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Stability of Einstein Metrics and Spin Structures

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Abstract

We survey the recent work on the stability of Einstein metrics and related topics, especially our joint work with Xiaodong Wang and Guofang Wei [12, 13]. Our work shows that spinors and Dirac operators play an important role.

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1 Introduction

One of the most fruitful approaches to finding the "best" (or canonical) metric on a manifold has been through the critical points of a natural geometric functional. In this approach one is led to the study of variational problems and it is important to understand the stability issue associated to the variational problem. Roughly speaking stability means that the second variation has a definite sign. Stability issue is also important in the study of geometric flows.

It is well known that the Einstein metrics are precisely the critical points of the total scalar curvature functional (a.k.a. Hilbert-Einstein action in general relativity)

$$\mathcal{S}(g) = \int_M S(g) d\mathrm{vol}(g)$$

on the space of Riemannian metrics with normalized volume on a closed manifold M. It behaves in opposite ways along the conformal changes and transversal directions. The variational problem in the conformal class of a metric is the famous Yamabe problem, which was finally resolved by Schoen in the seminal work [36], after Trudinger and Aubin's work. Thus the question here is to study the stability for the total scalar curvature functional when we restrict to the transversal directions, that is, the changes of conformal structures.

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This has to do with the second variation of the total scalar curvature functional restricted to the traceless transverse symmetric 2-tensors, which has the Jacobi operator given by

$$\mathcal{L}_g h = \nabla^* \nabla h - 2\overset{\circ}{R}h,$$

where $(\overset{\circ}{R}h)_{ij} = R_{ikjl}h_{kl}$ denotes the action of the curvature on symmetric 2-tensors. It is eseentially the Lichnerowicz Laplacian [1] (exactly when the metric g is Ricci flat; otherwise, they differ by a term involving the Ricci curvature). For Ricci flat metrics, the stability problem is raised by Kazdan-Warner [22] about thirty years ago:

Problem Are compact Ricci flat manifolds stable? In other words, if $\mathrm{Ric}(g) = 0$, is the Lichnerowicz Laplacian $\mathcal{L}_g$ positive semi-definite?

This problem is referred as the "*positive mass problem for Ricci flat manifolds*" in [7] (see also [12] for a connection with the positive mass theorem in [11]). There are positive results of Bourguignon and Koiso [1] in the case when the sectional curvature is sufficiently pinched. However, not much else was known.

In [12], we solve this problem in the positive for a large class of Ricci flat metrics, namely those which admit parallel spinors up to a cover. Manifolds with parallel spinors are necessarily Ricci flat. Moreover, for simply connected irreducible manifolds, the existence of parallel spinors is equivalent to special honolomy; namely, Calabi-Yau, hyperKähler, G_2 and $Spin(7)$. The problem in the case of $K3$ surfaces is also solved in [17]. Our class of Ricci flat metrics encompass all the known examples. However, it is still an open problem whether these are all the Ricci flat metrics.

Lichnerowicz Laplacian appears naturally in many geometric variational problems. In [7], Cao-Hamilton-Ilmanen studied the stability problem for Ricci solitons and Ricci shrinking solitons using the functionals introduced by Perelman [35] and showed that they are governed by the Lichnerowicz Laplacian. Thus, as an application of our result, Cao-Hamilton-Ilmanen deduce that compact manifolds with nonzero parallel spinors are also stable as Ricci soliton [7]. The application of our result to Ricci flow is discussed in our work [13] based on [38].

The existence of parallel spinors forces the metric to be Ricci flat. In order to deal with general Einstein metrics, in [13], we extended our approach using spin^c structures. We show that, if a compact Einstein manifold (M, g) with nonpositive scalar curvature admits a nonzero parallel spin^c spinor, then it is stable.

Since a Kähler manifold with its canonical spin^c structure has nonzero parallel spin^c spinors, this implies any compact Kähler-Einstein manifold with non-positive scalar curvature is stable. This also follows essentially from Koiso's work [27], [1], although it does not seem to have been noticed before.

