RESEARCH STATEMENT

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1. Overview

The braid groups B_n ($n \geq 1$) arise in the study of mapping class groups, knot theory, and robotics among other fields. My research focuses on the study of how braid group representations can be used to determine certain properties of braids. One very interesting property of the braid groups was announced in 1992 by Patrick Dehornoy in [7]. He proved that the braid groups were left-orderable, but he used methods that were foreign to most topologists. A short time later, a 5 author paper [8] gave a completely topological proof of braid group orderability, and in fact, the order was the same as the order given by Dehornoy.

I use the Burau representation to determine positivity of 3-braids in the Dehornoy order and whether certain 3-braids are right-veering. Positive braids and right-veering braids are closely related fields of study. In fact, right-veering braids are positive in the Dehornoy order. A few, but not all definitions are given here.

1.1. **Definitions.** Let D^2 be the unit disk in $\mathbb{C}$, and let $Q = \{p_1, p_2, \ldots, p_n\}$ be a set of n interior points of D^2. We denote by D_n the set $D^2 \setminus Q$ with basepoint p_0 in its boundary, ∂D_n. Regard B_n as the group of isotopy classes of orientation preserving self-homeomorphisms of D_n which fix ∂D_n. The group B_n is isomorphic to the group having generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ when $|i-j| = 1$.

The *unreduced Burau representation* is the homomorphism $\rho : B_n \to Aut(H_1(\tilde{D}_n, \tilde{p}_0))$ which is given by the induced action of a braid representative on the n-dimensional $\mathbb{Z}[t, t^{-1}]$-module $H_1(\tilde{D}_n, \tilde{p}_0)$, where $\tilde{D}_n$ is a $\mathbb{Z}$-fold covering space of D_n. Given a basis, we denote by $\mathrm{M}(\beta)$ the $n \times n$ Burau matrix that represents the braid β. The matrix $\mathrm{M}(\beta)$ can, in fact, be found by working very concretely in D_n. If we choose our basis to be lifts of the set $\{y_1, y_2, \ldots, y_n\}$ shown in Figure 1, then entry $p_{i,j}(t)$ in $\mathrm{M}(\beta)$ is $\langle \delta_i, \beta(y_j) \rangle = \sum_{k\in\mathbb{Z}} (t^k\tilde{\delta}_i, \tilde{\beta}(y_j))t^k \in \Lambda$ where $(t^k\tilde{\delta}_i, \tilde{\beta}(y_j))$ is the algebraic intersection number of the two arcs in $\tilde{D}_n$.

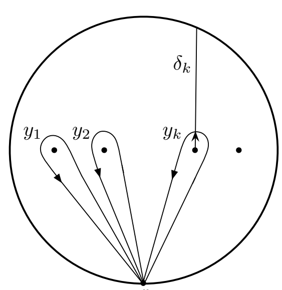

FIGURE 1

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2. Current Research

A solution to the word and conjugacy problems for B_n was discovered in 1968 by Garside in [1] and expanded upon by many others over the years. However, it still remains an interesting endeavor to find distinguished representative words for elements of the braid groups and the braid monoids. For the $n = 3$ case, we use the matrix $\mathrm{M}(\beta)$ (ρ is known to be faithful for this case) to find a distinguished representative word for the 3-braid β. If e is a straight edge from p_1 to p_3, we start by extracting geometric intersection numbers of $\beta(e)$ and arcs $\alpha_1, \alpha_2, \alpha_3$ shown in Figure 2 from the entries of $\mathrm{M}(\beta)$ by specializing at $t = -1$.

Lemma 1. *The geometric intersection number of $\beta(e)$ and α_i is $|p_{i,3}(-1) - p_{i,1}(-1)|$, for $i = 1, 2, 3$.*

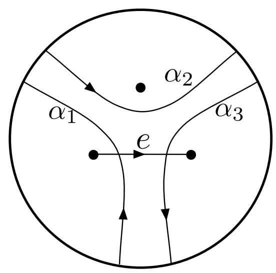

FIGURE 2

These intersection numbers completely determine the image of e under β. The key observation is that for $j = 0, 1$ or 2, $(\sigma_2\sigma_1)^j\beta(e)$ is an arc that after finitely many applications of σ_1^{-1} and σ_2 is isotopic to e. Repeated applications of σ_1^{-1} and σ_2 "unravels" $\beta(e)$. However, the image of e does not completely determine a word representing β. The arc e is left invariant by the map $\Delta^2 = (\sigma_2\sigma_1)^3$ which is a full twist about ∂D_3 and by the map $\tau = \sigma_1^{-1}\sigma_2\sigma_1$ which is a counterclockwise exchange of p_1 and p_3. The arc e is, of course, also invariant under the inverses of these maps. Thus, we have the following theorem.

Theorem 1. *Every 3-braid β is represented by a word of the form $(\sigma_2\sigma_1)^m w(\sigma_1, \sigma_2^{-1})\tau^r$, where $\tau = \sigma_1^{-1}\sigma_2\sigma_1$, $w(\sigma_1, \sigma_2^{-1})$ is a word in only σ_1 and σ_2^{-1}, and m, r are integers .*

I have a way of putting a braid β into the form given in Theorem 1 by using only its Burau matrix $\mathrm{M}(\beta)$. This is done by using Lemma 1, the unraveling processing mentioned above, and the determinant of $\mathrm{M}(\beta)$. This allows us to determine if β is positive or not. Roughly speaking, a braid β is positive in the Dehornoy order if and only if the curve diagram associated to β is to the right of the curve diagram associated to the identity map. A detailed description of this order can be found in [2] or [8]. In summary, in most cases it is not the element τ that determines the positivity of β, but it is the fractional twist coefficient m that dominates in determining positivity. To a lesser degree, the rightmost letter α in the word $w(\sigma_1, \sigma_2^{-1})$ plays a role in positivity. The details for the trivial case that $w(\sigma_1, \sigma_2^{-1})$ is the empty word are omitted here.

