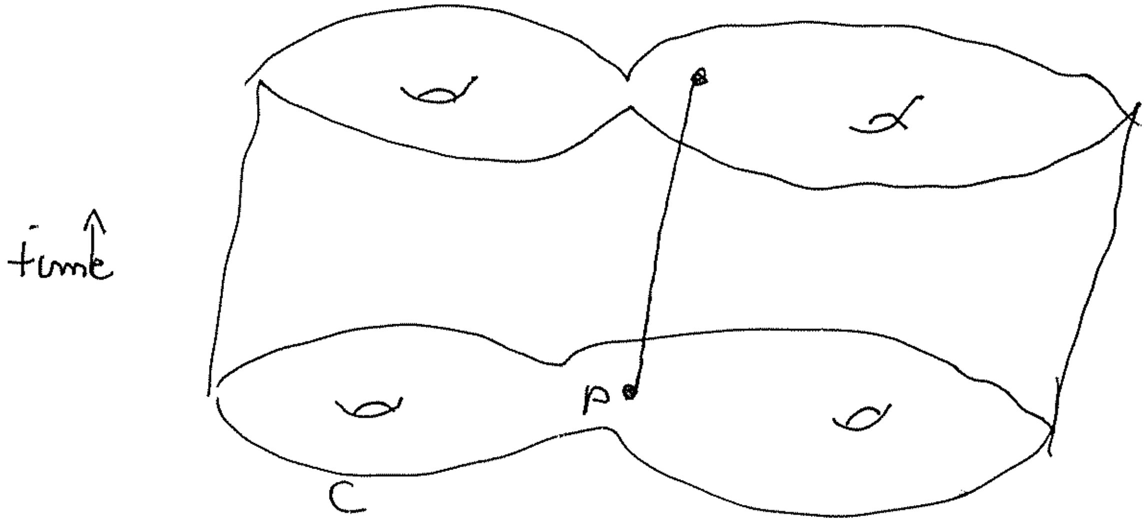


$\leadsto$ line operator supported on γ
Wilson line $W_R(\gamma)$
R rep. of G

non-local operators in 4d theory parameterized by data very different from representations
($\leadsto$ $(\alpha, \beta, \gamma, \eta)$ in work of Gukov/Witten '06)

Generalization:
take $W = \mathbb{R} \times C$ C: Riem surface
time ↑   ⟂⟂

and $\gamma = \mathbb{R} \times \{pt\}$



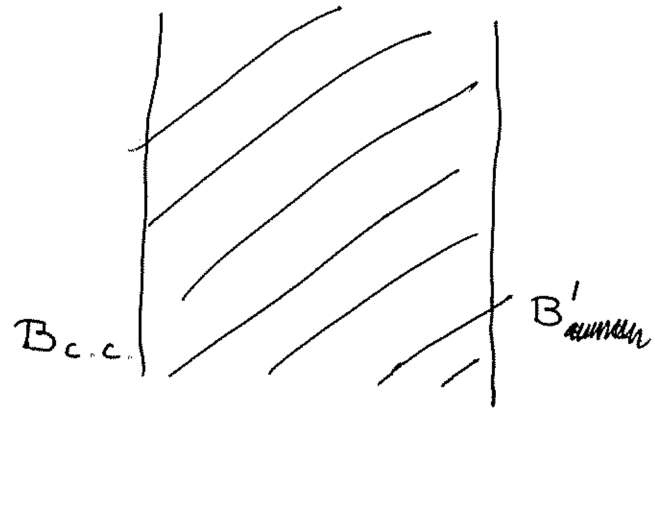
hamiltonian approach $\Rightarrow$ Hilbert space $\mathcal{H}$

replace C by disc w/ puncture at origin $\Rightarrow$ $\mathcal{H}$ = representation space

in 4d gauge theory $M = \Sigma \times C$

here $\Sigma = \mathbb{R} \times I$

time



4d gauge theory on $M = \Sigma \times C$

$\wr\wr$

2d topol. σ-model $\Sigma \rightarrow \mathcal{M}_H(G, C)$

moduli space of sol. to Hitchin equ. on C

$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{c.c.}, \mathcal{B}')$

= space of open string states between $\mathcal{B}_{c.c.}$ and $\mathcal{B}'$ on $\mathcal{M}_H$

for applications to representation theory $C = D^*$ (punctured disc) with a surface operator $\mathcal{M}_H \simeq T^*(G/\pi)$

Eg. for $G = SU(2)$ $\mathcal{M} = T^*S^2$

$\mathcal{B}' = \{\phi = 0\}$ = brane supported on $(G/\pi) \subset \mathcal{M}_H$

$\mathcal{B}_{c.c.}$ = canonical coisotropic brane

$\mathcal{M}_H = T^*(G/\pi)$ hyperkähler : I, J, K
Kähler-forms : $\omega_I, \omega_J, \omega_K$

σ-model could be A-model or B-model (related by mirror-sym) focus on A-model for $\omega = \omega_K$

topol. A-branes:

- Lagrangian A-branes on $\mathcal{M}_H$ ($\mathcal{B}'$ is such a Lag. A-brane)

- A-branes w/ curvature F of Chern-Paton bdl.

$$(F \cdot \omega^{-1})^2 = -\mathbb{1}$$

(⇝ Kapustin, Orl.)