Our approach of using spin^c structure is new and gives more general result. Moreover, by using interesting geometric variational problems, we give applications to Yamabe invariant and prove a surprising volume comparison for scalar curvature. Let us also mention the well known result for compact Einstein manifold with negative sectional curvature [25], [44], [1]. In this case the manifold is strictly stable in the sense that the operator $\mathcal{L}_g$ is in fact positive definite. In contrast, there are many Einstein manifolds with positive scalar curvature which are unstable [26], [7] (see also [3]).

In the case of Kähler-Einstein manifold with Einstein constant c we obtain the following interesting Bochner type formula:

$$\langle \mathcal{D}\Phi(h), \mathcal{D}\Phi(h)\rangle = \langle \mathcal{L}_g h, h\rangle + 2c\langle h_H, h_H\rangle,$$

where h_H denotes the Hermitian part of h. This shows that the eigenvalues of the twisted Dirac operator $\mathcal{D}$ comes into play when dealing with Kähler-Einstein manifolds with positive scalar curvature.

2 Einstein metrics from variational problems

A Riemannian metric g on M^n is called an Einstein metric if $\mathrm{Ric} = \lambda g$ for λ some constant.

If M^n is a compact Riemannian manifold, then (M^n, g) is an Einstein metric of volume 1 iff it's a critical point of the total scalar curvature functional (Hilbert action) $\mathcal{S}$ restricted to $\mathcal{M}_1 = \{ g \mid \mathrm{vol}_g = 1 \}$,

$$\mathcal{S} : \mathcal{M} \to \mathbb{R}, \quad \mathcal{S}(g) = \int S_g dvol.$$

When (M^n, g) is an Einstein metric of total volume 1, one can consider the second variation of the total scalar curvature functional at g. Thus, for any symmetric 2-tensor h on M, let $g(t)$ for $t \in (-\epsilon, \epsilon)$ be a smooth family of metrics in $\mathcal{M}_1$ with $g(0) = g$ and $\frac{d}{dt}g(t)|_{t=0} = h$. Then [1]

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{S}(g(t))|_{t=0} = \int_M \langle h, -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h + \frac{1}{2}(\Delta tr_g h)g \\ -(S/n)(tr_g h)g + \delta(\delta h)g + \delta^*\delta h\rangle,\end{aligned}$$

where the action of curvature tensor on the symmetric 2-tensors is given by $(\overset{\circ}{R}h)_{ij} = R_{ikjl}h_{kl}$, $\delta = \delta_g$ denotes the divergence operator on the tensors and δ^* its adjoint .

In the direction orthogonal to diffeomorphism and conformal changes, i.e. when $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$, it simplifies considerably:

$$\frac{d^2}{dt^2}\mathcal{S}(g(t))|_{t=0} = \int_M \langle h, -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h\rangle = -\frac{1}{2}\int_M \langle h, \mathcal{L}_g h\rangle.$$

The infinitesimal Einstein deformations in $\mathcal{M}_1$ modulo diffeomorphism are solutions of the system

$$\delta_g h = 0, \quad tr_g h = 0, \quad \mathcal{L}_g h = \nabla^*\nabla h - 2\overset{\circ}{R}h = 0.$$

Definition: Let g be an Einstein metric on M.

1). g is (infinitesimally) stable if

$$\langle \mathcal{L}_g h, h\rangle \geqslant 0$$

for any symmetric 2-tensor $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$.

2). g is strictly stable if

$$\langle \mathcal{L}_g h, h\rangle > 0$$

for any symmetric 2-tensor $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$.

3). g is strongly stable if there exists $\lambda > 0$ such that

$$\langle \mathcal{L}_g h, h\rangle \geqslant \lambda \|h\|^2$$

for any symmetric 2-tensor $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$.

For compact manifolds, the last two notions are equivalent. In this paper we restrict our attention to compact manifolds. Clearly, g is stable if the total scalar curvature achieves a local maximum at g among the space of constant scalar curvature metrics of volume 1.[1]

Early study concerns the strong/strict stability of Einstein metrics [1]. The following is the result of [24].

