



# Comparison between two analytic torsions on orbifolds

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**Abstract** In this paper, we establish an equality between the analytic torsion introduced by Dar (Math Z 194(2): 193–216, 1987) and the orbifold analytic torsion defined by Ma (Trans Am Math Soc 357(6): 2205–2233, 2005) on an even dimensional manifold with isolated conical singularities which in addition has an orbifold structure. We assume the orbifold flat vector bundle is an honest vector bundle, although the metric on the flat bundle may not be flat.

## 1 Introduction

Let  $X$  be a closed Riemannian manifold and  $F$  be a flat real vector bundle over  $X$  carrying the flat connection  $\nabla^F$ . As a geometric invariant, the analytic torsion of  $X$ , was first introduced by Ray and Singer [20,21] in searching for an analytic interpretation of the combinatorially defined Reidemeister torsion. The latter is the first topological invariant which is not a homotopy invariant, introduced by Reidemeister [22] and Franz [12].

The Ray–Singer analytic torsion is a weighted linear combination of the determinants of the Laplacian acting on the space of differential forms with values in  $F$ , and depends on the the metrics on  $F$  and on the Riemannian manifold. We explain this in greater detail. Let  $g^{TX}$  be a Riemannian metric on  $X$  and  $g^F$  be an Euclidean metric on  $F$ . We denote by  $\Omega^i(X, F)$  the space of smooth  $i$ -forms on  $X$  with values in  $F$ , and set  $\Omega(X, F) = \bigoplus_i \Omega^i(X, F)$ . The flat connection  $\nabla^F$  induces naturally a differential  $d^F$  on  $\Omega(X, F)$ . Let  $d^{F*}$  be the (formal) adjoint

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operator of  $d^F$  associated with  $g^{TX}$  and  $g^F$ . The Laplacian operator  $\Delta := d^F d^{F*} + d^{F*} d^F$  acts on  $\Omega(X, F)$  and preserves its  $\mathbb{Z}$ -grading. Let  $P$  be the orthogonal projection operator from  $\Omega(X, F)$  to  $\ker \Delta$ . Let  $e^{-t\Delta}$  be the heat semi-group operator of  $\Delta$ .

As usual  $\Gamma$  denotes the gamma function. The torsion zeta function is defined as follows. For  $s \in \mathbb{C}, \Re s > \frac{\dim X}{2}$ ,

$$\zeta(s) = \frac{-1}{\Gamma(s)} \sum_{i=0}^{\dim X} (-1)^i i \int_0^\infty t^{s-1} \text{Tr} \Big|_{\Omega^i(X,F)} [e^{-t\Delta}(1 - P)] dt. \tag{1.1}$$

By the standard elliptic theory and the asymptotic expansion of heat kernel,  $\zeta(s)$  extends to a meromorphic function of  $s \in \mathbb{C}$  which is holomorphic at  $s = 0$ . The Ray–Singer analytic torsion of  $X$  with coefficients in  $F$  is defined as

$$T(X, g^{TX}, g^F) = \exp \left\{ \frac{1}{2} \frac{\partial \zeta}{\partial s}(0) \right\}. \tag{1.2}$$

The celebrated Cheeger–Müller theorem [7, 17] establishes the Ray–Singer conjecture. Namely, if  $g^F$  is flat, i.e.  $(F, \nabla^F)$  is induced by an orthogonal representation of the fundamental group of  $X$ , then the analytic torsion coincides with the classical Reidemeister torsion. There are various generalizations of this result. Müller [18] extended his result to the case of unimodular representation, where only the metric induced on  $\det F$  is required to be flat. Simultaneously, Bismut and Zhang [4] reformulated the above Cheeger–Müller theorem as an equality between the Reidemeister and Ray–Singer metrics defined on the determinant of cohomology, and proved an extension to arbitrary flat vector bundles with arbitrary Euclidean metrics.

