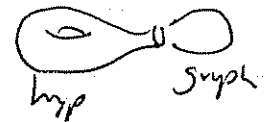


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Dimensional reduction and the long-time behavior of Ricci flow

M^3 compact $(M, g(\cdot)) \quad \frac{dg}{dT} = -2Ric$

Put $\hat{g}(T) = \frac{g(T)}{T}$

Perelman: for large T , $(M, \hat{g}(T)) \sim$ 

In known examples

1. finite # of surgeries

2. After surgery, $\|Rm\|_\infty = O(\frac{1}{T})$

What to expect for

$$X \stackrel{G-H}{=} \lim_{T \rightarrow \infty} (M, \hat{g}(T))$$

(Gromov-Hausdorff limit)

	X	topological Thurston type
M^3 $\sim O(T^{1/2})$	pt.	\mathbb{R}^3, Nil
	S^1 or $[0,1]$	Sol
	compact 2-orbifold, $\chi(X) < 0$	$H^2 \times \mathbb{R}, SL_2(\mathbb{R})$
	compact 3-dim ^{nl} hyperbolic noncompact limit	H^3

H^3 / Γ joined to H^2 / Γ_2
 $H^3, H^2 \times \mathbb{R}, SL_2(\mathbb{R})$ plane

Some expanding solutions on \mathbb{R}^3 :

- 1. \mathbb{R}^3 , g_{flat}
- 2. \mathbb{H}^3 , $g(t) = 4t g_{hyp}$
- 3. $\mathbb{H}^2 \times \mathbb{R}$, $g(t) = 2t g_{hyp} + g_{\mathbb{R}}$
- 4. $e^{-2z} dx^2 + e^{2z} dy^2 + 4t dz^2$, Sol.
- 5. $\frac{1}{3} \frac{1}{t^{1/3}} (dx + \frac{1}{2} y dz - \frac{1}{2} z dy)^2 + t^{1/3} (dy^2 + dz^2)$ Nil.

Def $g_s(t) = \frac{1}{s} g(st)$ is also a Ricci flow solution

Thm Suppose $(M^3, g(\cdot))$ is a Ricci flow solution on a compact 3-manifold, defined for $t \in [1, \infty)$

Suppose $\|R_m\|_{\infty} = O(T^{-1})$
and $diam = O(\sqrt{T})$

Thm $\lim_{s \rightarrow \infty} \tilde{g}_s(\cdot)$ exists on the universal cover \tilde{M}

and is one of the previous solutions, if M has topology type \mathbb{R}^3 , \mathbb{H}^3 or Nil

Also true for ~~Sol and Nil~~ if the $\mathbb{H}^2 \times \mathbb{R}$ and Sol
 $\mathbb{H}^2 \times \mathbb{R}$, $SU(2)$,
Sol

expanders are "stable".

Dimensional Reduction

M Principal \mathbb{R}^N bundle over B^n compact.
 $\pi \downarrow$

B Let \bar{g} be an \mathbb{R}^N -invariant metric on M .

$x^i, x^j =$ coords in \mathbb{R}^N direction
 $x^\alpha, x^\beta =$ coords in B direction.

$$\bar{g} = G_{ij} (dx^i + A^i)(dx^j + A^j) + g_{\alpha\beta} dx^\alpha dx^\beta$$

$$G_{ij} = G_{ij}(b), \quad A^i = A^i_\alpha(b) dx^\alpha$$

$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha$$

Ricci flow equation

$$\begin{aligned} \frac{\partial}{\partial t} G_{ij} &= g^{\alpha\beta} G_{ij;\alpha\beta} + \frac{1}{2} g^{\alpha\beta} C^{kl} G_{klj\alpha} G_{i\beta} \\ &\quad - g^{\alpha\beta} C^{kl} G_{ik,\alpha} G_{j\beta} \\ &\quad - \frac{1}{2} g^{\alpha\delta} g^{\beta\gamma} G_{ik} G_{jl} F^k_{\alpha\beta} F^l_{\gamma\delta} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} A^i_\alpha &= -g^{\alpha\delta} F^i_{\alpha\gamma;\delta} - g^{\alpha\delta} C^{ij} G_{jk,\alpha} F^k_{\alpha\delta} \\ &\quad - \frac{1}{2} g^{\alpha\delta} C^{kl} G_{kl,\alpha} F^i_{\alpha\delta} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\beta} &= -2R_{\alpha\beta} + C^{ij} G_{ij;\alpha\beta} - \frac{1}{2} C^{ij} G_{jk,\alpha} C^{kl} G_{l\alpha\beta} \\ &\quad + g^{\gamma\delta} G_{ij} F^i_{\alpha\gamma} F^j_{\beta\delta} \end{aligned}$$

Follow Perelman to modify these equations:

Define $\mathcal{F}(G, A, g, f) =$

$$\int_B \left[|\nabla f|^2 + R - \frac{1}{4} g^{\alpha\beta} G^{ij} G_{ij, \alpha} G^{kl} G_{kl, \beta} - \frac{1}{4} g^{\alpha\beta} g^{\rho\delta} G_{ij} F_{\alpha\beta}^i F_{\rho\delta}^j \right] e^{-f} \text{dvol}$$

$$W(G, A, g, f, \tau) = (4\pi\tau)^{-\frac{n}{2}} \left[\tau \mathcal{F} + \int_B (f-n) e^{-f} \text{dvol} \right]$$

$$W_+(G, A, g, f, T) = (4\pi T)^{-\frac{n}{2}} \left[T \mathcal{F} - \int_B (f-n) e^{-f} \text{dvol} \right]$$

(Defined, in original case, by —, Ilmanen + Lee)

Thm If e^{-f} satisfies a conjugate Heat equation

then $\frac{dW_+}{dT} \geq 0$. If W_+ is constant in T ,

then $F_{\alpha\beta}^i \equiv 0$, $\det(G_{ij})$ is constant, and

$$(*) \begin{cases} g^{\alpha\beta} G_{ij, \alpha\beta} - g^{\alpha\beta} G^{kl} G_{kl, \alpha} G_{\beta\gamma, \delta} = 0 \\ R_{\alpha\beta} - \frac{1}{4} G^{ij} G_{jkt, \alpha} G^{kl} G_{\alpha, \beta} + \frac{1}{2T} g_{\alpha\beta} = 0 \end{cases}$$

i.e.