Theorem 1 (Koiso). *Let g be an Einstein metric on M^n. Denote by K_{min} and K_{max} the minimum and the maximum of its sectional curvature. If*

$$\min\left\{(n-2)K_{max} - \frac{S}{n},\ \frac{S}{n} - nK_{min}\right\} < \max\left\{-\frac{S}{n},\ \frac{S}{2n}\right\}, \tag{2.1}$$

then g is strongly stable.

It follows then that an Einstein metric with sufficiently pinched positive sectional curvature, $K_{min} > \frac{n-2}{3n} K_{max}$, is strongly stable. (This is an unpublished result of Bourguignon.) Also, an Einstein metric with negative sectional curvature is strongly stable. Other examples of strongly stable Einstein metrics include all compact irreducible symmetric spaces except $Sp(n), n \geqslant 2$, $Sp(n)/U(n), n \geqslant 3$, and $SU(2n)/Sp(n)$,
$n \geqslant 3$ [26]. In particular $S^n, \mathbb{C}P^n, \mathbb{H}P^n$ are strongly stable.

An important consequence of strong stability is the local rigidity of such Einstein metrics as the strong stability prevents infinitesimal Einstein deformation. Hence by [24] they are not locally deformable.

For an Einstein metric to be stable, one allows infinitesimal Einstein deformations. Thus, such Einstein metrics could be sitting in a moduli space of positive dimension. Koiso's result also gives us a class of stable Einstein metrics when one replaces the strict inequality of (2.1) by the weak inequality ($\leqslant$). It is also interesting to note that one of the examples above, $SU(2n)/Sp(n)$ for $n \geqslant 3$, although not strongly stable, is actually stable [3].

[1]Some authors use this as the definition of stability, Cf. [26] also [3].

Let us mention several examples of unstable Einstein metrics: 1). product of two Einstein metrics with positive Einstein constants; 2). Jensen spheres S^{4q+3}; 3). Kähler-Einstein metrics with positive scalar curvature and $\dim H^{1,1}(M) \geqslant 2$. In particular $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}, 3 \leqslant k \leqslant 8$; 4). many homogenous Einstein metrics ([4], [3]).

Einstein metrics are also critical points of

- the first eigenvalue of the conformal Laplacian $\Delta_g + \frac{n-2}{4(n-1)}S_g$ restricted to $\mathcal{M}_1$.
- Perelman invariant $\lambda(g)$, namely, the first eigenvalue of $4\Delta_g + S_g$, restricted to $\mathcal{M}_1$.
- the Yamabe functional $Y(g) = \frac{\int_M S_g dV_g}{\mathrm{Vol}(g)^{1-\frac{2}{n}}}$.
- $K(g) = \int_M |S_g|^{n/2} dV_g$.

Moreover, their second derivatives at an Einstein metric for $h \in \delta_g^{-1}(0) \cap tr_g^{-1}(0)$ are the same as the second derivative of the total scalar curvature (up to a constant).

Also, we note the functionals K and Y are scale invariant.

3 Parallel spinors and Bochner type formulas

In the work of [24], Bochner type formulas play a crucial role. One of such formulas is obtained by viewing symmetric 2-tensors as 1-forms with values in the cotangent bundle. In [12], we introduce spinors and Dirac operators into this circle of ideas which enables us to deal with Ricci flat metrics.

Theorem 2. *If a compact Riemannian manifold (M, g) has a cover which is spin and admits nonzero parallel spinors, then g is stable.*

If (M, g) has a cover which is spin and admits nonzero parallel spinors, then g is necessarily Ricci flat. In [13], we generalize our results to manifolds with nonzero parallel spin^c spinor. The motivation here is to extend our method to deal with nonzero scalar curvature and we found spin^c to be a good framework to work with.