Based on the work of Goresky, MacPherson [13, 14] and Cheeger [8–10], Dar [11] described a possible extension of the Cheeger–Müller theorem to singular spaces. Assume now that  $X$  is a Riemannian manifold with isolated conical singularities  $\Sigma$  and  $F$  be a flat bundle over  $X \setminus \Sigma$ . Let  $\Omega_0^i(X \setminus \Sigma, F)$  be the space of smooth  $i$ -forms with values in  $F$ , compactly supported on  $X \setminus \Sigma$ , and set  $\Omega_0(X \setminus \Sigma, F) = \oplus_i \Omega_0^i(X \setminus \Sigma, F)$ . Choose an ideal boundary condition in the sense of Cheeger [8, 9], which corresponds to a closed extension of the de Rham complex  $(\Omega_0(X \setminus \Sigma, F), d^F)$  into a Hilbert complex. Let  $\Delta_c$  be the Laplacian associated to the Hilbert complex, and  $e^{-t\Delta_c}$  be the heat semi-group operator of  $\Delta_c$ .

Although the asymptotic expansion of the trace of the heat kernel of  $e^{-t\Delta_c}$  can contain logarithmic terms [10], Dar [11] made a crucial observation that after taking the weighted linear combination as in (1.1), the contribution from the logarithmic term drops out. Thus, one can define the analytic torsion  $T_c(X, g^{TX}, g^F)$  as in (1.2).

On the other hand, Ma [16] extended the Quillen metric to compact complex orbifolds. In this setting, he established the anomaly formula and calculated the behavior of the Quillen metric by orbifold immersions, which generalized the corresponding results in [2] and [3].

We assume in addition that  $X$  carries an orbifold structure and the relevant geometric data are all in the orbifold category. Following Ma [16], one can define the orbifold torsion  $T_0(X, g^{TX}, g^F)$  in this real setting.

The main result of this paper is an equality between these two analytic torsions on a special type of singular space, which are defined from the rather distinct perspectives to the singularities.

**Theorem 1.1** *Let  $X$  be an even dimensional compact manifold with isolated conical singularities, which has an orbifold structure, and  $g^{TX}$  be an orbifold metric on  $X$  which is also conical near the singularities. Let  $F$  be a flat real honest orbifold vector bundle with an Euclidean metric  $g^F$  over  $X$ . If near the singularities  $g^F$  is flat, then*

$$T_c \left( X, g^{TX}, g^F \right) = T_o \left( X, g^{TX}, g^F \right). \tag{1.3}$$

By an honest orbifold vector bundle we mean that locally it defines a vector bundle on the quotient space. The proof of Theorem 1.1 consists of the comparison of the two kinds of cohomology and of the two heat kernels.

The isomorphism between the two kinds of cohomology can be established by a straightforward calculation using the Mayer-Vietoris sequence and Cheeger’s result [8] on  $L^2$  cohomology.

The comparison of the two heat kernels is inspired by Cheeger’s approach [7] to the Ray–Singer conjecture. Our basic observation is that Cheeger’s constructions of the Green kernels on the annulus  $A_{u,1}^{m+1}$  and the corresponding estimates in [7, Section 6] can be used with little modification to establish a Sobolev inequality on  $C_{(0,1]}(S^m/G)$ , from which we get the estimate for the heat kernel constructed from the point of view of conical singularities. Combining this with the estimate for orbifold heat kernel [16] and applying the Duhamel principle [7, (3.9)], we obtain an equality between these two heat kernels outside the singularities.

The rest of this paper is organized as follows. In Sect. 2, we first recall the relevant facts about  $L^2$ -cohomology and the basic definitions of the two analytic torsions introduced by Dar [11] and Ma [16], respectively. In Sect. 3, we first prove a Sobolev inequality on the bounded cone, then for an even dimensional manifold with isolated conical singularities which carries an orbifold structure, we establish an isometry between the  $L^2$ -cohomology and the singular cohomology and an equality between  $K_c(t, \cdot, \cdot)$  and  $K_o(t, \cdot, \cdot)$ . As a corollary, we deduce our main result.

## 2 Two analytic torsions

### 2.1 The analytic torsion on a manifold with conical singularities

Let  $N$  be a closed manifold carrying a Riemannian metric  $g^{TN}$ . The model cone  $C(N)$  over  $N$  is the space  $(0, +\infty) \times N$  carrying the conical metric  $dr^2 + r^2g^{TN}$ , where  $r$  denotes the radial coordinate. For  $u > 0$ , set

$$\begin{aligned} C_{(0,u)}(N) &= \{(r, y) \in C(N) \mid 0 < r < u\}, \\ C_{(0,u]}(N) &= \{(r, y) \in C(N) \mid 0 < r \leq u\}. \end{aligned} \tag{2.1}$$

