Homework 2

Due: Friday, October 10

October 9, 2008

1. The function $\xi : S^1 - \{i\} \to \mathbb{R}$ given by $\xi(z) = \dfrac{Re(z)}{1 - Im(z)}$ is referred to as *steographic projection*. Consider the four points $z_k = e^{\frac{\pi k}{4} i}$, $k = 1, 3, 5, 7$ of S^1 that form the corners of a square in $\mathbb{C}$. Calculate their images under ξ.

2. Prove that the steographic projection function $\xi : S^1 - \{i\} \to \mathbb{R}$ is onto.

3. Steographic projection can be generalized to the n-dimensional sphere

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \,|\, x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$$

in $\mathbb{R}^{n+1}$. It is given by the function $\pi : S^n - N \to \mathbb{R}^n$ which is defined by the equation

$$\pi(x_1, x_2, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots, x_n),$$

where $N = (0, 0, \ldots, 0, 1)$ is the "north pole" of the sphere. Consider the 2-sphere, S^2. Show that steographic projection sends the equator of S^2 to the unit circle S^1 in $\mathbb{R}^2$.

4. Exercise 1.8: Prove that $\mathbb{H}$ is open in $\mathbb{C}$. For each point z of $\mathbb{H}$, calculate the maximum ϵ so that $B_\epsilon(z)$ is contained in $\mathbb{H}$.

5. Prove that $f : \mathbb{C} \to \mathbb{C}$ is continuous if and only if $f^{-1}(V)$ is open in $\mathbb{C}$ for every open set V in $\mathbb{C}$. Note: This is a very useful characterization of continuity and is true for any continuous function for which the domain and range are metric spaces.

6. Exercise 1.18: Let p and q be two distinct points of $\overline{\mathbb{R}}$. Prove that p and q determine a unique hyperbolic line whose endpoints at infinity are p and q.

7. Exercise 2.2: Show that the homeomorphism f of $\overline{\mathbb{C}}$ defined by setting

$$f(z) = az + b \text{ for } z \in \mathbb{C} \text{ and } f(\infty) = \infty,$$

where $a, b \in \mathbb{C}$ and $a \neq 0$, takes Euclidean circles in $\mathbb{C}$ to Euclidean circles in $\mathbb{C}$.