Theorem 3. *If a compact Einstein manifold (M, g) with nonpositive scalar curvature admits a nonzero parallel spin^c spinor, then it is stable.*

For the spinors to be well defined, the manifold needs to have spin (or spin^c) structures. Recall a manifold M has a spin structure if the second Stiefel-Whitney class $w_2(M) = 0$. It has a spin^c structure if $w_2(M) \equiv c \mod 2$ for some $c \in H^2(M, \mathbb{Z})$. Any manifold with an almost complex structure J has a (canonical) spin^c structure since $w_2(M) \equiv c_1(J) \mod 2$.

Since Kähler manifold with its canonical spin^c structure has a nonzero parallel spin^c spinor, we have

Corollary 4. *A compact Kähler-Einstein manifold with non-positive scalar curvature is stable.*

This also follows essentially from Koiso's work on deformation of Kähler-Einstein metrics.

The key idea in [12, 13] is to construct a Dirac operator, suitably twisted, whose square is the Lichnerowicz Laplacian plus some curvature terms. More precisely, let (M, g) be a compact Riemannian manifold with a spin^c structure. Thus, $w_2(M) \equiv c$, where $c \in H^2(M, \mathbb{Z})$ is the canonical class of the spin^c structure. Let $\mathcal{S}^c \to M$ denote the spin^c spinor bundle and $L \to M$ the complex line bundle with $c_1(L) = c$. Then

$$\mathcal{S}^c = \mathcal{S} \otimes L^{1/2}$$

Here, on the right hand side, the spinor bundle $\mathcal{S}$ and the square root $L^{1/2}$ may exist only locally. Since $\mathcal{S}$ has a natural connection induced by the Levi-Civita connection on TM, a unitary connection ∇^L on L gives rise to a connection ∇^c on $\mathcal{S}^c$: $\nabla^c = \nabla \otimes 1 + 1 \otimes \nabla^{L^{1/2}}$. The curvature of this connection is given by

$$R_{XY}\sigma = \frac{1}{4} R(X, Y, e_i, e_j) e_i e_j \cdot \sigma - \frac{1}{2} F(X, Y)\sigma,$$

for a spin^c spinor σ. Here F is the curvature form of ∇^L.

If σ_0 is a parallel spin^c spinor, i.e., σ_0 is a section of $\mathcal{S}^c$ such that $\nabla^c_X \sigma_0 = 0$ for all X, then $R_{XY}\sigma_0 = 0$. Hence we have

$$R_{klij} e_i e_j \cdot \sigma_0 = 2F_{kl}\sigma_0.$$

This implies

$$R_{kl} e_l \cdot \sigma_0 = F_{kl} e_l \cdot \sigma_0.$$

In the case when the spin^c structure comes from a spin structure, the line bundle L is trivial; consequently $F = 0$. Thus $\mathrm{Ric} \equiv 0$ for manifolds with nonzero parallel spinor.

Given σ_0 a nonzero parallel spin^c spinor (normalized to be of unit length), we define a linear map $\Phi : S^2(M) \to \mathcal{S}^c \otimes T^*M$ by

$$\Phi(h) = h_{ij} e_i \cdot \sigma_0 \otimes e^j.$$

This enables us to view a symmetric 2-tensors as a twisted spin^c spinor.

The map Φ satisfies the following properties:

1. $\mathrm{Re}\,\langle \Phi(h), \Phi(\tilde{h}) \rangle = \langle h, \tilde{h} \rangle$,
2. $\nabla^c_X \Phi(h) = \Phi(\nabla_X h)$.

Here Re denotes the real part. The following Bochner type formula is crucial.

Lemma 5. *Let h be a symmetric 2-tensor on M. Then*

$$\mathcal{D}^*\mathcal{D}\Phi(h) = \Phi(\nabla^*\nabla h - 2\overset{\circ}{R}h - h \circ F + Ric \circ h).$$

Here $(h \circ F)_{ij} = h_{ip}F_{pj} = -h_{ip}F_{jp}$ and $(Ric \circ h)_{ij} = R_{ip}h_{jp}$.

Once again, when the spin^c structure comes from a spin structure, the formula above becomes

$$\mathcal{D}^*\mathcal{D}\Phi(h) = \Phi(\nabla^*\nabla h - 2\overset{\circ}{R}h).$$

This formula was also found by M. Wang [43] in a different form.