Let  $X$  be a  $m + 1$  dimensional compact Riemannian manifold with isolated conical singularities  $\Sigma$ . By this we mean that  $\Sigma$  is a finite set,  $X \setminus \Sigma$  is a smooth manifold of dimension  $m + 1$  with a Riemannian metric  $g^{TX}$ , and for  $p \in \Sigma$ , there exists an open neighborhood  $\mathcal{U}_u(p)$  ( $u > 0$ ) in  $X$  and a closed Riemannian manifold  $(N_p, g^{TN_p})$  such that  $(\mathcal{U}_u(p) \setminus \{p\}, g^{TX}|_{\mathcal{U}_u(p) \setminus \{p\}})$  is isometric to the cone  $(C_{(0,u)}(N_p), dr^2 + r^2g^{TN_p})$ . For  $u > 0$  small enough, set

$$C_{(0,u)}^*(N_p) = C_{(0,u)}(N_p) \cup \{p\}, \quad C_{(0,u]}^*(N_p) = C_{(0,u]}(N_p) \cup \{p\}. \tag{2.2}$$

Then  $X_u := X \setminus (\sqcup_{p \in \Sigma} C_{(0,u)}^*(N_p))$  is a compact manifold with boundary,

$$X = X_u \cup \left( \sqcup_{p \in \Sigma} C_{(0,u]}^*(N_p) \right), \tag{2.3}$$

where the union is along the boundary  $\sqcup_{p \in \Sigma} \{u\} \times N_p$ .

Let  $F$  be a flat real vector bundle over  $X \setminus \Sigma$  carrying the flat connection  $\nabla^F$ . Let  $\Omega_0^i(X \setminus \Sigma, F)$  be the space of smooth  $i$ -forms with values in  $F$ , compactly supported on

$X \setminus \Sigma$ , and set  $\Omega_0(X \setminus \Sigma, F) = \bigoplus_i \Omega_0^i(X \setminus \Sigma, F)$ . Let  $L^2(\Omega^i(X \setminus \Sigma, F))$  be the Hilbert space which consists of square integrable  $i$ -forms on  $X \setminus \Sigma$  with values in  $F$ .

We recall the main features of the  $L^2$ -cohomology of  $X$  with coefficients in  $F$ . Let  $(\Omega_0(X \setminus \Sigma, F), d^F)$  be the de Rham complex, where  $d^F$  is the differential induced from the flat connection  $\nabla^F$  in a natural way. Recall that an ideal boundary condition (cf. [9, Section 2], [5, p. 105]) for the complex  $(\Omega_0(X \setminus \Sigma, F), d^F)$  is a choice of closed extensions  $d_{c,i}^F$  of  $d_i^F$  in  $L^2(\Omega^i(X \setminus \Sigma, F))$  for  $0 \leq i \leq m + 1$ , such that

$$d_{c,i}^F \left( \text{dom} \left( d_{c,i}^F \right) \right) \subset \text{dom} \left( d_{c,i+1}^F \right), \quad d_{c,i+1}^F \circ d_{c,i}^F = 0. \tag{2.4}$$

We then get a Hilbert complex in the sense of Brüning-Lesch [5, p. 90],

$$0 \rightarrow \text{dom} \left( d_{c,0}^F \right) \xrightarrow{d_{c,0}^F} \cdots \xrightarrow{d_{c,m}^F} \text{dom} \left( d_{c,m+1}^F \right) \rightarrow 0. \tag{2.5}$$

Let  $\delta^F$  denote the formal adjoint of  $d^F$ , then the minimal and maximal extensions of  $d^F$ ,

$$d_{\min}^F := \text{closure of } d^F, \quad d_{\max}^F := \text{adjoint of the closure of } \delta^F,$$

are examples of the ideal boundary conditions. A priori there may be many distinct ideal boundary conditions.

