Regularization of σ-models

Standard example / "φ^3 theory" scalar field in D-dims

$$S = \int \left(\tfrac{1}{2}(\nabla\varphi)^2 - \tfrac{m^2}{2}\varphi^2 + \tfrac{g}{3!}\varphi^3 \right) d^D x$$

idea / $\int \mathcal{D}\varphi \; \mathcal{O}_1(\varphi)\cdots\mathcal{O}_N(\varphi)\, e^{-\int \mathcal{L}(\varphi)}$ path integral

expand as a function of g
asymptotic expansion of integral (Feynman)

- Fourier xform to momentum variables
- Σ (terms labeled by Feynman diagrams)

typical term $\displaystyle\int \frac{dq}{(q^2+m^2)((q-k)^2+m^2)}$

3 Regularization methods

1) Replace $\mathbb{R}^D$ with lattice $\mathbb{Z}^D$
   → Discrete analogues of quantities in theory
   scale / lattice spacing $a \to 0$
2) Momentum cutoff $\displaystyle\int_{\mu}^{\Lambda} \frac{dq}{(q^2+m^2)\cdots}$ $\Lambda \to \infty$ divergent, μ sets scale
3) Dimensional regularization - applies term by term in Feynman expansion

$$\int_{\mu}^{\Lambda} \frac{d^D q}{|q|^2+m^2} = \int_0^{\infty} d\sigma \int_{\mu\to 0}^{\Lambda\to\infty} e^{-\sigma(|q|^2+m^2)} d^D q$$

$$= \int_0^{\infty} 2^{-D}\pi^{-D/2} e^{-\sigma m^2} d\sigma \quad \text{diverges}$$

$$= \frac{1}{2\pi}\frac{1}{2-D} + (\text{regular @ } D=2)$$

alter quantities m, g

egs $m(\mu) = \mu^{D-2}\left(m + \frac{\hbar}{D-2}\square + \cdots\right)$

$\rightsquigarrow$ cancellation

σ-model

Divergences in Feynman expansion are where?

$\varphi : \mathbb{R}^D \to M$ ~ has geometric data: g_{ij} (2-form, scalars)

(~ or other mfld: Σ^D)

work near $D=2$

$$S(\varphi) = \int \|\varphi^*(g_{ij})\|^2 + \int \varphi^*(\omega) + c\int \varphi^*(R) + \int \varphi^*(u)$$

Assume $g_{ij} = g^{(0)}_{ij} + \varepsilon\delta g_{ij}$

divergences come from

$$-R\partial\varphi\partial\varphi \qquad \nabla\partial u - [\nabla\partial u]_{\varphi=\varphi_0}$$

$$\frac{1}{4\pi}\frac{1}{D-2}\int\left(-R_{iab}(\varphi)\partial_\nu\varphi^i\partial^\nu\varphi^j + \frac{1}{2}(\Delta_g u(\varphi) - \Delta_g u(\varphi_0))\right)$$
$$+ (\text{regular @ } D=2)$$

$$g_0 = \mu^{D-2}\left(g + \frac{\hbar}{2-D}\delta g_1 + \mathcal{O}(\hbar^2)\right) \quad @ \text{ 1-loop}$$
$$u_0 = \mu^D\left(u + \frac{\hbar}{2-D}\delta u_1 + \mathcal{O}(\hbar^2)\right)$$
$$\Rightarrow (\delta g_1)_{ij} = R_{ij} \qquad \delta u_1 = -\frac{1}{2}\Delta_g u$$

Renormalization quantities | keep g_0, u_0 "fixed," calculate $\mu \frac{d}{d\mu}$

$\mu \frac{d}{d\mu} g \propto R_{ij}$ $\qquad$ $\mu \frac{d}{d\mu} u \propto \frac{-1}{2} \Delta_g u$

$\frac{d}{dt} g_{ij}(t) = -R_{ij} + \cdots$

$\frac{d}{dt} u(t) = \frac{-1}{2} \Delta_{g(t)} u$