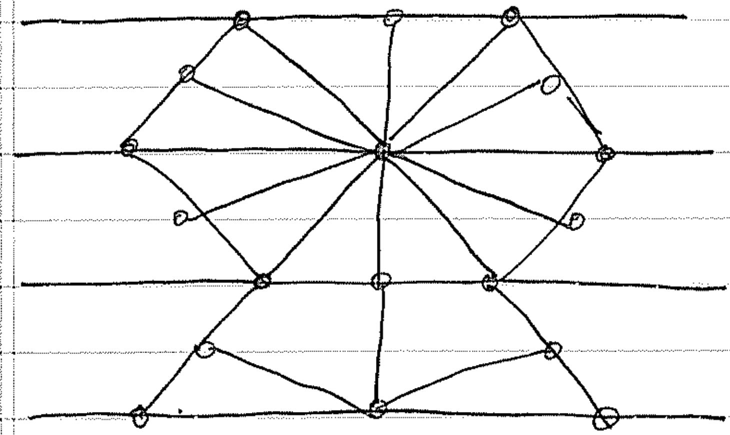
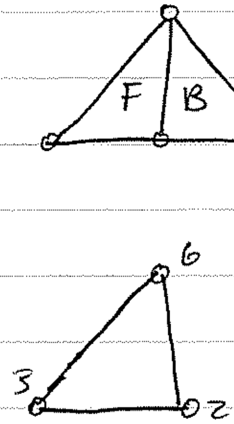
Hyperbolic Structures

Euclidean Wallpaper





Fundamental Domain

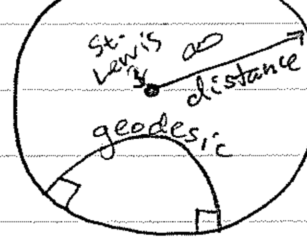
The Hyperbolic Plane

unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$

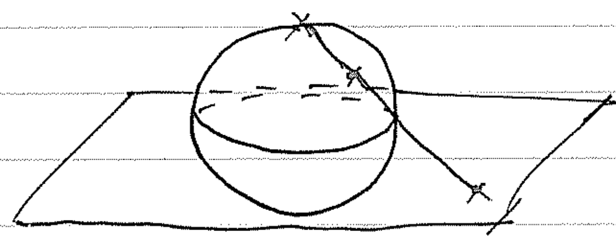
+ metric

"the TWA space"

"the sphere of radius $\sqrt{-1}$"



Transport metric on sphere of radius R onto plane by stereographic proj.

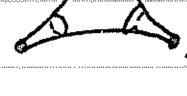


$$ds = dx / (1 + \lambda \|x\|^2)$$

Gaussian curvature $\lambda = 1/R^2$

$\lambda = -1$ gives hyperbolic plane

$3 \sim \mathbb{Z}_3$ rot. sym.



$\longleftarrow S^2(3,4,4)$

$4 \sim \mathbb{Z}_4$ rot. sym.

ie $\mathbb{Z}_4$ rot. sym.

Guess for smallest volume manifold is
Weeks manifold, vol $= 0.9427$
recently, Agol-Dunfield showed vol > 0.67

transcendental integral need to calculate:
$$\pi(\theta) = -\int_0^\theta \log|2\sin\theta|\,d\theta$$

note) if have 2 distinct manifolds, if they have different volumes, calculating, you will eventually find a difference, but if they are the same, you will never know.

General Remark 8 Thurston geometries. Even if structure is not unique, the Moduli space is finite dimensional.

Finite dimensional moduli space

G = Lie group, finite dim
X^n = homogeneous space, G acts on X
M^n = closed manifold

(G, X) structure $M = X/\Gamma$ ~ discrete subgroup of G

deformation space is open subset of Alg. Variety

$$\rho_0 : \underset{\substack{\cup \\ g_1, \dots, g_n}}{\pi_1(M)} \longrightarrow \Gamma \subset G = \text{matrix group}$$

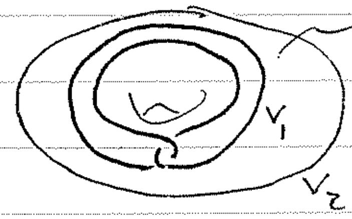
$$g_1, \dots, g_n \longmapsto M_1, \dots, M_n$$

$$\mathrm{Hom}(\pi_1(M), G) = \langle M_1, \ldots, M_n ; \; M_{i_1}^{\pm 1} \cdots M_{i_k}^{\pm 1} = I \rangle$$

Whitehead 3-Manifold

$W \neq \mathbb{R}^3$, but $W \times \mathbb{R} = \mathbb{R}^4$

$W = \bigcup_{n>0} V_n$



note: this is a thin solid torus

W is not tame

def n-mfld M is *tame* if $\exists$ compact n-mfld N st $M \setminus \partial M = \mathrm{int}(N)$.

tameness thm (Agol, Calegari-Gabai)
If M^3 hyperbolic, $\pi_1(M)$ finitely generated then M is tame.

