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Ricci Flow & Renormalization II

Setup M = Riemannian mfld.

$\{(G_{ij}, \varphi) : \text{Riemannian metric on } M,\ \varphi : M \to \mathbb{R}\}$

σ-Models 2D Quantum Field Theory

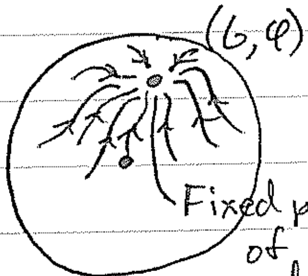
$X : \Sigma \longrightarrow M$ $\qquad\qquad (\Sigma, g_{ab})$

Worldsheet, 2-mfld.

$$I(X) = \frac{1}{4\pi\alpha'}\left(\|dX\|^2_{L_2} + \alpha' \int \varphi \circ X \ \underbrace{R}_{R(g_{ab})} \ d\mathrm{vol}_\Sigma \right)$$

$$= \frac{1}{4\pi\alpha'} \int d^2x \sqrt{g} \left(g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) \right) + \alpha' R \varphi(X)$$

Want $\displaystyle\int_{\text{Maps}} F(X) e^{-I(X)}$ this is divergent, regularize and renormalize, approximation scale t ("RG time")



$t \mapsto t + \Delta t \Rightarrow G_{ij}(t) \longmapsto G_{ij}(t+\Delta t)$

$\varphi(t) \longmapsto \varphi(t + \Delta t)$

$$\frac{dG_{\mu\nu}}{dt} := -\beta^G_{\mu\nu} \qquad\qquad \frac{d\varphi}{dt} := -\beta^\varphi$$

result $\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma} + \mathcal{O}((\alpha')^3 R^3)$

$\beta^\varphi = C_0 - \frac{\alpha'}{2} \nabla^2 \varphi + \frac{(\alpha')^2}{16} R_{\lambda\mu\nu\rho} R^{\lambda\mu\nu\rho} + \mathcal{O}((\alpha')^3 R^3)$

C_0: bosonic $\frac{1}{6}(D-26)$

Add a diffeo at same time as flow in t,

$$\bar{\beta}^G_{\mu\nu} := \beta^G_{\mu\nu} + \nabla_\mu M_\nu + \nabla_\nu M_\mu$$

$$\bar{\beta}^\varphi := \beta^\varphi + M^\mu \partial_\mu \varphi$$

where $M_\mu = \alpha' \partial_\mu \varphi + W_\mu(G)$, $\quad W_\mu = \frac{(\alpha')^3}{2} \partial_\mu (R_{\lambda\nu\gamma\rho} R^{\lambda\nu\gamma\rho}) + \dots$

$$S : \{(G,\varphi)\} \longrightarrow \mathbb{R}$$

$$\frac{\delta S}{\delta G_{\mu\nu}} = K_{\mu\nu}{}^{\lambda\rho} \bar{\beta}^G_{\lambda\rho} + \dots \qquad \frac{\delta S}{\delta \varphi} = K^{\mu\nu} \bar{\beta}^G_{\mu\nu} + \dots$$

"Central Charge Action"

for G

$$S = \int e^{-2\varphi} \left(C_0 - \alpha' \left(\frac{R}{4} + \partial_\mu \varphi \partial^\mu \varphi \right) + \mathcal{O}(\alpha'^2) \right) d\mathrm{vol}_M$$

Shorthand notation: $\varphi_i = (G_{\mu\nu}, \varphi)$

$$\frac{\delta S}{\delta \varphi_i} = K_{ij} \bar{\beta}^j$$

invertible $\Rightarrow \bar{\beta}^i = K^{ij} \frac{\delta S}{\delta \varphi_j}$

<u>problem</u>) $-K_{ij}$ <u>not</u> positive definite

so don't have monotonic flow

eg) $\frac{\delta S}{\delta \varphi} = -2\sqrt{G}\, e^{-2\varphi} \bar{\beta}^\varphi$

Perelman's idea: $S(G,\varphi)$ should be extremized relative to volume $\equiv 1$

$$\int d^D x \sqrt{G}\, e^{-2\varphi} = 1$$

add a Lagrange multiplier to implement this

$$\hat{S}(G,\varphi) = S(G,\varphi) + \lambda\left(\int d^D x \sqrt{G}\, e^{-2\varphi} - 1\right)$$

$$= \int d^D x \sqrt{G}\, e^{-2\varphi}(\bar{\beta}^{\varphi}) + \lambda(\cdots)$$

$$\hat{S} = S(c_0 \mapsto c_0 + \lambda) - \lambda$$

$$\frac{\delta \hat{S}}{\delta \varphi} = 0 \Rightarrow \bar{\beta}^{\varphi} + \lambda = 0 \quad \text{ie.} \quad \lambda = -\bar{\beta}^{\varphi}$$

so now $\frac{\delta \hat{S}}{\delta \varphi_i} = \hat{K}_{ij}\bar{\beta}^j$, $-\hat{K}_{ij}$ positive definite

so have a gradient flow now

this is an alternative proof of Zamolodcikov C-thm.