D. Morrison: Poincaré - lecture 1

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(1)

Introduction

• Subjects: • Topologic. classif. of compact 3-man.

late 70's: Thurston conjecture

{ nb: motivated by D. Sullivan, led, conn. w/ Rocek }

• Ricci flow: defines paths in space of all metrics on n-manifold (Euclidean)

(Hamilton early 80s)

applied to above

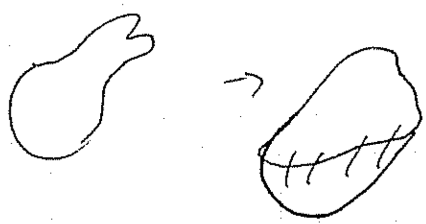
$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(g_{ij}(t))$$

aka $Ric[g]_{ij}$

not time!

∃ local sol-ns only w/ this sign; opp sign ⇒ no sol'ns

- nonlinear heat eq'n
- dispersive: curvature tends to spread itself out



• Reformulation of Ricci flow as gradient flow (Perelman)

math.dg/0211159

$$\mathcal{L} = \int_M (R + |\nabla f|^2) e^{-f} dV$$

where f is a smooth function on M

R = scalar curvature

$\delta \mathcal{F}[v_{ij}, h]$ $\qquad v_{ij} = \delta g_{ij}$

$\| \qquad h = \delta f$

$\qquad v = g^{ij} v_{ij}$

$$\int_M e^{-f}\Big[-v_{ij}(R_{ij} + \nabla_i\nabla_j f) + (v/2 - h)(2\Delta f - |\nabla f|^2 + R)\Big]$$

$$\boxed{\text{NOTE if } dm = e^{-f}dV \text{ is held fixed, then } v/2 = h}$$

<u>Fix dm</u>: set $\quad f = +\log\left(\frac{dV}{dm}\right)$

$\Downarrow$

$$\mathcal{F}^m = \int_M (R + |\nabla f|^2)\, dm$$

$\Downarrow$

$$\frac{d}{dt} g_{ij} = -2(R_{ij} + \nabla_i\nabla_j f)$$

this can be undone by diffeo

modulo diffeos, this <u>is</u> Ricci flow

Existence of solutions is <u>not</u> always guaranteed (depends on dm), but if they do, are indep. of dm, up to diffeomorphism

- <u>Connection to physics</u> (D. Friedan)

2d QFTs based on arbitrary Riemannian manif

M, g_{ij} : look at $\quad Maps(\Sigma \to M) \ni \varphi$

$$S[\varphi] = \int_\Sigma \|\varphi^*(g_{ij})\|$$

$$\int_{\mathrm{Maps}(\Sigma, M)} e^{-S[g]} (\cdots)\, dg$$ ← (hard to define!)

Physicists - want to know how all this depends on distance scale ← wrt metric on Σ!

↓

<u>renormalization</u> for the theory

[integral has short-distance sing: introduce a scale param: renorm corrects for leading divergence]

Friedan: renorm. <u>classically</u> governed by Ricci flow on M

[Wilson: redefine quantities in integral as we change our distance scale]

- Hamilton: originally interested in $\dim(M) = 2$ (mid-80s)

<u>Aside</u>: curvatures

- g_{ij} gives a pairing

(Tangent vector fields) $\otimes$ (Tangent v.f.) $\to$ (Tangent v.f.)

$$\nabla_X Y = \nabla_Y X = Z$$

$$\nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]} = R(X,Y)$$

- Riemannian curvatures

$$\langle R(X,Y)Z, W\rangle = Rm(\underbrace{X, Y}_{\text{skew}}, \underbrace{Z, W}_{\text{skew}})$$

symmetric under changing pairs

Ric = contract middle variables

$$\mathrm{Ric}(e_i, e_j) = \sum_{k=1}^{m} \mathrm{Rm}(e_i\, e_k\, e_k\, e_j)$$

Scalar curvature $R = \sum_{ij} g^{ij} R_{ij}$

If g_{ij} evolves by Ricci flow,

$$\boxed{\partial/\partial_t R = \Delta R + 2R^2} \rightarrow \text{(nonlinear heat eq'n!)}$$

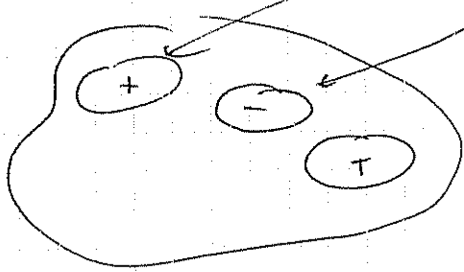
Laplacian on M

dim. 2: $R_{ij} = R g_{ij}$

technical problem:

pos curv. regions shrink

neg. curv. regions grow



does this happen in a nice, uniform way?

