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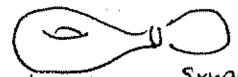
<u>Dimensional reduction and the long-time behavior of Ricci flow</u>

$M^3$ compact $\qquad (M, g(\cdot)) \qquad \frac{dg}{dT} = -2\,Ric$

Put $\hat{g}(T) = \frac{g(T)}{T}$

Perelman: for large $T$, $\quad (M, \hat{g}(T)) \sim$ 



In known examples
1. Finite # of surgeries
2. After surgeries, $\|Rm\|_\infty = O(\frac{1}{T})$

What to expect for

$$X \overset{GH}{=} \lim_{T \to \infty} (M, \hat{g}(T))$$

(Gromov-Hausdorff limit)

diam $\sim O(T^{1/2})$

| $X$ | Thurston topological type |
|---|---|
| pt. | $\mathbb{R}^3$, Nil |
| $S^1$ or $[0,1]$ | Sol |
| compact 2-orbifold, $\chi(X) < 0$ | $H^2 \times \mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$ |
| compact 3-dim'l hyperbolic non-compact limit | $H^3$ |

$H^3/\Gamma_1$ joined to $H^2/\Gamma_2$
$H^3$, $H^2 \times \mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$ pieces

Some expanding solitons on $\mathbb{R}^3$.

1. $\mathbb{R}^3$, $g_{flat}$
2. $\mathbb{H}^3$, $g(t) = 4t\, g_{hyp}$
3. $\mathbb{H}^2 \times \mathbb{R}$, $g(t) = 2t\, g_{hyp} + g_{\mathbb{R}}$
4. $e^{-2z} dx^2 + e^{2z} dy^2 + 4t\, dz^2$, Sol.
5. $\frac{1}{3t^{1/3}} \left( dx + \frac{1}{2} y\, dz - \frac{1}{2} z\, dy \right)^2 + t^{1/3} (dy^2 + dz^2)$ Nil.

Def $g_s(t) = \frac{1}{s} g(st)$ is also a Ricci flow solution

Thm Suppose $(M^3, g(\cdot))$ is a Ricci flow solution on a compact 3-manifold, defined for $t \in [1, \infty)$

Suppose $\| Rm \|_\infty = O(T^{-1})$

and $\mathrm{diam} = O(\sqrt{T})$

Then $\lim_{s \to \infty} \tilde{g}_s(\cdot)$ exists on the universal cover $\tilde{M}$

and is one of the previous solutions, if $M$ has topological type $\mathbb{R}^3$, $\mathbb{H}^3$ or Nil

Also true for ~~Sol and Nil~~ if the $\mathbb{H}^2 \times \mathbb{R}$ and Sol
$\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$, Sol

expanders are "stable".

## Dimensional Reduction

$$\begin{array}{c} M \\ \pi \downarrow \\ B \end{array}$$

Principal $\mathbb{R}^N$ bundle over $B^n$ compact.

Let $\bar{g}$ be an $\mathbb{R}^N$-invariant metric on $M$.

$x^i, x^j$ = coords in $\mathbb{R}^N$ direction

$x^\alpha, x^\beta$ = coords in $B$ direction.

$$\bar{g} = G_{ij}\left(dx^i + A^i\right)\left(dx^j + A^j\right) + g_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$G_{ij} = G_{ij}(b), \quad A^i = A^i_\alpha(b)\, dx^\alpha.$$

$$F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha$$

Ricci flow equation

$$\begin{aligned} \frac{\partial}{\partial t} G_{ij} = {} & g^{\alpha\beta} G_{ij;\alpha\beta} + \frac{1}{2} g^{\alpha\beta} G^{kl} G_{kl,\alpha} G_{ij,\beta} \\ & - g^{\alpha\beta} G^{kl} G_{ik,\alpha} G_{lj,\beta} \\ & - \frac{1}{2} g^{\alpha\gamma} g^{\beta\delta} G_{ik} G_{jl} F^k_{\alpha\beta} F^l_{\gamma\delta} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} A^i_\alpha = {} & -g^{\alpha\delta} F^i_{\alpha\gamma;\delta} - g^{\alpha\delta} G^{ij} G_{jk,\gamma} F^k_{\alpha\gamma} \\ & - \frac{1}{2} g^{\alpha\delta} G^{kl} G_{kl,\gamma} F^i_{\alpha\delta} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial t} g_{\alpha\beta} = {} & -2R_{\alpha\beta} + G^{ij} G_{ij;\alpha\beta} - \frac{1}{2} G^{ij} G_{jk,\alpha} G^{kl} G_{li,\beta} \\ & + g^{\gamma\delta} G_{ij} F^i_{\alpha\gamma} F^j_{\beta\delta} \end{aligned}$$

Follow Perelman to modify these equations:

Define $\mathcal{F}(G,A,g,f) =$

$$\int_B \left[ |\nabla f|^2 + R - \frac{1}{4} g^{\alpha\beta} G^{ij} G_{jk,\alpha} G^{kl} G_{li,\beta} - \frac{1}{4} g^{\alpha\gamma} g^{\beta\delta} G_{ij} F^i_{\alpha\beta} F^j_{\gamma\delta} \right] e^{-f} \, dvol$$

$$\mathcal{W}(G,A,g,f,\tau) = (4\pi\tau)^{-\frac{n}{2}} \left[ \tau \mathcal{F} + \int_B (f-n) e^{-f} \, dvol \right]$$

$$W_+(G,A,g,f,T) = (4\pi T)^{\frac{n}{2}} \left[ T\mathcal{F} - \int_B (f-n) e^{-f} \, dvol \right]$$

(Defined, in original case, by —, Ilmanen + Lee).

