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The Cross-Curvature Flow on Locally Homogeneous 3-manifolds

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Outline

- Notation
- Intro to cross curvature flow (XCF)
- Intro to locally homog. 3-manifolds
- XCF on locally homog 3-manifolds

Notation

M = closed Riemannian manifold of $\dim M = 3$

$g = \sum g_{ij}\, dx^i \otimes dx^j$ Riemannian metric

g^{ij} = inverse of g_{ij}

R_{ij} = Ricci curvature

R = Scalar curvature $= g^{ij} R_{ij}$

$E_{ij} \colon E_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$

(Einstein tensor) $P^{ij} = g^{ik} E_{kl} g^{jl}$

V_{ij} = inverse of P^{ij}

X_{ij} = cross curvature tensor $= \left(\frac{\det P^{kl}}{\det g^{kl}}\right) V_{ij}$

$X = g^{ij} X_{ij}$ trace of X_{ij}

f_i = orthogonal frame

e_i = orthonormal frame.

Assume orthonormal frame s.t. Ricci and cross are both diagonal. $R_{2323} = k_1$, $R_{3131} = k_2$, $R_{1212} = k_3$

$\Rightarrow R_{11} = k_2 + k_3$ etc.

$$(X_{ij}) = \begin{pmatrix} k_2 k_3 & & \\ & k_1 k_3 & \\ & & k_1 k_2 \end{pmatrix}$$

(works even with vanishing sectional curvature.)

Cross-curvature flow:

$$\frac{\partial}{\partial t} g_{ij} = -2X_{ij}$$

if sectional curvature is positive

$$\frac{\partial}{\partial t} g_{ij} = 2X_{ij}$$

if sectional curvature is negative.

Normalized

Rescale in space: $\tilde{g} = \psi g$ fix volume

Reparametrize time: $\tilde{t} = \int \psi^2(t)\,dt$

NXCF (normalized cross curvature flow):

$$\frac{\partial}{\partial t} g_{ij} = -2X_{ij} + \frac{2}{3}\bar{X} g_{ij}$$

average of X

if sectional curvature is positive

$$\frac{\partial}{\partial t} g_{ij} = 2X_{ij} - \frac{2}{3}\bar{X} g_{ij}$$

if sectional curvature is negative

$$\bar{X} = \frac{\int X\,dv}{\int dv}.$$

Steady solutions:

For XCF: flat metric or product metric

for NXCF: Xinstein $X_{ij} = \frac{1}{3} X g_{ij}$

Xinstein must be Einstein.

Compact gradient solution:

$$X_{ij} + f_{ij} = c g_{ij}$$

X pos-def, $c > 0$.

Cross curvature flow and Ricci flow

Lemma: (M^3, g) positive or negative sectional curvature

$\Rightarrow (X_{ij}) > 0$ and identity map is harmonic $(M, X) \to (M, g)$.

Lemma (M^3, g) positive or negative Ricci curvature,

then $(M, g) \to (M, \pm Rc)$

is harmonic.

Short time existence

<u>Thm</u> (Chow-Hamilton 2004, Buckland 2005)
(M, g) negative or positive sectional curvature, (dim = 3).
Then for any initial metric a solution to cross-curvature flow exists for a short time.

<u>Conjecture</u> (Chow - Hamilton)
NXCF exists for all time, converges to constant sectional curvature as $t \to \infty$.

Positive curvature

Hamilton, 1982, dim 3, pos. Ricci curvature
$\Rightarrow$ diffeo to a standard sphere

Klingenberg - Rauch - Berger, 1960 : $\frac{1}{4} < K \leq 1$

$\Rightarrow$ homeomorphic to sphere.

Hamilton 1986, dim 4, if $\exists$ metric with positive curvature operator then diffeo to a standard sphere.

Bohm-Wilking 2006: any compact Riemann manifold with positive curvature operator must be a space form.

Negative curvature

Gao-Yau 1986: every closed 3-manifold admits a metric with negative Ricci curvature.

Lohkamp 1994: every closed n-manifold admits metric with negative Ricci curvature for all $n \geq 3$.

Problem Is the negativity of sectional curvature preserved under the cross curvature flow?

Def
$$\mathrm{vol}(E) = \int_M \sqrt{\frac{\det P^{ij}}{\det g^{ij}}}\, d\mu.$$

is scale invariant.

Prop $\mathrm{vol}(E)$ is nondecreasing under cross curvature flow, for negative sectional curvature

Thm (Chow - Hamilton 2004)

$(M^3, g(t))$, $t \in [0,T)$ solution to XCF

where $g(0)$ has negative sectional curvature

if $T < \infty$ and $\inf_{M \times [0,T)} \frac{\det \rho_{ij}}{\det g_{ij}} = 0$

Then $g(t)$ has negative sectional curvature for $t \in [0,T)$

and $\exists$ sequence (x_i, t_i) with $t_i \to T$

$|\lambda_1(x)| \to \infty$, $|\lambda_3(x)| \to 0$

$\lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x)$

eigenvalues of Einstein tensor.

