HOMEWORK 3

SOLUTIONS

1. (a) T
   (b) T
   (c) F
   (d) T
   (e) F
   (f) F
   (g) T
   (h) F
   (i) T
2.

$$\begin{aligned} B &= (-2, 2) \\ C &= [0, 2) \\ D &= \{0\} \\ E &= \{1\} \end{aligned}$$

   Hence C, D, E are ALL subsets of B; D, E are both subsets of C.
   (a) D and E have the property that neither is contained in the other ("is contained in" means "is a subset of" here).
   (b) X could be B, C or E. X could NOT possibly be D.
3. (a) Not valid.
   (b) P: I eat chocolate. Q: I am depressed.
   $P \Rightarrow Q$ true implies that $\bar{Q} \Rightarrow \bar{P}$ is true.
   (c) P: A movie is worth seeing. Q: A movie is made in England. R: Smallbrain reviews a movie.
   $\bar{P} \Rightarrow \bar{Q}$ and $P \Rightarrow R$ are both true. Thus, $\bar{R} \Rightarrow \bar{Q}$ is true, since $\bar{R} \Rightarrow \bar{P}$ (contrapositive to $P \Rightarrow R$) and $\bar{P} \Rightarrow \bar{Q}$ are both true.
5. (a) T
   (b) F
   (c) F
   (d) F
   (e) T
6. (a) *Proof.* We prove the result by contradiction. Assume $\sqrt{6} - \sqrt{2} \leq 1$. Then squaring both sides yields $8 - 2\sqrt{12} \leq 1$, and hence $-2\sqrt{12} \leq -7$. Now multiplying both sides by -1 reverses the inequality to give $2\sqrt{12} \geq 7$. Now, squaring both sides gives $48 \geq 49$. This is a contradiction! Hence, the assumption $\sqrt{6} - \sqrt{2} \leq 1$ must have been false, therefore $\sqrt{6} - \sqrt{2} > 1$ must be true. □

(b) *Proof.* We prove the result by contraposition. Assume that n is an odd integer. Then $n = 2k+1$ for some $k \in \mathbb{Z}$. Thus, $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2+2k)+1$. We know that $2k^2 + 2k$ is an integer because k is an integer. Thus, $2(2k^2+2k)$ is even which implies that $n^2 = 2(2k^2+2k)+1$ is odd. □

(c) *Proof.* Assume that $n = m^3 - m$ for some $m \in \mathbb{Z}$. Then factoring yields $n = m(m-1)(m+1)$. Notice that $m-1, m$, and $m+1$ are 3 consecutive integers. We claim that contained in three consecutive integers is a multiple of 2 and a multiple of 3. Note that if this claim is true, then the product of these three integers will be a multiple of 6.

Consider the integer m. The integer m is either a multiple of 3 or it's not. First suppose that m is a multiple of 3, i.e. $m = 3k$ for some integer k. If m is even, then k must be even (k could not be odd because this would imply that m was odd). So $k = 2s$ for some integer s, and hence $m = 3(2s) = 6s$. Thus, m is a multiple of 6, and hence the product $m(m-1)(m+1)$ is a multiple of 6. If m is odd, then $m+1$ is even, and hence the product $m(m-1)(m+1) = 3k(m-1)(2r) = 6kr(m-1)$ is a multiple of 6. Now, suppose that m is NOT a multiple of 3. Then $m = 3k+1$ or $m = 3k+2$ for some integer k. But notice in either of these cases one of the other factors $m+1$ and $m-1$ is a multiple of 3, and the argument above can be repeated (For example, if $m = 3k+1$, then $m - 1 = 3k$. Now repeat the argument above for $m-1$ as it is a multiple of 3. If $m = 3k+2$, then $m+1 = 3(k+1)$. Now repeat the argument above for $m+1$ as it is a multiple of 3). □

7. (a) Counterexample: $n = 2$ and $n = 4$. Check that these choices make the conclusion false!

(b) Counterexample: the positive integer 7 cannot be written as the sum of three squares (check this).

8. (a) $P : \forall n \in \mathbb{Z}$ such that n is a prime number, n is odd.
$\bar{P} : \exists n \in \mathbb{Z}$ such that n is prime and n is even.
$\bar{P}$ is true.

Proof. $n = 2$ is an integer that is prime and even. □

(c) $P : \exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$, $x \neq n^2 + 2$.
$\bar{P} : \forall x \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that $x = n^2 + 2$.
P is true.

Proof. Let $x = 0$. Then $x - 2 = -2 \neq n^2$ for all $n \in \mathbb{Z}$. □

(d) $P : \exists x \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z}$, $x \neq 2$.
$\bar{P} : \forall x \in \mathbb{Z}$, $\exists n \in \mathbb{Z}$ such that $x = n + 2$.
$\bar{P}$ is true.

Proof. Let $x \in \mathbb{Z}$. Then $n = x - 2$ is an integer with the property that $x = n + 2$. □

(f) $P : \forall y \in \{x \mid x \in \mathbb{Z}, x^2 < 0\}, 5y^2 + 5y + 1$ is a prime number.
$\bar{P} : \exists y \in \{x \mid x \in \mathbb{Z}, x^2 < 0\}$ such that $5y^2 + 5y + 1$ is not a prime number.
P is vacuously true because the set mentioned is empty.