20 October 2006
D. Cooper

## Further Aspects ---

(slide on Euclidean wallpaper)

(slide on the Hyperbolic Plane)

(slide on Escher hyperbolic wallpaper)

---

? Weeks manifold  Vol = 0.9427

Agol - Dunfield  Vol > 0.67

$$Л(\theta) = -\int_0^\theta \log|2\sin\theta|\, d\theta$$

## Finite dimensional moduli space

G = Lie group, fin dim'l

$X^n$ = homog. space

M = closed manifold

(G, X) Structure $\quad M = X/\Gamma$ ← discrete

deformation space is open subset of alg. variety

$\rho_0 : \pi_1 M \longrightarrow \Gamma \subset G$ = matrix group

/ conjugacy

$g_1, \dots, g_n \longmapsto M_1, \dots, M_n$

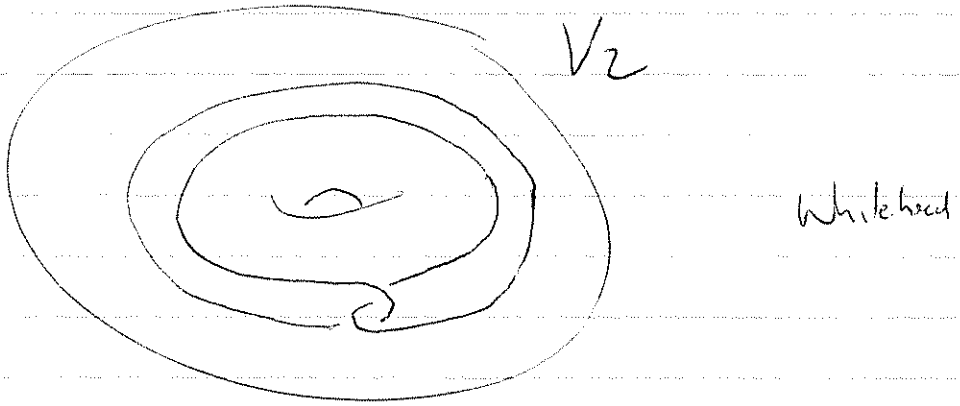
$\mathrm{Hom}(\pi_1 M, G) = \langle M_1, \dots, M_n,\ M_{\lambda_1}^{\pm} \cdots M_{\lambda_k}^{\pm} = I \rangle$

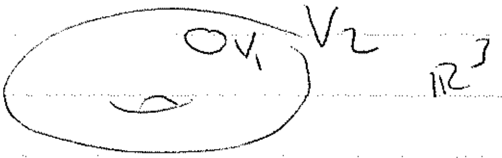
## Whitehead 3-manifold

$W \neq \mathbb{R}^3$

$W \times \mathbb{R} = \mathbb{R}^4$

$W = \bigcup_{n>0} V_n \qquad S^1 \times \mathbb{R}^2$





W NOT TAME

An $n$-manifold $M$ is tame if there compact $n$-manifold $N$ s.t. $M - \partial M = \text{interior}(N)$

$\mathbb{R}^3 = \text{int}(B^3)$ is tame.

Tameness Theorem (Agol, Caligari-Gabai)

If $M^3$ hyperbolic, $\pi_1 M$ is finitely generated then $M$ is tame

Recognition Problem Solved

Can decide if 2 closed orientable 3-manifolds are the same or not.

$(S^3)$

J. Manning
Thompson
T. Li
Jaco
Rubenstein

dim 2 closed Compact manifolds, commutative monoid
(add = connect sum)

$= \langle S, P, T : S$ is identity, $S + X = X$
$3P = P + T \rangle$

### High dim'l

$$S^7 = M_1 \# M_2 \leftarrow \text{exotic 7-spheres}$$

$$= (M_1 \# M_2) \# \cdots \# (M_1 \# M_2)$$

### Kneser (30's)

Closed 3manifold has prime decomposition

**prime** not nontrivial connect sum

**irreducible** every sphere bounds a ball

irreducible ⇒ prime

prime ⇒ irreducible or $S^1 \times S^2$ or $S^1 \tilde{\times} S^2$.

### Milnor (60)

Prime decomposition (closed $M^3$)

pieces are unique except

$$(S^1 \times S^2) + (S^1 \tilde{\times} S^2) = 2(S^1 \tilde{\times} S^2)$$

**not orientable.**

(spheres to cut along are not unique)

**Torus decomposition** cut along tori into pieces which are either (1) SFS

or (2) hyperbolic

[or (3) Solv: orbifold bundle with generic fiber $T^2$]

**$J^{\text{aco}}S^{\text{halen}}J^{\text{ohannsen}}$** use least number of tori

**Thm** $\overset{M \text{ prime}}{\wedge}$ JSJ unique up to isotopy

**Graph manifolds**

**def** (Waldhausen 68)

$$\underset{\text{glued along boundary tori}}{\bigcup} (\text{circle bundles})$$

$$S^1\times D^2,\quad S^1\times \underset{\uparrow\ \text{pants}}{P},\quad S^1\tilde{\times}M$$

**Thm** $M^3$ closed $\Rightarrow M = \underset{\nearrow \text{ graph}}{G} \underset{\uparrow \text{ tori, incompr.}}{\cup} \underset{\nwarrow \text{ hyperbolic}}{N}$

$G, N$ are unique.