$G: B \rightarrow \text{SL}(n, \mathbb{R}) / \text{SO}(n, \mathbb{R})$ is harmonic

and the Einstein eqn is modified by the energy of this harmonic map

Thm If there is a blowdown limit

$$\bar{g}_\infty(\cdot) = \lim_{j \rightarrow \infty} \bar{g}_{S_j}(\cdot)$$

then it is a sphere (S^2).

<u>Example</u>	N	$\dim(B)$	
	0	3	H^3/Γ expander
	1	2	$H^2/\Gamma \times \mathbb{R}$ expander
	2	1	Solo
	3	0	\mathbb{R}^3 , flat

Example of collapsing



$M = S^1 \times \Sigma$, unit sphere bundle

has an $SO_2(\mathbb{R})$ structure

$$\pi: M \rightarrow \Sigma$$

$$\hat{g}(t) = \frac{g(t)}{t}$$

with limit $\lim_{t \rightarrow \infty} (M^3, \hat{g}(t)) \stackrel{Gromov}{=} \Sigma$

Take $B \subset \Sigma$ a small ball

$\pi^{-1}(B) \subset M$, topologically $B \times S^1$

$\widetilde{\pi^{-1}(B)}$ is topologically $B \times \mathbb{R}$

$$\lim_{t \rightarrow \infty} (\pi^{-1}(B), \widetilde{\hat{g}}(t)) = B \times \mathbb{R} \text{ isometrically.}$$

$$\lim_{t \rightarrow \infty} (\pi^{-1}(B), \tilde{g}(t)) = (B \times \mathbb{R}) / \mathbb{R}$$

$$\lim_{t \rightarrow \infty} (M, \tilde{g}(t)) = \Sigma \times \mathbb{R} / \mathbb{R}$$

(NB - in response to question - running Ricci flow on $SL_2(\mathbb{R})$ itself, it straightens out to $H \times \mathbb{R}$.)

Thm Say $\{M_i^n, P_i, g_i(\cdot)\}_{i=1}^{\infty}$ are pointed Ricci flow solutions. Say $\exists -\infty \leq A < 0$ and $0 \leq \Omega \leq \infty$ s.t.

1. $g_i(t)$ is defined for $T \in (A, \Omega)$ and is complete

2. \forall compact intervals $I \subseteq (A, \Omega) \exists K_I < \infty$ s.t.

$$\| \text{Rm}(g_i(t)) \|_{\infty} < K_I \text{ whenever } t \in I$$

Then a subsequence converges to a Ricci flow solution on a pointed n -dimensional étale groupoid.

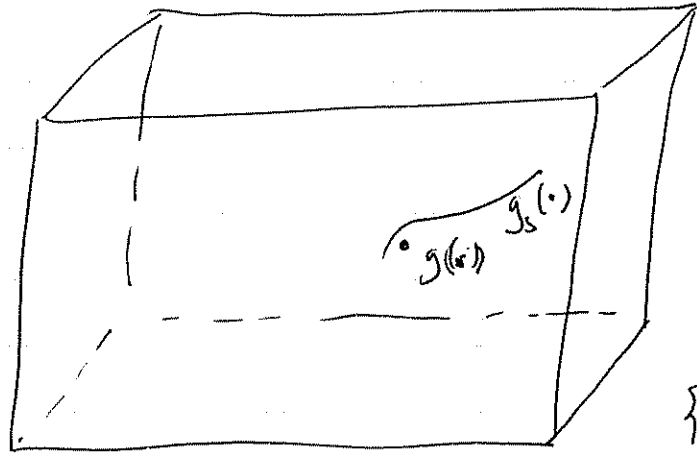
(earlier work: Perelman-Tuschmann, Flickenstein)

Cor $\forall K < \infty$, get precompactness of n -dim'l pointed Ricci flow solutions on $(1, \infty)$ with

$$\| \text{Rm}(g(t)) \|_{\infty} \leq \frac{K}{T}.$$

("Type III Ricci flow solution")

For $s \geq 1$, $g(s)$ get another solution $g_s(t) = \frac{1}{s} g(st)$.
Same curvature bound



{ 3-D Ricci flow solution
with $|Rm| \leq \frac{K}{t}$
 $D_{min} \leq D_0(t)$

condition starts: Ricci flow solution with g local symmetries

eg. local \mathbb{R} -symmetry on face

local \mathbb{R}^2 -symmetry on edge

local \mathbb{R}^3 or Nil symmetry at vertex

If limit $s \rightarrow \infty$ stays away from boundary, then have a non-collapsed limit: hyperbolic.

Claim: If it approaches boundary, it does so uniformly.
(uses stability of hyperbolic solution)

There is an ϵ in face which this breaks \rightarrow get the same analysis for face $\rightarrow \mathbb{H}^3$ or \mathbb{R}^3 or, if to edge, get Sol.

Current \mathbb{R}^3 or Nil.