By [33], a compact simply connected manifold with nonzero parallel spin^c spinor is the product of a Kähler manifold with a manifold with parallel spinor. Moreover, the spin^c structure is the product of the canonical spin^c structure on the Kähler manifold with the spin structure on the other factor.

For Kahler manifolds we have

Theorem 6. *If (M, g) is a compact Kähler manifold with nonpositive Ricci curvature, then $\mathcal{L}_g h = \nabla^*\nabla h - 2\overset{\circ}{R}h$ is positive semi-definite on $S^2(M)$. That is,*

$$\langle \mathcal{L}_g h, h\rangle \geqslant 0,$$

for any $h \in S^2(M)$. Moreover, in the case of negative Ricci curvature, $\mathcal{L}_g h = 0$ iff $\mathcal{D}\Phi(h) = 0$ and h is skew-hermitian.

In general, the Licherowicz Laplacian is given by

$$\Delta_L(h) = \nabla^*\nabla h - 2\overset{\circ}{R}h + Ric \circ h + h \circ Ric.$$

For manifolds with nonnegative Ricci curvature, one can actually say something about the Licherowicz Laplacian.

Theorem 7. *If (M, g) is a compact Kähler manifold with nonnegative Ricci curvature, then the Licherowicz Laplacian Δ_L is positive semi-definite on $S^2(M)$:*

$$\langle \Delta_L h, h\rangle \geqslant 0,$$

for any $h \in S^2(M)$. Moreover, in the case positive Ricci curvature, $\Delta_L h = 0$ if and only if $\mathcal{D}\Phi(h) = 0$ and h is Hermitian.

In the case of Kähler-Einstein manifold with Einstein constant c we have the following interesting Bochner-Lichnerowics formula:

$$\langle \mathcal{D}\Phi(h), \mathcal{D}\Phi(h)\rangle = \langle \nabla^*\nabla h - 2\overset{\circ}{R}h, h\rangle + 2c\langle h_H, h_H\rangle,$$

where h_H denotes the Hermitian part of h.

We finish the section with the following question, which is an analog of Kazdan-Warner's question.

Questions: Are all compact Einstein manifolds with nonpositive scalar curvature stable?

4 Applications to Yamabe invariant and volume

Recall that the Yamabe invariant is a conformal invariant of g given by $\mu(g) = \inf Y(g)$ with the infimum taking over the conformal class of g. By the solution to Yamabe problem, the infimum is always achieved.

Theorem 8. *Let (M,g) be a compact Einstein manifold with nonpositive scalar curvature which admits a parallel spin^c spinor. Suppose all infinitesimal Einstein deformations of g are integrable. Then g is a local maximum of the Yamabe invariant.*

This property can be used to distinguish such Einstein metrics from the more general ones; see the next section for some discussion.

In the case of when the spin^c structure comes from a spin structure, the integrability condition is automatic by the Bogomolov-Tian-Todorov theorem, [5], [41], [42], see also [21].

For Kähler-Einstein metrics, the integrability is equivalent to that of the complex structure [27].

Theorem 9 (Koiso). *Let (M,g,J) be a Kähler-Einstein metric. If all infinitesimal complex deformations of J are integrable and $c_1(J) \leqslant 0$, then the premoduli space of Einstein metrics around g is smooth at g. Moreover, any metric $\tilde{g}$ in it is Kahler with respect to some complex structure $\tilde{J}$ close to J.*

From this result and the Kodaira-Spencer theory, one can find many examples of Kähler-Einstein manifolds with negative scalar curvature which satisfy the integrability condition, e.g., all hypersurfaces of degree $d \geqslant m+2$ in $\mathbb{C}P^m (m \geqslant 3)$.

Remark In general these Kähler-Einstein metrics are not global maximum of the Yamabe invariant since the manifolds could have metrics with $S > 0$ (e.g. when $\dim = 6 \mod 8$ by using [39]). Moreover, by [34], the Yamabe constant of compact simply connected manifold of dimension $\geqslant 5$ is nonnegative.