Hamilton, B. Chow, etc. proved this

<u>Theorem</u>: · given Riemannian metric on compact surface

· consider rescaled Ricci flow (RF)

$$\dot{g}_{ij}(t) = -2R_{ij} + g_{ij}\overline{R}$$

← <u>avg</u> of scalar curvature.

concocted to have constant volume

change "time" parameterization

This flow exists $\forall$ time $t \to \infty$ limit is metric of constant curvature.

Non-rescaled : $\begin{cases} g=0; & \text{vol} \to 0 \text{ in finite time} \\ g=1; & \text{vol} \to \text{finite} \\ g>1; & \text{vol} \to \infty \end{cases}$

genus

Entire Ricci flow: $g_{ij}(t) = e^{f(t)} g_{ij}(0)$

$\Downarrow$

{ there is a unique constant curvature metric }

Classical theorem:

{ The conformal class of any metric of a compact surface has a unique constant curvature representative }

$M = X/\Gamma$

X = simply connected surface w/ const. curv. metric

$= S^2, \mathbb{R}^2, \mathbb{H}^2$ — these three admit compact quotients

geometric (Klein): $X = G/H$

$G = \mathrm{Isom}(X)$ acting transitively

$$\begin{array}{lll} e(M) < 0 & \text{if} & S^2 \\ = 0 & & \mathbb{R}^2 \\ > 0 & & \mathbb{H}^2 \end{array}$$

Three-manifolds (Thurston)

$M = X/\Gamma$; M compact

$X = H\backslash G \longleftarrow G = \mathrm{Isom}(X)$; acting transitively

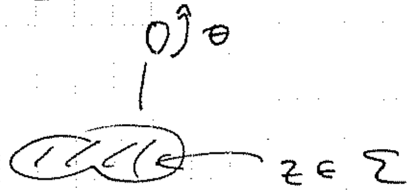
more generally: look at X/Γ w/ $\mathrm{vol}(X/\Gamma) < \infty$: more general than compact manifolds

$\boxed{\text{Thurston: there are 8 geometries!}}$

- Six of these geometries have circle fibrations

$$\begin{array}{ccc} S^1 & \longrightarrow & M \\ & & \downarrow \\ & & \Sigma \end{array}$$

actually these are "Seifert fibrations"



$z \to e^{2\pi i/n} z$

$\theta \to \theta + 2\pi/n$ $\leftarrow$ (on the central fiber this wraps circle $\underline{n}$ times; wraps once nearby.)

- These 6 can be classified by

  e = top. class of S^1 bundle

  χ = Euler char. of Σ

	$\chi < 0$	$\chi = 0$	$\chi > 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2 \times \mathbb{R}$
$e \neq 0$	S^3	Nil	$\widetilde{SL_2(\mathbb{R})}$ ← univ cover.

↑ all other top. classes reached by quotients

n^{th} quot → ne

- Two of the geometries have T^2 fibrations over 1-d object 

  Nil, Sol ← (corresp. Lie algebras nilpotent & solvable)

- Final geometry: $\mathbb{H}^3$

  $\left\{ \text{eg 3 geometries have constant curvature: } S^3, \mathbb{R}^3, \mathbb{H}^3 \right\}$

$$\left\{ \begin{array}{l} \text{Nil: } \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \\[2em] \text{Sol: } \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \end{array} \right\}$$

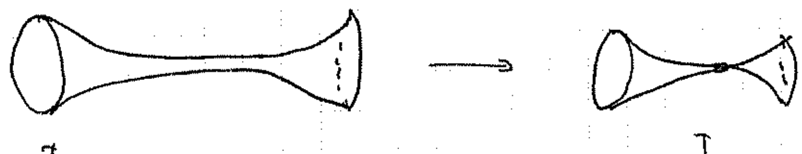
- **Summary** of outcome of RF

(M, g_{ij}) M compact, 3d

let gradient flow commence!

