Homework 4

Due: Friday, October 24

October 21, 2008

1. Exercise 2.18.
2. Exercise 2.19.
3. Let
$$Rz = STS^{-1}z = \lambda z$$
be the normal form of a Möbius transformation T with two fixed points, p, q, where $Sz = \dfrac{z-p}{z-q}$. Prove that $\lambda = [Tz, p, z, q]$. *Note: This provides a convenient means for computing λ.*
4. Let
$$Rz = STS^{-1}z = z + \beta$$
be the normal form of a parabolic Möbius transformation with fixed point $p \neq \infty$, where $Sz = \dfrac{1}{z-p}$. Let z_0 be the point that T sends to ∞. Prove that
$$\beta = -\frac{1}{z_0 - p}$$
and also
$$\beta = \frac{1}{T(\infty) - p}.$$
Note: This provides a means for computing β.
5. Analyze each of the following Möbius transformations by finding the fixed points, finding the normal form (note this may only be defined up to taking its inverse), stating whether it is elliptic, hyperbolic, loxodromic, or parabolic, and finally sketching the appropriate coordinate system of Steiner circles indicating the motion of the transformation.
   (a) $Tz = \dfrac{z}{2z-1}$
   (b) $Tz = \dfrac{3z-4}{z-1}$
6. For what values of θ and b is the Euclidean transformation $Tz = e^{i\theta}z + b$ elliptic? hyperbolic? parabolic? loxodromic?
7. What kind of transformation (elliptic parabolic, hyperbolic, or loxodromic) is the inversion $Tz = \dfrac{1}{z}$?
8. Let R, S, and T be transformations such that $T = S^{-1}RS$. Prove that z is a fixed point of T if and only if Sz is a fixed point of R. Conclude that T and R have the same number of fixed points.
9. A transformation such that $T^2 = I$ (the identity transformation) is called an *involution*. Prove that an involutory Möbius transformation must be elliptic.
Hint: Given the normal for of T, what is the normal for of T^2?