On the other hand, LeBrun [28] showed that Kähler-Einstein surfaces with negative scalar curvature are global maximum of the Yamabe invariant.

In general, scalar curvature only give infinitesimal information on the volume of geodesic balls. Using the functional $K(g)$ introduced in §2 we have

Theorem 10. *Let (M,g_0) be a compact Einstein manifold with $S_{g_0} < 0$ which admits a parallel spin^c spinor. Suppose all infinitesimal Einstein deformations of g_0 are integrable. Then there exists a neighborhood $\mathcal{U}$ of g_0 such that for any metric $g \in \mathcal{U}$ with scalar curvature $S_g \geqslant S_{g_0}$*

$$\mathrm{Vol}(M,g) \geqslant \mathrm{Vol}(M,g_0)$$

and equality holds iff g is also an Einstein metric which admits a parallel spin^c spinor and $S_g = S_{g_0}$.

This class of metrics is essentially the same as that of Kähler-Einstein metrics. Besson-Courtois-Gallot [2] proved the same result for Einstein metrics with negative sectional curvature. Note that if (M,g_0) is metric with negative constant scalar curvature and satisfies the volume comparison above, then g_0 is an Einstein metric.

5 Positive scalar curvature and scalar flat metrics on Calabi-Yau manifolds

In this section we discuss the implications of our stability result to metrics with positive scalar curvature and scalar flat metrics on manifolds with special holonomy, especially Calabi-Yau manifolds.

The stability result suggests that around the metric with parallel spinors the total scalar curvature tends to be nonpositive. In fact, we have

Theorem 11. *If (M, g) is a compact spin manifold which admits a nonzero parallel spinor, then, there is no metrics of positive scalar curvature in a (scale invariant) neighborhood of g in the space of Riemannian metrics.*

This was conjectured by the physicists Hertog-Horowitz-Maeda [18]. Our proof makes use of the variational problem for the first eigenvalue of the conformal Laplacian $\Delta_g + \frac{n-2}{4(n-1)}S(g)$ whose second variation is also governed by $\mathcal{L}_g$. It is interesting to compare with [35] where the first eigenvalue of $\Delta_g + \frac{1}{4}S(g)$ is used.

The existence of metrics with positive scalar curvature is a well studied subject (see [40], [31], [19], [37], [16], [39]). A $K3$ surface does not admit any metric of positive scalar curvature, but a simply connected Calabi-Yau 3-fold does [39]. However, these metrics of positive scalar curvature on a Calabi-Yau 3-fold should not be too close to the Calabi-Yau metrics.

This result has the following interesting consequences.

Corollary 12. *Any scalar flat deformation of a Calabi-Yau metric on a compact manifold must be a Calabi-Yau deformation. In fact, any deformation with non-negative scalar curvature of a Calabi-Yau metric on a compact manifold is necessarily a Calabi-Yau deformation. The same is true for the other special holonomy metrics, i.e., hyperkähler, G_2, and $Spin(7)$.*

The following corollary is a special case of Theorem 8.

Corollary 13. *Let (M, g_0) be a compact, simply connected Riemannian spin manifold of dimension n with a parallel spinor. Then g_0 is a local maximum of the Yamabe invariant.*

Corollary 13 gives a geometric way of distinguishing Calabi-Yau (or other special holonomy) metrics from the general Ricci flat or scalar flat metrics. For example, we use it to show that there are scalar flat metrics on certain Calabi-Yau (or G_2) manifolds which are not Calabi-Yau (or G_2) [12]. Here we need to use the theorem of Stolz [39] on the existence of positive scalar curvature metrics and the scalar flat metric is found along a path connecting a positive scalar metric to a negative scalar curvature metric.

If one thinks the Kazdan-Warner's problem as the infinitesimal stability, Theorem 11 mentioned above is then a local stability. In this view, one is led to the question of global stability for Calabi-Yau manifolds.