We assume that

$$H^{\frac{m}{2}}(N_p, F|_{N_p}) = 0, \quad p \in \Sigma. \tag{2.6}$$

As shown in [6,9], in this case the ideal boundary condition is unique, i.e.,

$$d_{\min,i}^F = d_{\max,i}^F, \quad \text{for } 0 \leq i \leq m + 1. \tag{2.7}$$

We denote by  $(C, d_m)$  the unique extension of  $(\Omega_0(X \setminus \Sigma, F), d^F)$  into a Hilbert complex as in (2.5). The  $L^2$ -cohomology of  $X$  with coefficients in  $F$  is the cohomology of the complex  $(C, d_m)$ ,

$$H_{(2)}^i(X, F) := \ker d_{m,i} / \text{im } d_{m,i-1}, \quad 0 \leq i \leq m + 1. \tag{2.8}$$

Let  $\Delta_c$  be the Laplacian associated to the Hilbert complex  $(C, d_m)$ ,

$$\Delta_c := d_{\min}^F \delta_{\min}^F + \delta_{\min}^F d_{\min}^F. \tag{2.9}$$

We denote by  $\Delta_{c,i}$  the restriction of  $\Delta_c$  to  $C^i$  for  $0 \leq i \leq m + 1$ . By the  $L^2$ -Hodge theorem [9, Section 1], all the  $L^2$ -cohomology groups  $H_{(2)}^i(X, F)$  are finite dimensional and the complex  $(C, d_m)$  is Fredholm in the sense of Brüning-Lesch [5, p. 90]. Moreover, the canonical maps

$$\ker \Delta_{c,i} \longrightarrow H_{(2)}^i(X, F), \quad 0 \leq i \leq m + 1. \tag{2.10}$$

are isomorphisms (see also [5, Corollary 2.5]).

Let  $e^{-t\Delta_{c,i}}$  be the heat semi-group operator of  $\Delta_{c,i}$ , and  $K_{c,i}(t, \cdot, \cdot)$  be the heat kernel of  $e^{-t\Delta_{c,i}}$ . Let  $P_{c,i}$  be the orthogonal projection operator from  $L^2(\Omega^i(X \setminus \Sigma, F))$  to  $\ker \Delta_{c,i}$ . Set  $P_{c,i}^\perp = 1 - P_{c,i}$ .

One defines the  $i$ -th torsion zeta function as follows.

$$\zeta_{c,i}(s) := \frac{-1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr} \left[ e^{-t\Delta_{c,i}} P_{c,i}^\perp \right] dt. \tag{2.11}$$

The asymptotic expansion of the trace of the heat kernel  $K_{c,i}(t, \cdot, \cdot)$  yields a meromorphic extension of  $\zeta_{c,i}(s)$  to the whole complex plane, and determines its behavior near  $s = 0$ . In

particular, it need not be regular at  $s = 0$  because of the appearance of a logarithmic term [10, Theorem 2.1]. The crucial observation made by Dar [11, Theorem 4.4] is that

$$\operatorname{Res}_{s=0} \left( \sum_{i=0}^{m+1} (-1)^i i \cdot \zeta_{c,i}(s) \right) = 0. \tag{2.12}$$

Thus, the full torsion zeta function  $\zeta_c(s) := \sum_{i=0}^{m+1} (-1)^i i \cdot \zeta_{c,i}(s)$  is indeed holomorphic at  $s = 0$ . The analytic torsion  $T_c(X, g^{TX}, g^F)$  of  $X$  with coefficients in  $F$  is defined as in (1.2) (cf. [11, p. 215]),

$$T_c \left( X, g^{TX}, g^F \right) = \exp \left\{ \frac{1}{2} \frac{\partial \zeta_c}{\partial s}(0) \right\}. \tag{2.13}$$

### 2.2 The analytic torsion on an orbifold

We refer to [15] for relevant definitions of orbifolds and to [16] for notations used here. In [15] orbifolds were called V-manifolds.

Let  $(X, \mathcal{U})$  be a compact orbifold endowed with a Riemannian metric  $g^{TX}$ . Let  $F$  be a flat real orbifold vector bundle over  $X$  equipped with the flat connection  $\nabla^F$  and an Euclidean metric  $g^F$ .