<u>Thm</u> If $e^{-f}$ satisfies a conjugate Heat equations

then $\frac{dW_+}{dT} \geq 0$. If $W_+$ is constant in $T$,

then $F^i_{\alpha\beta} \equiv 0$, $\det(G_{ij})$ is constant, and

$$(*) \begin{cases} g^{\alpha\beta} G_{ij;\alpha\beta} - g^{\alpha\beta} G^{kl} G_{ik,\alpha} G_{lj,\beta} = 0 \\ R_{\alpha\beta} - \frac{1}{4} G^{ij} G_{jk,\alpha} G^{kl} G_{li,\beta} + \frac{1}{2T} g_{\alpha\beta} = 0 \end{cases}$$

i.e. $G: B \to SL(n,\mathbb{R})/SO(n,\mathbb{R})$ is harmonic

and the Einstein eqn is modified by the energy of this harmonic map

Thm If there is a blowdown limit

$$\bar{g}_\infty(\cdot) = \lim_{j\to\infty} \bar{g}_{s_j}(\cdot)$$

then it satisfies (*).

Example

| N | $\dim(B)$ | |
|---|---|---|
| 0 | 3 | $\mathbb{H}^3/\Gamma$ expander |
| 1 | 2 | $\mathbb{H}^2/\Gamma \times \mathbb{R}$ expander |
| 2 | 1 | Sol |
| 3 | 0 | $\mathbb{R}^3$, flat |

Example of collapsing

$\Sigma =$ 

$M = S^1\Sigma$, unit sphere bundle has an $\widetilde{SL_2(\mathbb{R})}$ structure

$\pi : M \to \Sigma$

$$\hat{g}(t) = \frac{g(t)}{t}.$$

GH limit:

$$\lim_{t\to\infty} (M^3, \hat{g}(t)) \overset{GH}{=} \Sigma$$

Take $B \subseteq \Sigma$ a small ball

$\pi^{-1}(B) \subseteq M$, topologically $B \times S^1$

$\widetilde{\pi^{-1}(B)}$ is topologically $B \times \mathbb{R}$

$$\lim_{t\to\infty} \left(\widetilde{\pi^{-1}(B)}, \widetilde{\hat{g}(t)}\right) = B \times \mathbb{R} \text{ isometrically}$$

$$\lim_{t\to\infty} (\pi^{-1}(B), \hat{g}(t)) = (B\times\mathbb{R})/\mathbb{R}$$

$$\lim_{t\to\infty} (M, \hat{g}(t)) = \Sigma\times\mathbb{R}/\mathbb{R}$$

(NB - in response to question - running ~~the~~ Ricci flow on $SL_2(\mathbb{R})$ itself, it straightens out to $H\times\mathbb{R}$).

Thm Say $\{M_i^n, p_i, g_i(\cdot)\}_{i=1}^{\infty}$ are pointed Ricci flow solutions. Say $\exists$ $-\infty \le A < 0$ and $0 \le \Omega \le \infty$ s.t.

1. $g_i(t)$ is defined for $t\in(A,\Omega)$ and is complete
2. $\forall$ compact intervals $I\subseteq(A,\Omega)$ $\exists$ $K_I<\infty$ s.t.

$$\|Rm(g_i(t))\|_\infty < K_I \text{ whenever } t\in I$$

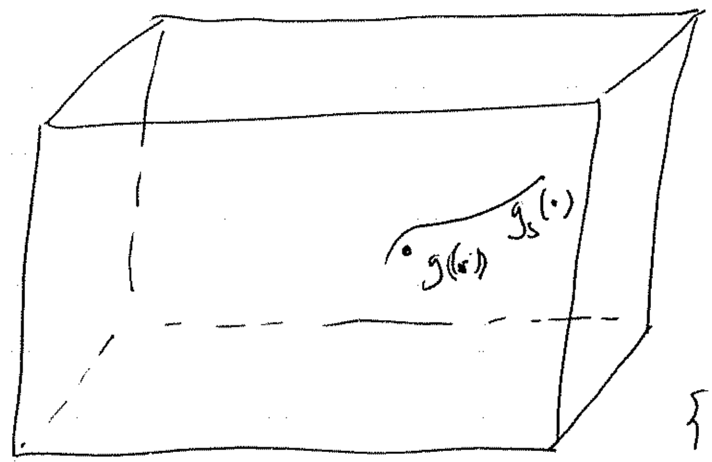
Then a subsequence converges to a Ricci flow solution on a pointed $n$-dimensional étale groupoid.

(earlier work: Petrunin-Tuschmann, Fleckenstein).

Cor $\forall$ $K<\infty$, get precompactness of $n$-dim'l pointed Ricci flow solutions on $(1,\infty)$ with

$$\|Rm(g(t))\|_\infty \le \frac{K}{t}.$$ ("Type III Ricci flow solution")

For $s \geq 1$, $g(\cdot)$ get another solution $g_s(t) = \frac{1}{s} g(st)$.
same curvature scale




{3-D Ricci flow solution with $|Rm| \leq \frac{k}{t}$, $D_{vim} \leq D_1(1)$

Cooling strata: Ricci flow solution with ≥ local symmetries

e.g. local $\mathbb{R}$-symmetry on face
local $\mathbb{R}^2$-symmetry on edge
local $\mathbb{R}^3$ or Nil symmetry at vertex

If limit $s \to \infty$ stays away from boundary, then have a non-collapsed limit: hyperbolic.

Claim: If it approaches boundary, it does so uniformly.
(uses stability of hyperbolic solution)

There is an orbit in face which this tracks → get the same analysis for face → $H^2 \times \mathbb{R}$ or, if to edge, get Sol.

Corner: $\mathbb{R}^3$ or Nil.