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The topology and geometry of the Seiberg-Witten Curves

1) Relevance of the SW curve for Donaldson theory
2) Jacobian rational elliptic surface
3) BPS states, 1/2 BPS states
4) DET $\bar{\partial}$

1) (M, g) Donaldson invariants

$$\phi : \mathrm{Sym}_*\left(H_0(M) \oplus H_2(M)\right) \to \mathbb{Q}$$

$M = \mathbb{CP}^2$

$$S = \int_\Sigma \omega \qquad \phi(p^n, S^m)$$

$$Z = \sum \frac{p^n}{n!}\frac{S^m}{m!}\phi(p^n S^m)$$

1998: Ellingsrud, Goettsche

$$= -\frac{3}{2}S + S^5 - pS^3 - \frac{13}{8}p^2 S + \cdots$$

1988, Witten: $\underline{\Phi}$ = expectation values of twisted $N=2$ SUSY YM-theory

1994, Seiberg-Witten:

$$Z = \underbrace{Z_u}_{\text{u-plane integral}} + Z_{SW}$$

$N=2$ abelian theory (= low-energy effective field theory)

for CP^2: $Z_{SW} = 0$, $Z = Z_u$.

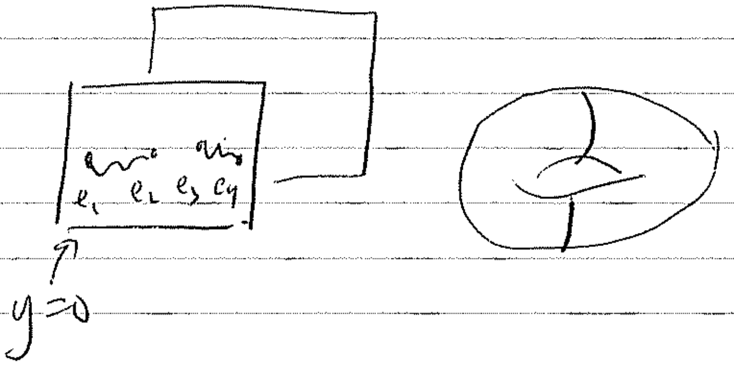
1996: Moore-Witten: SW-curve

1987: Atiyah: The logarithm of η-fct
1998: Freed: Special Kähler geometry
1988: Bismut-Bost
Seeley-Singer

2) Jacobian elliptic surface

$$E: \quad y^2 = x^3 + Ax + B$$

$$\Delta = 4A^3 + 27B^2 \neq 0$$



$P = $ pt at ∞.

E_u: $u \in \mathbb{C}$, $[u:1] \in \mathbb{CP}^1$

$$A(u) = -\frac{u^2}{3} + \frac{1}{4}, \quad B(u) = \frac{u}{12} - \frac{2}{27} u^3.$$

$P_u = $ pt at ∞

$$\Delta \sim (u^2 - 1)$$

Jacobian rational elliptic surface

$$e_{1,2} = \frac{u}{6} \pm \frac{\sqrt{u^2-1}}{2}$$

$$e_3 = -\frac{u}{3} \qquad e_4 = \infty$$

go around $u = \pm 1$, e_1, e_2 are interchanged

$u = \pm 1$ node I_1 $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 1950's Kodaira

$u = \infty$ cusp I_4^* $-T^4$

$H_1(E_u)$, $\mathcal{A}_u = \{\alpha_u, \beta_u\}$ $\alpha \cdot \alpha = \beta \cdot \beta = 0$

$\alpha \cdot \beta = 1$

$M \in SL(2, \mathbb{Z})$

$$\pi_1\Big(\mathbb{CP}^1 \setminus \bigcup_{u \in \{\pm 1, \infty\}} B_\varepsilon(u)\Big) \to SL(2, \mathbb{Z}).$$

Monodromy in this example: $\in \Gamma_0(4)$.

Let $Z \to UP$ be the total space $UP = \mathbb{CP}^1 \setminus \bigcup_{u \in \{\pm 1, \infty\}} B_\varepsilon(u)$

Meyer, 1964:

$$\mathrm{Sign}(Z) = \ldots$$

Meyer class function $\Phi : SL(2, \mathbb{Z}) \to \mathbb{Q}$.

$$\Phi(T) = +2/3$$

$$\Phi(-T^k) = -k/3.$$

$$\mathrm{Sign}(Z) = -\sum_{u \in \{\pm 1, \infty\}} (M_u)$$

$$= -2 \cdot \frac{2}{3} + \frac{4}{3} = 0.$$

(5)

- if you add N_f massive multiplets, get

N_f+2 nodes I_1 $\qquad T$

1 cusp $I^*_{4-N_f}$ $\qquad T^{4-N_f}$

$$\mathrm{Sign}(Z) = (N_f+2)\left(-\frac{2}{3}\right) + \left(\frac{4-N_f}{3}\right) = -N_f$$

$$\begin{array}{ccc} Z & \longleftarrow & \coprod \left(\tau, z \in \mathbb{C}/\langle 2w, 2w' \rangle\right) \\ \downarrow & & \\ UP & & \tau \in H/\Gamma_0(4) \end{array}$$