Theorem 2. *Suppose that $\beta = (\sigma_2\sigma_1)^m w(\sigma_1, \sigma_2^{-1})\tau^r$ where $w(\sigma_1, \sigma_2^{-1})$ is not the empty word, and let α be the rightmost letter of $w(\sigma_1, \sigma_2^{-1})$. If $m \geq 2$, then β is positive. If $m \leq -2$, then β is negative. If $|m| \leq 1$, then β is positive if α is σ_1 and negative if α is σ_2^{-1}.*

3. Thesis Research

If we let T^2 denote the torus and S denote the once-punctured torus, there is a natural surjection $\delta : \mathrm{Mod}(S, \partial S) \to \mathrm{Mod}(T^2) \cong SL_2\mathbb{Z}$ from the mapping class group of S to the mapping class group of T^2. Moreover, there is an isomorphism between $\mathrm{Mod}(S, \partial S)$ and B_3 given by the two-fold branched cover of D_3 by S. Thus, the kernel of the map δ is generated by the element Δ^4.

Recall the Thurston classification of mapping classes as *reducible, periodic* or *pseudo-Anosov*. Reducible elements of $SL_2(\mathbb{Z})$ leave some essential simple closed curve invariant, and therefore a reducible 3-braid β is a conjugate of some power of σ_1 plus possibly some boundary twisting. That is, $\beta = \Delta^{2k}\mu\sigma_1^m\mu^{-1}$ for some $\mu \in B_3$. Periodic elements of $SL_2(\mathbb{Z})$ have order dividing 12.

In my thesis, I used the $(n-1)$-dimensional Burau representation (referred to as the *reduced* Burau representation) to classify reducible and periodic homeomorphisms of the once-punctured torus S as right-veering. *Right-veering* homeomorphisms of surfaces with boundary mostly show up in 3-dimensional contact topology. In particular, Honda, Kazez, and Matić prove in [3] that a contact 3-manifold (M, ξ) is tight if and only if all of its open book decompositions (Σ, h) have right-veering monodromy h. The following lemma gives necessary and sufficient conditions for a reducible map of S to be right-veering. In the statement of the lemma, T_a is a positive Dehn twist about a curve parallel to the boundary of S, and T_b is a Dehn twist about a nonseparating curve in S.

Lemma 2. *(Honda, Kazez, Matić [3]) Let $h = T_a^k T_b^m$ represent a reducible mapping class of S. Then h is right-veering if and only if either $k > 0$ or $k = 0$ and $m \geq 0$.*

We extract the values k and m from the reduced Burau matrix $\mathrm{M}_r(\beta)$, then apply Lemma 2 to determine whether β is right-veering. It is easily verified that the trace of $\mathrm{M}_r(\beta)$ is $t^{3k} + (-1)^m t^{3k+m}$. Several immediate observations can be made from this nice form of the trace, but it is not enough to determine k and m completely. For example, if the trace is $t^6 + t^{-6}$, it is impossible to determine if $k = 2$ and $m = -12$ or if $k = -2$ and $m = 12$. Moreover, this distinction is necessary since the former implies β is right-veering and the latter that β is left-veering. If $\widehat{\beta}$ is the image of β in $SL_2(\mathbb{Z})$, the following theorem gives us sufficient information to determine k and m given the eigenvalue λ for $\widehat{\beta}$.

Theorem 3. *Suppose that β is a nontrivial reducible map, and let $\hat{\beta}$ be its image in $SL_2(\mathbb{Z})$. Then value $|m|$ is the greatest common divisor of the entries off the diagonal of $\widehat{\beta}$. Moreover, for $\lambda = 1$, the sign of m is the sign of the $(1,2)$-entry of $\widehat{\beta}$, and for $\lambda = -1$, the sign of m is the sign of the $(2,1)$-entry of $\widehat{\beta}$.*

For the periodic case, β^{12} must be in $\ker \delta$, and hence β^{12} is a power of a Dehn twist about the boundary of S. I proved that a map h is right-veering if and only if h^n is right-veering for $n \geq 0$, thus we have the following theorem.

Theorem 4. *Let $\beta \in B_3$ be a nontrivial periodic map. Then $\mathrm{M}_r(\beta^{12}) = \begin{pmatrix} t^{6k} & 0 \\ 0 & t^{6k} \end{pmatrix}$ for some integer k, and hence, β is right-veering if and only if $k \geq 0$.*

4. Future Endeavors

The next step in my research plan is to generalize my results for $n = 3$. The Burau representation is not known to be faithful for $n = 4$ and is known to be unfaithful for $n \geq 5$ (see [4] and [5]). However, Bigelow proves in [6] that the Lawrence-Krammer representation is faithful for all n, thus this representation could be used as a tool for finding distinguished forms for braids. There are also many open problems relating to the Dehornoy order that I find interesting. Many of these involve generalizations of braid groups. For example, it is unknown if surface braid groups are orderable. It is also not known if finite index subgroup of the mapping class group of a surface is orderable.

Keeping current on topics in areas which may be accessible to undergraduates is also an important factor in my future plans. Undergraduate research can make all the difference when a student is deciding if he or she would like to study mathematics at the graduate level. Moreover, it can create a deeper appreciate and love for the subject. In particular, I have noticed that many undergraduates seem to take a liking to the study of knots and links. This area has the potential to be presented in a geometric and hands-on way for beginners, but they also have deep implications throughout mathematics.

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