Kapustin <u>topol. - holomorphic σ-models and duality</u> 10/12/07



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0710 2097 (w/ Natalia Saulina)

1) <u>Topological Twisting and TFT's in 4d</u>

SuSy 4d gauge theory on $M_4$

$\int \mathcal{D}A\, \mathcal{D}\lambda\, e^{-S/\hbar}$ : path integral

study limit $\hbar \to 0$

generically: no SuSy as there are no covariantly const. spinors

$\to$ need to modify the theory to preserve some fermionic sym.

$\rho$: Spin(4) $\to$ internal sym.

$\Rightarrow$ get a fermionic sym. $Q$

1) $Q$ preserves the integrand of path int.

2) $\{Q, Q\} = 0$

3) Can restrict observables to those which are $Q$-invariant

$$\int \mathcal{D}A\, \mathcal{D}\lambda\, e^{-S/\hbar}\, F(A, \lambda, \dots)$$

if $F = Q(f)$, $F$ is $Q$ invariant $\forall f$

$\Rightarrow$ observables live in $Q$-cohomology

$Q$: BRST operator

4) Such correlators are metric independent and often are independ of $\hbar$

$\rightsquigarrow$ action $S = \{Q, \cdot\} + \begin{pmatrix}\text{metric}\\ \text{indep.}\end{pmatrix}$

2) <u>Application</u>: $N=4$ SYM

$\mathrm{Spin}(4) \longrightarrow \mathrm{Spin}(6)$

three possible twists $\to$ "Donaldson" twist

$\to$ "Vafa-Witten" twist

$\to$ GL twist

$N=4$ SYM w/ gauge group $G$

$\xrightarrow{\text{conj.}} \mathbb{R}$ (electr. - mag. duality)

$N=4$ SYM w/ gauge group ${}^L G$

- Wilson loop observables $\mathrm{Tr}\, R\left(\mathrm{Hol}_{\gamma \subset M_4}(\nabla)\right)$
  irred. represent.

- Wilson loop observ.

- t' Hooft observ

- t' Hooft observabls labelled by ${}^L R$ and $\gamma \subset M_4$

- $M_4 = M_3 \times \mathbb{R}$ in this case
  it can be checked

$\rightsquigarrow$ leads to the statement of geom. Langlands duality

- other situation $M_4 = C \times \Sigma$
  $C, \Sigma$: Riem. surfaces,
  limit in which $C$ is small ($\mathrm{vol}\, C \to 0$)

$\rightsquigarrow$ effective theory on $\Sigma$ (TFT)
  $\sigma$-model on $\Sigma$:

$$\int \mathcal{D}\phi \; e^{-\cdots}$$

$$\phi \quad \Sigma \to \mathcal{M}_C$$

$\mathcal{M}_C$ → moduli space of vacua for 4d theory on $C \times \mathbb{R}^2$

$\rightsquigarrow$ 4d TFT on $C \times \Sigma$

$\quad\quad\quad\quad \wr\wr$

2d TFT on $\Sigma$ (depends on topol. of $C$)

* $N = 2$ SYM "with matter" 

Bosonic fields : $A$ : connection on a $G$-bdl $E$

$\phi \in \Gamma(\mathrm{ad}(E_{\mathbb{C}}))$

$q \in \Gamma(R(E_{\mathbb{C}}))$

$\tilde{q} \in \Gamma(R^{\vee}(E_{\mathbb{C}}))$

$\psi$ $R = \mathrm{ad}$ $\rightsquigarrow$ theory is $N=4$ SYM

( for $N=4$, $\beta$-fct vanishes $\rightsquigarrow$ finite theory (no diverg. in Greens-fcts))

$$\mathrm{tr}_R\, T^a T^b = \underset{\substack{\uparrow \\ \text{index of rep.}}}{C(R)}\, \delta^{ab}$$

$C(R) = C(\mathrm{ad}) \Longrightarrow$ finite $(\beta = 0)$

R-symmetry : $SU(2) \times U(1)$

- one way to turn this into topological theory (like the Donaldson-twist before)
- what about GL-twist ?

special mfld $M_4 = C \times \Sigma$

Holonomy is $U(1)_C \times U(1)_\Sigma$

"nice" twist : • $\mathcal{U}(1)_C$ is identified w/ $\mathcal{U}(1)_R \subset SU(2)$ (5)

• $\mathcal{U}(1)_\Sigma$ is identified w/ $\mathcal{U}(1)$ (in R-Sym.)