We denote by  $\Omega^i(X, F)$  the space of smooth sections of  $\Lambda^i(T^*X) \otimes F$  over  $X$ , and set  $\Omega(X, F) = \bigoplus_i \Omega^i(X, F)$ . The flat connection  $\nabla^F$  induces naturally a differential  $d^F$  on  $\Omega(X, F)$ . Let  $H(X, F) = \bigoplus_{i=0}^{\dim X} H^i(X, F)$  be the singular cohomology group of  $X$  with coefficients in  $F$ . The de Rham theorem for orbifolds [15, p. 78] gives us a canonical isomorphism,

$$H^i(\Omega(X, F), d^F) \simeq H^i(X, F), \quad \text{for } 0 \leq i \leq \dim X. \tag{2.14}$$

Let  $\langle \cdot, \cdot \rangle_{\Lambda(T^*X) \otimes F}$  be the metric on  $\Lambda(T^*X) \otimes F$  induced from  $g^{TX}, g^F$ , and  $dv_X$  be the Riemannian volume form on  $X$  associated to  $g^{TX}$ . As in [16, (2.8)], one defines the  $L^2$ -scalar product  $\langle \cdot, \cdot \rangle$  on  $\Omega(X, F)$ , for  $s, s' \in \Omega(X, F)$ ,

$$\langle s, s' \rangle := \int_X \langle s, s' \rangle_{\Lambda(T^*X) \otimes F}(x) dv_X(x). \tag{2.15}$$

Let  $\delta_0^F$  be the (formal) adjoint operator of  $d^F$  with respect to (2.15). Set

$$\Delta_0 := d^F \delta_0^F + \delta_0^F d^F. \tag{2.16}$$

Then  $\Delta_0$  is a second order differential operator, which acts on  $\Omega(X, F)$  and preserves its  $\mathbb{Z}$ -grading, with  $\sigma_2(\Delta_0) = |\xi|^2$  ( $\xi \in T^*X$ ).

Using the same proof as in [16, Proposition 2.2], one deduces the Hodge decomposition.

**Proposition 2.1** *There is an  $L^2$ -orthogonal direct sum decomposition,*

$$\Omega^i(X, F) = \ker \Delta_{0,i} \oplus \operatorname{im} d_{i-1}^F \oplus \operatorname{im} \delta_{0,i+1}^F, \quad \text{for } 0 \leq i \leq \dim X. \tag{2.17}$$

From (2.14) and (2.17), one has the canonical identification

$$\ker \Delta_{0,i} \simeq H^i(X, F), \quad \text{for } 0 \leq i \leq \dim X. \tag{2.18}$$

Let  $K_0(t, \cdot, \cdot)$  be the heat kernel of the heat semigroup operator  $e^{-t\Delta_0}$  with respect to  $dv_X$ .

**Proposition 2.2** (cf. [16, Proposition 2.1]) *For each  $U \in \mathcal{U}$ , there exists a smooth section  $\Phi_j \in \Gamma(\tilde{U} \times \tilde{U}, \text{pr}_1^* \tilde{F} \otimes \text{pr}_2^* \tilde{F})$  such that for every  $k > \dim X$ ,  $x, y \in U$ , as  $t \rightarrow 0$ ,*

$$K_o(t, x, y) = \frac{(4\pi t)^{-\frac{\dim X}{2}}}{|K_U^F|} \sum_{g \in G_U^F} \sum_{j=0}^k e^{-\frac{\tilde{d}^2(g\tilde{x}, \tilde{y})}{4t}} g^{-1} \Phi_j(g\tilde{x}, \tilde{y}) t^j + O\left(t^{k-\frac{\dim X}{2}}\right). \tag{2.19}$$

On  $\{(x, y) \in X \times X \mid d^X(x, y) > c > 0\}$ , we have

$$\partial_x^\alpha \partial_y^\beta K_o(t, x, y) = O\left(e^{-\frac{c^2}{4t}}\right), \text{ as } t \rightarrow 0. \tag{2.20}$$

From (2.20), one has the following estimate for the pointwise norm of  $K_o(t, \cdot, \cdot)$ .

**Corollary 2.3** *Given  $T > 0$ ,  $\varepsilon > 0$  and  $n \in \mathbb{N}$ , there exists a constant  $K(T, \varepsilon, n) > 0$  such that for  $x, y \in X$  with  $d^X(x, y) > \varepsilon$ , and  $0 < t \leq T$ ,*

$$\|K_o(t, x, y)\| \leq K(T, \varepsilon, n) t^n. \tag{2.21}$$

The same estimates hold for  $d_x^F K_o(t, x, y)$  and  $\delta_{o,y}^F K_o(t, x, y)$ .

Let  $P_o$  be the orthogonal projection operator from  $\Omega(X, F)$  on  $\ker \Delta_o$  with respect to the  $L^2$ -scalar product. Set  $P_o^\perp = 1 - P_o$ .