$\mathcal{B}_{c.c.}$ = space-filling A-brane on $\mathcal{M}_H = T^*(G/T)$
$F = \omega_J$

in A-model on $\mathcal{M}_H = T^*(G/T)$
$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{c.c.}, \mathcal{B}')$

in this example (for these $\mathcal{B}_{c.c.}$ and $\mathcal{B}'$) $\mathcal{B}_{c.c.}$ and $\mathcal{B}'$ are both of type (A, B, A)

⟹ space of open string states
$\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{c.c.}, \mathcal{B}')$
= hol. sections of vector bundle on G/T
(Borel-Weil-Bott-theory)

in general, consider A_K-model (pick ω_K) on $\mathcal{M}_H = T^*(G/\mathbb{T})$

A_K-branes $\longleftrightarrow$ representations of $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$

<u>Remark</u>: $\mathcal{M}_H \cong T^*(G/\mathbb{T}) \simeq$ reg. coadjoint orbit of $G_{\mathbb{C}}$

G acts on $G/\mathbb{T} = SU(2)/U(1)$

all A_K-branes: too Large set

$\Rightarrow$ instead consider:

A-branes (Lagrangian) invariant under $K_{\mathbb{R}} \subseteq G_{\mathbb{R}}$

Ex: for $sl(2)$.

representations of $G_{\mathbb{R}} = Sl(2;\mathbb{R})$

$\mathcal{M}_H = T^*S^2$ A-branes under the maximal cpt $K_{\mathbb{R}} = SO(2)$

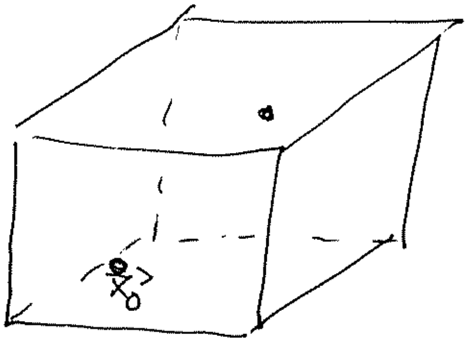
Gibbons-Hawking:

$$ds^2 = H \cdot (d\vec{x})^2 + H^{-1}(dx + \vec{A})$$

$$\vec{x} \in \mathbb{R}^3, \quad x \in [0; 2\pi)$$

$$H = \frac{1}{|\vec{x} - \vec{x_0}|} + \frac{1}{|\vec{x} + \vec{x_0}|}$$

$$\vec{\nabla} \times \vec{A} = \vec{\nabla} H$$



$\vec{x}_0 = (\alpha, \beta, \gamma)$

$$\vec{\omega} = (\omega_I, \omega_J, \omega_K)$$
$$= (d\chi + \vec{A}) \cdot d\vec{x} - \frac{1}{2} H \, d\vec{x} \times d\vec{x}$$

$\omega_K |_{B'} = 0$ iff contained in a plane transverse to z – direction (k – direction) $\mathbb{R}^2 \subset \mathbb{R}^3$

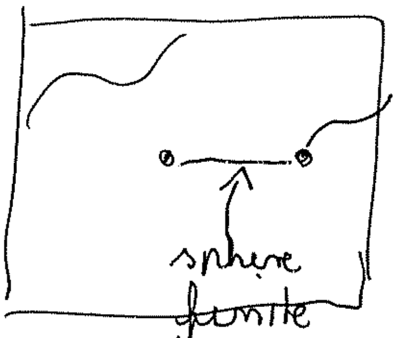
$\Rightarrow$ classification of $Sl(2;\mathbb{R})$ irreducible reps

$\Updownarrow$

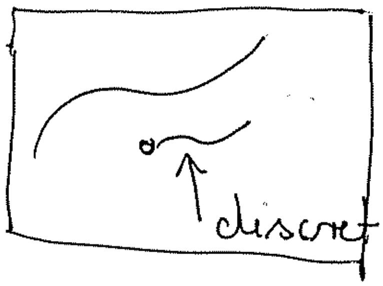
classifications of $K_{\mathbb{R}}$ – inv. A – branes

$\mathbb{R}^2$

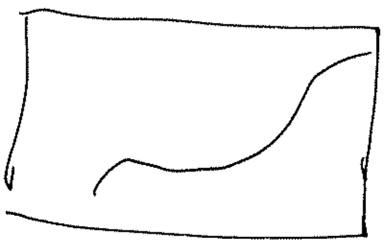
possibilities



2 centers
contained in plane



1 center
contained



no center

($\to$ principle series
representations)

obtained by quant.
of cylinder

(up to symplectomorph.)