(a) $S^3/\Gamma \rightarrow (pt)$ in finite time

(b)



topol. $S^2 \times I$

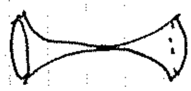
pinches off at finite time

$\partial(S^2 \times I) = S^2 \times S^0 \leftarrow$ 2 points!

$= \partial(D^3 \times S^0)$

$D^3 \times S^0$

$\leftarrow$ glue along ∂: $\rightarrow S^3$

$\Rightarrow$ replace  w/ $(D^3 \times S^0)$

Perelman: Ricci flow before, after cutting out, flow continues

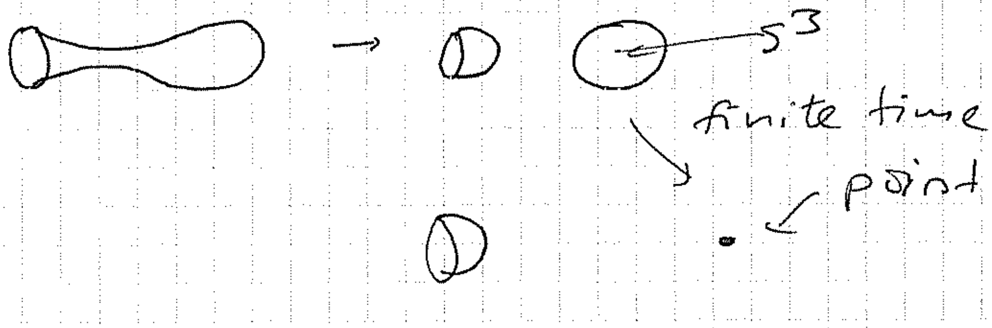
{ Hypothesis: no embedded $\mathbb{RP}^2$-s
Morgan–Tian }

(needed f/ proof of Poincaré?)

(P. makes surgery at some discrete time before singularity)

Surgeries/extinctions occur at discrete evolution times $t_1, t_2, \ldots$

- Not known if # is finite.
- Known: for $t \gg 0$, all that happens is

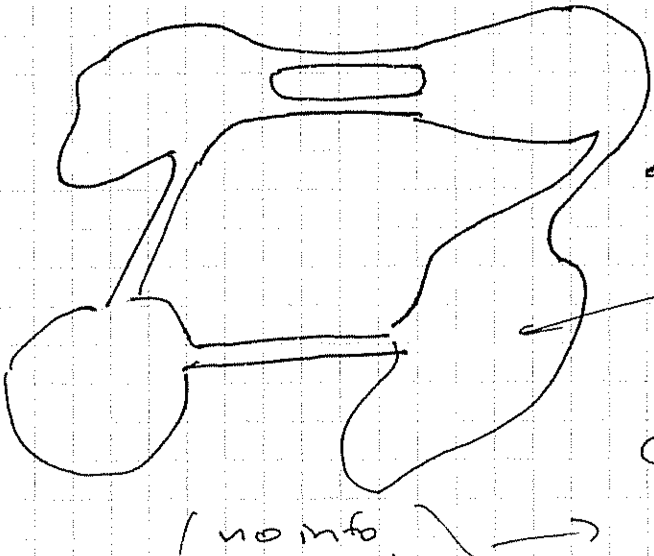


These decompose 3-manif into pieces

- Unique decomp. of 3-man. into "prime summands" by cutting along S^2's:
- This is the endpoint of RF (up to "bubbling" of S^3's)

- 3-manifold divides into regions:

(rescaling of T^2 shrinking, & finite)



$I \times T^2$: (interval gets ∞ long).

"geometrize" as $t \to \infty$

Caveat: if geometric piece is not $\mathbb{H}^3/\Gamma$, collapses along S^1, "fibration"

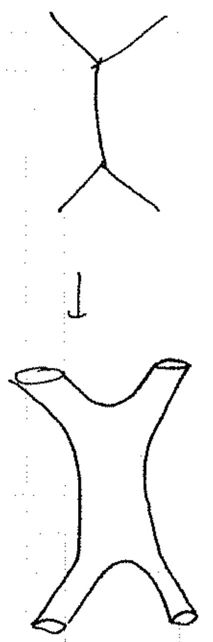
(no info about this w/ Perelman)

~~[illegible]~~

→ in other words, S^1 collapses to the base.

Closing comments

In 2d: · string theory takes diagrams like



· 3d: collapse on S^1's: string theory has a "strong coupling" limit:

$$\Sigma \longrightarrow (\text{3-manif.})$$

possible approach to fundamental found. of M theory.