Recognition Problem Perelman $\Rightarrow$ solved
can decide if 2 closed orientable 3-manifolds are the same or not.

[for S^3: A. Thompson, for hyperbolic: J. Manning, fibration: T. Li
algorithmic: Jaco, Rubenstein

Uniqueness of decomposition

<u>dim 2</u>

closed Cpct manifolds form commutative monoid, add = connected sum

$$= \left\langle S, P, T : \begin{array}{l} S \text{ is identity } S + X = X \\ 3P = P + T \end{array} \right\rangle$$

<u>high dim</u> $S^7 = M_1 \# M_2 \sim$ exotic 7-spheres
$$= (M_1 \# M_2) \# \dots \# (M_1 \# M_2)$$

<u>Kneser (30s)</u> closed 3-manifold has a prime decomposition

<u>prime</u> not a non-trivial connected sum

<u>irreducible</u> every sphere bounds a ball

irreducible $\Rightarrow$ prime $\Rightarrow \left(\begin{array}{l} \text{irreducible or} \\ S^1 \times S^2 \text{ or } S^1 \tilde{\times} S^2 \end{array} \right)$

<u>Milner (60)</u> Prime decomp (closed M^3) pieces are unique

$$(S^1 \times S^2) + (S^1 \tilde{\times} S^2) = 2(S^1 \tilde{\times} S^2)$$

not orientable

not unique up to isotopy

Torus decomposition| cut along tori into pieces that are either

(1) Siefert Fibration Spaces

(2) Hyperbolic

(or (3) Solv: orbifold bundle, generic fiber T^2)

J S J decomposition| use the least number of tori

Jaco Shalen Johansson

thm| M prime $\Rightarrow$ JSJ unique up to istopy

Graph Manifolds

def| (Waldhausen '68)

Graph 3-mfld $= \bigcup$ (circle bundles) glued along boundary tori

in fact, can restrict to

$S^1 \times D^2$, $\quad S^1 \times$ (Pants), $\quad S^1 \tilde{\times}$ (Mobius band)

thm| M^3 closed $\Rightarrow$ $M = G \cup N$

graph, tori incompres., hyperbolic

G, N are unique

If M irreducible $\Rightarrow$ isotopy uniqueness

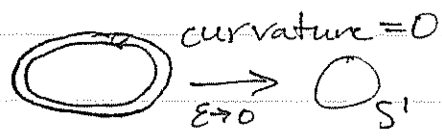
thm| M Graph $\Leftrightarrow$ M has Thurston decomp. w/ no hyperbolic pieces $\Leftrightarrow$ M collapses w/ bounded curvature

Geometrization Conj / Thm (Thurston / Hamilton, Perelman, ...)
M closed orientable ⇒ M has a prime decomp. into pieces, which have a JSJ decomp. into pieces each of which has Thurston geometries. this is called Thurston decomp.

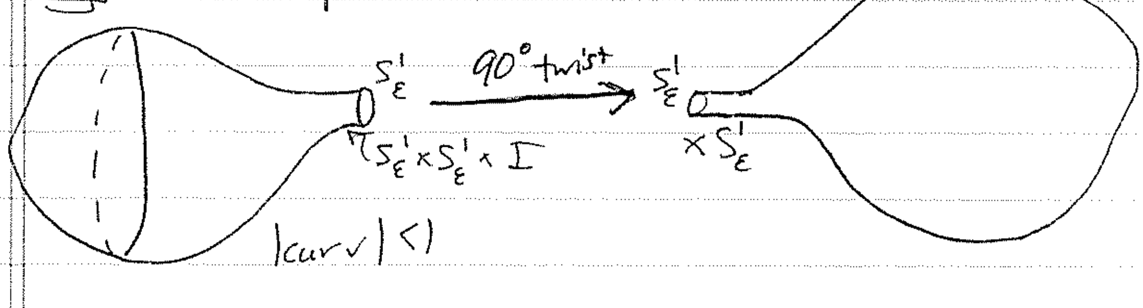
Uniqueness of Geometry? No
of topology? Yes

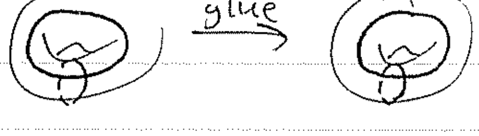
Collapse w/ bounded curvature
M^n collapses w/ bounded curvature: $\exists$ a sequence of Riemannian metrics on M st
(1) $|\text{all curvatures}| \leq 1$
(2) $\forall x \in M \quad \text{inj}(x) \longrightarrow 0$

eg Torus $S^1 \times S^1_\varepsilon$



eg S^3 collapses





$|\text{inj}| = \varepsilon$

this is the Genus 1 - Heegaard split of S^3

Same idea works in general w/ more complicated cycles, long interval

Cheeger - Gromov | (collapsing thm)
If M^3 closed collapses, then it is graph