Question: Is there any scalar flat but not Ricci flat metric on a compact Calabi-Yau manifold? In other words, must a scalar flat metric on a compact Calabi-Yau manifold Ricci flat?

A closely related question is

Question: Are Ricci flat metrics necessarily Calabi-Yau on a compact Calabi-Yau manifold?

We note that the deformation analogue of this question has a positive answer by the work of [27], [5], [41], [42]. Moreover, in (real) dimension four, the answer is positive by the work of Hitchin [20] on the rigidity case of Hitchin-Thorpe inequality. However, one of these two questions will have a negative answer by our result.

In another direction, Futaki [14] characterized scalar flat closed manifolds not admitting positive scalar curvature metrics. Namely they are product of special holonomy manifolds with nonzero α invariant. One can think of this result as characterizing scalar flat metrics which achieve global maximal in Yamabe invariant.

6 Stability of Ricci flow, Perelman invariant

There has been a lot of work recently concerning the stability of Ricci flow [17], [38], [10], see also [7]. The general question can be phrased as follows. If g_0 is a metric such that the (normalized) Ricci flow $g(t)$ starting from g_0 converges, is it true that the (normalized) Ricci flow $\tilde{g}(t)$ starting from all metrics $\tilde{g}_0$ that are sufficiently close to g_0 also converges? Using the method of Natasa Sesum [38] (see also [9]), we obtain [13]

Theorem 14. *Let (N, g_0, J_0) be a compact Kähler-Einstein manifold with non-positive scalar curvature. Suppose all infinitesimal complex deformations of J_0 are integrable. Then the (normalized) Ricci flow starting from any Riemannian metric sufficient close to g_0 converges exponentially to a Kähler-Einstein metric.*

The difference between this theorem and the well known result for Kähler-Ricci flow on Kähler-Einstein manifolds with nonpositive first Chern class [6] is that the Ricci flow here starts with any metric nearby, rather than just the ones in a given Kähler class. On the other hand, the result of [6] is a global result in the sense that the initial metric is any metric in a given Kähler class. However, even though we start with any nearby Riemmannian metric, the (normalized) Ricci flow still converges to a Kähler-Einstein metric.

In particular, we have

Corollary 15. *The Calabi-Yau metrics and G_2 metrics are stable under the Ricci flow.*

We finally make some remarks about Perelman invariant. Recall that Perelman invariant, $\lambda(g)$, is the lowest eigenvalue of the operator $4\Delta + S$.

The first variation formula for $\lambda(g)$ is given by [35]

$$\frac{d}{dt}|_{t=0}\lambda(g(t)) = -\int e^{-f}\langle Ric + D^2 f, h\rangle,$$

where $e^{-f/2}$ is the eigenfunction of $\lambda(g)$ with $\int_M e^{-f} dvol = 1$. So f satisfies

$$-2\Delta f + |Df|^2 + S = \lambda(g).$$

Letting $h = -2(Ric + D^2 f)$ one gets [35]

Theorem 16 (Perelman). *$\lambda(g(t))$ is nondecreasing along any Ricci flow.*

By restricting the variation to $\mathcal{M}_1$, we deduce that g is a critical point of $\lambda(g)$ iff g satisfies the equation

$$Ric + D^2 f - \frac{R + \Delta f}{n} g = 0.$$

In particular, Einstein metrics are critical points.

If g is Einstein, the second variation is [7]

$$\frac{d^2}{dt^2}\lambda(g(t))|_{t=0} = \int_M \langle -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h, h\rangle + |\delta h|^2 - \frac{1}{2}|Dv_h|^2 + \frac{S}{2n}(trh)^2 dV_g,$$

where v_h satisfies $\Delta v_h = \delta^2 h - \frac{S}{n} trh$.

Since $\lambda(g)$ is diffeomorphism invariant, we can restrict the variation to h with $\delta h = 0$. Then we deduce that

$$\frac{d^2}{dt^2}\lambda(g(t))|_{t=0} \leqslant 0$$

if g is Kähler-Einstein with $S \leqslant 0$.

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