If $M$ is irreducible then get isotopy uniqueness

**Thm** Graph $\Leftrightarrow$ $M$ has a Thurston decomposition no hyperbolic pieces

$\Leftrightarrow$ $M$ collapses with bounded curvature

Geometrization Conj / Thm (Thurston / Hamilton, Perelman, ---)

$M$ closed orientable $\Rightarrow$ $M$ has a prime decomposition into pieces, which have a JSJ decomposition into pieces, each of which has Thurston geometries.

Uniqueness of Genn? No.
of topology? Yes.

---

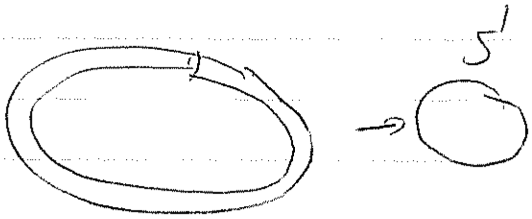
$M^n$ collapses with bounded curvature

there is a sequence of Riem. metrics on $M$ s.t.

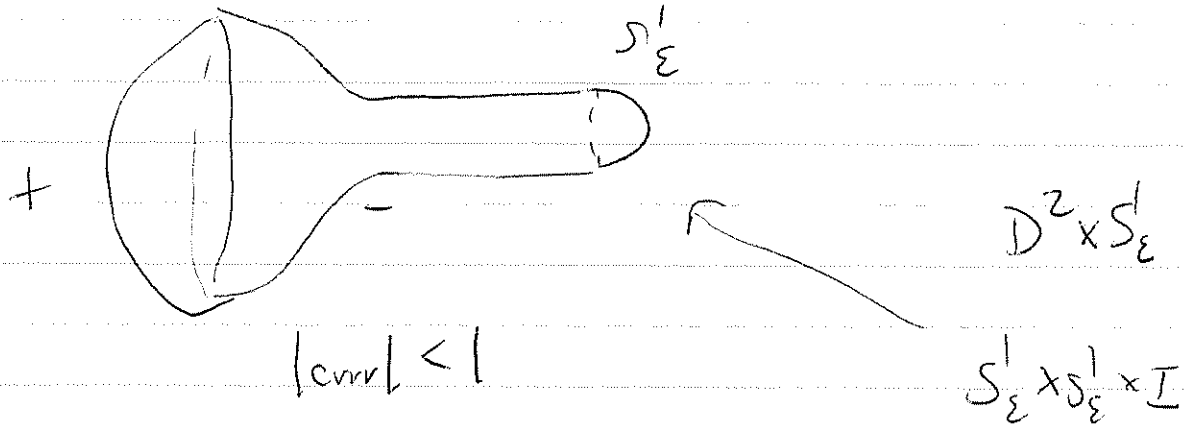
(1) $|\text{all curvatures}| \leq 1$

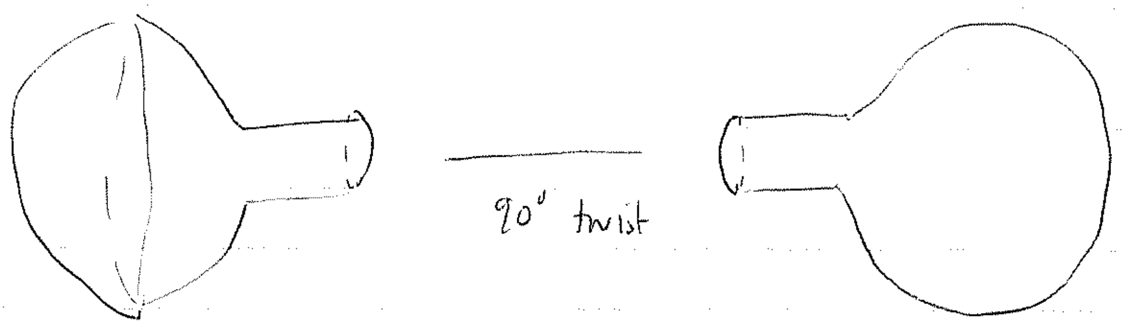
(2) $\forall x \in M$, $\text{inj}(x) \to 0$

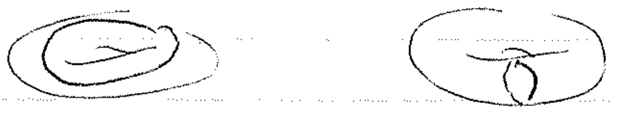
Example Torus $S^1 \times S^1_\varepsilon$



Example $S^3$ collapses

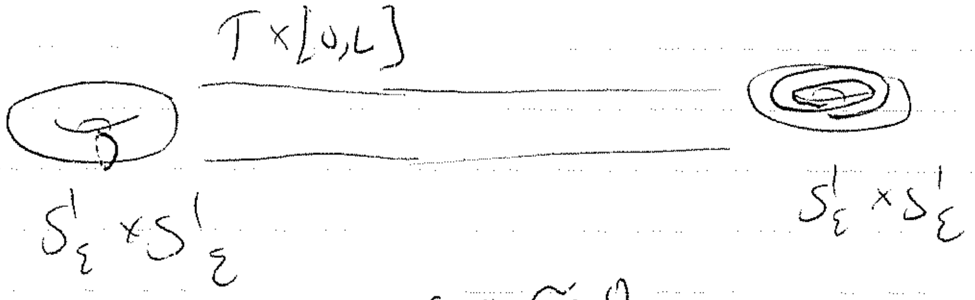






"genus 1 Heegard splitting of $S^3$ "

$|\mathrm{inj}| = \varepsilon$



curv $\approx 0$

<u>Cheeger-Gromov</u> (Collapsing thm)

If $M^3$ closed, collapses then it is graph.