⇝ fermionic sym.

Bosonic fields : $\phi \in \Gamma(\mathrm{ad}(E_{\mathbb{C}}) \otimes K_\Sigma)$

$A$, $q \in \Gamma(R(E_{\mathbb{C}}) \otimes \overline{K_C})$

$\tilde{q} \in \Gamma(R^{\vee}(E_{\mathbb{C}}))$

( difference in fields $q, \tilde{q}$ : I have identified $\mathcal{U}(1)_C$ w/ $\mathcal{U}(1)_R \times \mathcal{U}(1)_{B}$ )

examine transformations under fermionic sym ⇝ • cplx str on $\Sigma$ does <u>not</u> matter

not a topol. field theory ← • cplx str on $C$ <u>does</u> matter

⇝ action : $S = \{Q, .\} + \begin{pmatrix}\text{indep of} \\ \text{Kähler forms} \\ \text{on } C \text{ or } \Sigma\end{pmatrix}$

how does observables $\mathcal{O}$ change in TFT $d\mathcal{O} = \{Q, \mathcal{O}\}$

here: $d_\Sigma \mathcal{O} = \{Q, \mathcal{O}\}$

$\bar{\partial}_C \mathcal{O} = \{Q, \mathcal{O}\}$

* take $\mathrm{vol}(\Sigma) \to 0$ $\Rightarrow$ get a holomorphic field theory in 2d on $C$

(2,0) $\sigma$-models: only anti-holom. symmetry
$T_{\bar{z}\bar{z}} = \{Q, \cdot\}$

$\rightsquigarrow$ twisting gives a holom. field theory

theory: $\Phi : C \to \mathcal{M}_\Sigma$

fermions in a vct bdl over $\mathcal{M}_\Sigma$

$\mathcal{M}_\Sigma$: Hitchin moduli space (hyperkähler mfld)

= moduli space of flat $G_{\mathbb{C}}$-connection on $\Sigma$

bundle: push-forward of universal flat bdl. and take associated vct-bdl ($\rightsquigarrow$ anomaly cancellation)

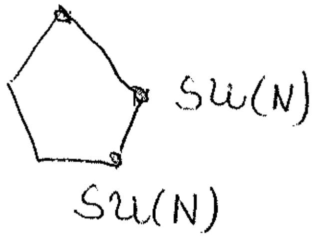
* $\mathrm{vol}(C) \to \infty \Rightarrow$ get a TFT on $\Sigma$
namely, the B-model
(because cplx str on $C$ enters)
of a target space
= "moduli space of generalized Higgs-bdls on $C$"

A gen. Higgs-bdl on $C$ is a pair $(E, \overline{q})$ w/ $E$: holom. $G_{\mathbb{C}}$-bdl
$\overline{q}$: holom section of $R(E) \otimes K_C$

anomaly cancellation of moduli space
$$c(ad) = c(R)$$

*) already in case $R = ad$ interesting.
What does the duality $G \leftrightarrow {}^LG$ say.
$$D^b(\mathrm{Coh}(\mathcal{M}_H(G, C)))$$
$$\cong D^b(\mathrm{Coh}(\mathcal{M}_H({}^LG, C)))$$

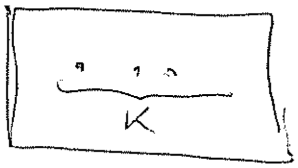
2) Quiver theories:
(labelled by Dynkin-diag.)




$$G = \underbrace{SU(N) \times \ldots \times SU(N)}_{K\text{-times}}$$

$$R = (N, \bar{N}, 1, \ldots)$$
$$+ (1, N, \bar{N}, 1, \ldots)$$
$$+ \ldots$$

$$\rightsquigarrow \quad c(R) = c(ad)$$




elliptic curves w/ K points