

Ricci Flow & Renormalization II

Setup | $M =$ Riemannian mfd.

$$\{(G_{ij}, \varphi): \text{Riemannian metric on } M, \varphi: M \rightarrow \mathbb{R}\}$$

σ -Models | 2D Quantum Field Theory

$$X: \Sigma \rightarrow M \quad (\Sigma, g_{ab})$$

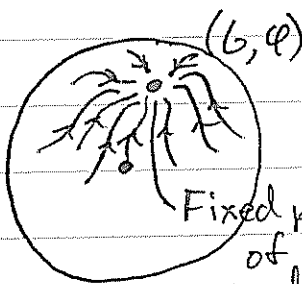
(Worldsheet, 2-mfd.)

$$I(X) = \frac{1}{4\pi\alpha'} \left(\|dx\|_{L_2}^2 + \alpha' \int \varphi_* X^{\flat} \overset{R(g_{ab})}{\curvearrowright} \text{dvol}_{\Sigma} \right)$$

$$= \frac{1}{4\pi\alpha'} \int d^2x \sqrt{g} (g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)) + \alpha' R \varphi(X)$$

Want $\int_{\text{Maps}} F(X) e^{-I(X)}$

this is divergent, regularize
and renormalize, approximation
scale t ("RG time")



$$t \mapsto t + \Delta t \Rightarrow G_{ij}(t) \mapsto G_{ij}(t + \Delta t)$$

$$\varphi(t) \mapsto \varphi(t + \Delta t)$$

$$\frac{dG_{\mu\nu}}{dt} := -\beta_{\mu\nu}^G$$

$$\frac{d\varphi}{dt} := -\beta^\varphi$$

result | $\beta_{\mu\nu}^G = \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\lambda\mu\rho\sigma} R_{\nu}^{\lambda\rho\sigma} + \mathcal{O}(\alpha'^3 R^3)$

$$\beta^\varphi = C_0 - \frac{\alpha'}{2} \nabla^2 \varphi + \frac{(\alpha')^2}{16} R_{\lambda\mu\rho\sigma} R^{\lambda\mu\rho\sigma} + \mathcal{O}(\alpha'^3 R^3)$$

(bosonic $\frac{1}{6}(D-26)$)

Add a diffeo at same time as flow in t ,

$$\bar{\beta}_{\mu\nu}^G := \beta_{\mu\nu}^G + \nabla_\mu M_\nu + \nabla_\nu M_\mu$$

$$\bar{\beta}^\varphi := \beta^\varphi + M^\mu \partial_\mu \varphi$$

where $M_\mu = \alpha' \partial_\mu \varphi + W_\mu(G)$, $W_\mu = \frac{(\alpha')^3}{2} \partial_\mu (R_{\lambda\nu\rho\sigma} R^{\lambda\nu\rho\sigma}) + \dots$

$$S: \{(G, \varphi)\} \rightarrow \mathbb{R}$$

$$\frac{\delta S}{\delta G_{\mu\nu}} = K_{\mu\nu}^{\lambda\rho} \bar{\beta}_{\lambda\rho}^G + \dots$$

$$\frac{\delta S}{\delta \varphi} = K^{\mu\nu} \bar{\beta}_{\mu\nu}^G + \dots$$

"Central Charge Action"

$$S = \int e^{-2\varphi} \left(C_0 - \alpha' \left(\frac{R}{4} + \partial_\mu \varphi \partial^\mu \varphi \right) + \mathcal{O}(\alpha'^2) \right) \text{dvol}_M$$

~ for G

Shorthand notation: $\varphi_i = (G_{\mu\nu}, \varphi)$

$$\frac{\delta S}{\delta \varphi_i} = K_{ij} \bar{\beta}^j$$

$$\text{invertible} \Rightarrow \bar{\beta}^i = K^{ij} \frac{\delta S}{\delta \varphi_j}$$

problem $-K_{ij}$ not positive definite

so don't have monotonic flow

$$\text{eg) } \frac{\delta S}{\delta \varphi} = -2\sqrt{G} e^{-2\varphi} \bar{\beta}^\varphi$$

Perelman's idea: $S(G, \varphi)$ should be extremized relative to volume $\equiv 1$

$$\int d^D x \sqrt{G} e^{-2\varphi} = 1$$

add a Lagrange multiplier to implement this

$$\begin{aligned}\hat{S}(G, \varphi) &= S(G, \varphi) + \lambda \left(\int d^D x \sqrt{G} e^{-2\varphi} - 1 \right) \\ &= \int d^D x \sqrt{G} e^{-2\varphi} (\bar{\beta}^\varphi) + \lambda (\dots)\end{aligned}$$

$$\hat{S} = S(c_0 \mapsto c_0 + \lambda) - \lambda$$

$$\frac{\delta \hat{S}}{\delta \varphi} = 0 \Rightarrow \bar{\beta}^\varphi + \lambda = 0 \quad \text{ie. } \lambda = -\bar{\beta}^\varphi$$

so now $\frac{\delta \hat{S}}{\delta \varphi_i} = \hat{K}_{ij} \bar{\beta}^j$, $-\hat{K}_{ij}$ positive definite

so have a gradient flow now
this is an alternative proof of Zamolodchikov C-thm.