$x = \wp(z, w, w')$
$y = \wp'(z, w, w')$

algebraic integrable system

1) Z complex symplectic manifold,
$\eta = du \wedge dz \qquad \Omega^{(2,0)}(Z)$

2) E_u, Lagrangian submanifolds

3) $dz \wedge d\bar{z}$: positive polarization

2) BPS-states

$\mathcal{L} \to E$
$= \mathcal{O}(Q-P)$

family $\rightsquigarrow$ section Q_u of the elliptic fibration

$Z = n_e\,\omega + n_m\,\omega'$
$\Omega = \pi i\,(n_m a - n_e b)$

- $\eta^{(2,0)} : \Gamma(\omega_{Z/UP}) \to \Gamma(\omega_{UP}^{-1})$
- $\Omega = \int_{E_u} \eta \wedge \bar{\eta} = dp \wedge dq \qquad \nabla\left(\frac{\partial}{\partial p}\right) = \nabla\left(\frac{\partial}{\partial q}\right) = 0$

(UP, Ω, ∇) special Kähler structure

$\{\text{flat real section of } TUP\} = \text{classical BPS states}$
$= \{ s \mid s = n_m \frac{\partial}{\partial p} + n_e \frac{\partial}{\partial q} \}$

$\nabla s = 0 \rightsquigarrow n_m, n_e$ constants check $\Sigma \to UP$, curvature $= -2\pi i \Omega$

(construction of meromorphic SW 1-form)

quantized states takes values in charge lattice

$\Omega(S, S') \in \mathbb{Z}$. (Dirac quant.)

$\Omega = dp \wedge dq$

$\Omega(S, S') = n_m n_e' - n_e n_m'$.

$Z = n_e \omega + n_m \omega'$

$Z = \omega$ go around $u = +1$ $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega' \\ \omega \end{pmatrix}$

$Z = \omega'$ is invariant. $\omega \to \omega' + \omega$ not invariant

Well-defined quantized states: $Z = 2l\omega + (1+2k)\omega'$

(semi classical stable BPS states)

$\omega = \dfrac{da}{du}$, $\omega' = \dfrac{da_D}{du}$.

$Z = \partial_u Z$, $Z = 2la + (1+2k)a_D$ Central charge

4) Determinant line bundles

$E_u \quad \bar\partial : \Omega^0(E_u) \to \Omega^{0,1}(E_u)$

$$\mathrm{DET}\ \bar\partial = (\mathrm{Ker}\ \bar\partial)^{-1} \otimes (\mathrm{coker}\ \bar\partial)$$

Coker = dual to $H^{(1,0)}(E_u)$.

Quillen: $\|\sigma\|_Q^2 = \det(\bar\partial^* \bar\partial)\, \|\sigma\|_{L^2}^2$

$$\sigma = (1 \otimes dz)^{-1}$$

$$\|\sigma\|_Q^2 = |\Delta|^{2/12}$$

$$\Omega = \partial\bar\partial \log \|\sigma\|_Q^2 = 0$$

(local anomaly vanishes)

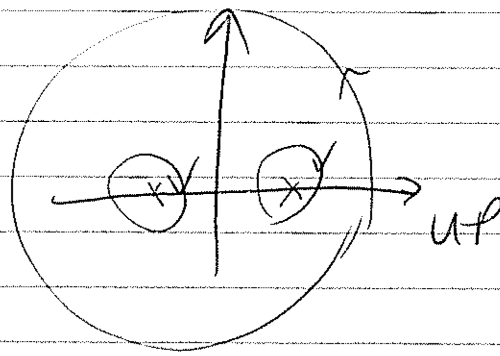
$\sigma \longrightarrow du \longrightarrow$ canonical trivialization of the det line bundle

$$\Delta = (u-1)(u+1)$$

$\sigma \to e^{\pi i \frac{\eta^0}{2}}\, \sigma$ around a singular fiber

$$\eta^0(u = \pm 1) = 1/3$$

$$\eta^0(u = \infty) = -2/3$$



∂ signature quotient

$(2 \text{ copies of } \bar{\partial})$

$$\eta^0_\infty(u = \pm 1) = 2/3 = \Phi(M_{\pm 1})$$

$$\eta^0_\infty(u = \infty) = -4/3 = \Phi(M_\infty)$$

extended DFT∂ across $u = \pm 1$, $\Omega = -\frac{1}{6}\,\delta(u=+1) - \frac{1}{6}\,\delta(u=-1)$

$$\mathrm{sign}(Z) = -\sum \Phi(M_i)$$
$$= \underbrace{\int_Z \frac{c_1(Z)^2}{2}}_{0} - \eta(\partial Z)$$
$$= -\eta(\partial Z)$$

$$\partial Z = \bigcup_{i=1}^{3} W_i$$

$$W_i \xrightarrow{\pi^2} S^1, \ M_i$$

$$\mathcal{D} = d* - *d : C^\infty(W) \oplus C^2(M) \circlearrowright$$

$$\{\lambda_j\}, \ \{-\mu_j\}$$

$$\eta(s) = \sum \lambda_j^{-2s} - \mu_j^{-2s}$$

$\eta(0)$ = spectral invariants

Z_u = partition function of nonlinear heterotic σ-model

E_u at $\infty \to S^1 \times \mathbb{CP}^1$