<u>Geometry of locally homogeneous 3-manifolds</u>

<u>Def</u> Given $x \in M$, $y \in M$ $\exists$ nbhds U of x and V of y

and isometry from $(U, g|_U)$ to $(V, g|_V)$. <u>locally homogeneous</u>

if isometry group acts transitively & can take $U = V = M$.

<u>homogeneous</u>

M simply connected $\Rightarrow$ locally homog $\Leftrightarrow$ homog (Singer 1960).

In dim 3, nine possibilities (labelled by minimal transitive isometry grps):

- $\mathbb{H}(3)$, $SO(3)\times\mathbb{R}$, $\mathbb{H}(2)\times\mathbb{R}$
- $\mathbb{R}^3$, $SU(2)$, $SL(2,\mathbb{R})$, Heisenberg, $E(1,1)=Sol$, $E(2)$)

Remarks: Ricci flow on these studied by Isenberg–Jackson 1992

XCF and Ricci flow are ODE on these spaces.

Milnor's theorem

$M=G$ = a 3-dim'l unimodular Lie grp.

$\mathfrak{g}$ = Lie algebra.

Milnor: $\exists$ orthonormal basis (e_1,e_2,e_3) and $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$ s.t.

$$[e_i,e_j]=\lambda_k e_k$$

Such a basis diagonalizes Ricci (and also cross curvature).

Moreover, $\exists$ f_1,f_2,f_3, f_i collinear to e_i

$$[f_i,f_j]=2\varepsilon_k f_k \qquad \varepsilon_k\in\{-1,0,+1\}$$

$(f_i)_1^3$ Milnor frame

$(\eta_i)_1^3$ dual frame of Milnor frame

$$g = A\eta^1\otimes\eta^1 + B\eta^2\otimes\eta^2 + C\eta^3\otimes\eta^3$$

$A, B, C > 0$.

Trivial cases

- $\mathbb{R}^3$: flat
- $H(3)$: constant negative curvature
  (NXCF leaves metric invariant)
- $SO(3)\times\mathbb{R}^1$: product metric $X_{ij}\equiv 0$.
- $H(2)\times\mathbb{R}^1$: Same.
- The rest of the talk will be about unnormalized negative XCF

Curvature on Nil

$$(R_{ij}) = \begin{pmatrix} \frac{2A}{BC} & & \\ & -2\frac{A}{BC} & \\ & & -2\frac{A}{BC} \end{pmatrix}$$

$$(X_{ij}) = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

XCF equation:
$$\frac{dA}{dt} = -2\frac{A^3}{BC^2}$$
$$\frac{dB}{dt} = 6\frac{A^2}{BC^2}$$
$$\frac{dC}{dt} = 6\frac{A^2}{B^2C}$$

<u>explicit solution</u>

Solution exists for all time,

$K(e_2 \wedge e_3) \to 0$

$K(e_1 \wedge e_2) = K(e_3 \wedge e_1) \to 0.$

pancake degeneracy:

Ricci flow on Nil also has pancake degeneracy (Isenberg-Jackson)

<u>Def</u> ε-flat if all sectional curvatures are bounded above in abs value by $\varepsilon\,(\text{diam})^{-2}$.

<u>Def</u> Almost flat if $\exists$ ε-flat for all small ε.

<u>Gromov</u> Almost flat $\Leftrightarrow$ covered by nilmanifold.

Curvature on $SU(2)$

$$Rc(e_1,e_1) = \frac{2}{ABC}\left(A^2 - (B-C)^2\right)$$

$$Rc(e_2,e_2) = \frac{2}{ABC}\left(C^2 - (A-B)^2\right)$$

$$Rc(e_3,e_3) =$$

$X_{\xi,\eta}$ is complicated; ODE is more complicated.

Solution exists for $0 \le t < T$.

As $t \to T^-$,

$$A, B, C \sim 2\sqrt{T-t}$$

Sectional curvature $K \sim \frac{1}{2\sqrt{T-t}}$

The NXCF has solution for all time, sectional curvature become constant, metric become round.

Ricci flow also converge to round metric (Isenberg-Jackson)

XCF on Sol

$$Rc\, f(e_1, e_x) = \underline{\hspace{3em}}$$

XCF equation even more complicated.

NXCF only exists up to time T.

Ricci flow develops eigen degeneracy.
(exist for all time)

XCF on $SL(2,\mathbb{R})$