For  $s \in \mathbb{C}$ ,  $\Re s > \frac{\dim X}{2}$ , set

$$\zeta_o(s) = \frac{-1}{\Gamma(s)} \sum_{i=0}^{\dim X} (-1)^i i \int_0^\infty t^{s-1} \text{Tr} \big|_{\Omega^i(X,F)} [e^{-t\Delta_o} P_o^\perp] dt. \tag{2.22}$$

Using Proposition 2.2,  $\zeta_o(s)$  extends to a meromorphic function of  $s \in \mathbb{C}$  which is holomorphic at  $s = 0$ . The orbifold analytic torsion of  $(X, \mathcal{U})$  with coefficients in  $F$  is defined as in (1.2),

$$T_o\left(X, g^{TX}, g^F\right) = \exp\left\{\frac{1}{2} \frac{\partial \zeta_o}{\partial s}(0)\right\}. \tag{2.23}$$

### 3 The equality between two torsions

In the remaining part, we assume  $m$  is odd.

#### 3.1 The Sobolev inequality on a bounded cone

Let  $S^m \subset \mathbb{R}^{m+1}$  be the sphere of center 0 and radius 1, which is endowed with the Euclidean metric and carries a free orientation preserving action of a finite group  $G$ . Then the quotient space  $N = S^m/G$  is a closed orientable Riemannian manifold. In this subsection, we establish the Sobolev inequality on the bounded cone  $C_{(0,1]}(N)$  following [7].

Let  $F$  be a flat real vector bundle over  $C_{(0,1]}(N)$ , which carries the flat connection  $\nabla^F$  and an Euclidean metric  $g^F$ . We assume that  $g^F$  is flat.

Let  $F^*$  be the dual bundle of  $F$  carrying the (dual) flat connection  $\nabla^{F^*}$  and the (dual) metric  $g^{F^*}$ .

Let  $\Omega^i(C(N), F)$  and  $\Omega^i(C(N), F^*)$  denote the spaces of smooth  $i$ -forms on  $C(N)$  with values in  $F$  and  $F^*$  respectively. Set

$$\Omega(C(N), F) = \bigoplus_{i=0}^{m+1} \Omega^i(C(N), F), \quad \Omega(C(N), F^*) = \bigoplus_{i=0}^{m+1} \Omega^i(C(N), F^*).$$

Let  $d^F$  and  $d^{F^*}$  be the exterior differentials on  $\Omega(C(N), F)$  induced by  $\nabla^F$  and on  $\Omega(C(N), F^*)$  induced by  $\nabla^{F^*}$  respectively.

We choose  $dr \wedge r^m dv_N$  as the oriented volume form of  $C(N)$ , where  $dv_N$  is the oriented volume form of  $N$ . Let  $\delta^F$  be the formal adjoint of  $d^F$  with respect to the natural  $L^2$ -metric on  $\Omega(C(N), F)$ . Set

$$\Delta = d^F \delta^F + \delta^F d^F. \tag{3.1}$$

Set

$$*^F: \Omega^i(C(N), F) \longrightarrow \Omega^{m+1-i}(C(N), F^*), \quad \beta \longmapsto \langle \cdot, * \beta \rangle_{g^F},$$

where  $*$  is the usual Hodge star operator. Then one has

$$\delta^F|_{\Omega^i(C(N), F)} = (-1)^i (*^F)^{-1} d^{F^*} *^F. \tag{3.2}$$

Let  $\Omega^i(N, F)$  denote the space of smooth  $i$ -forms on  $N$  with values in  $F|_N$ . Set  $\Omega(N, F) = \bigoplus_{i=0}^m \Omega^i(N, F)$ . As in [10, Section 3], operations on the cross section  $N$  are indicated by a tilde. Let  $\widetilde{d}^F$  be natural exterior differential on  $\Omega(N, F)$  with formal adjoint  $\widetilde{\delta}^F$ . Set

$$\widetilde{\Delta} = \widetilde{d}^F \widetilde{\delta}^F + \widetilde{\delta}^F \widetilde{d}^F. \tag{3.3}$$

From (3.1)–(3.3), it is a straightforward calculation to show that for

$$\beta = g(r)\phi + f(r)dr \wedge \psi \in \Omega^i(C(N), F),$$

where  $\phi \in \Omega^i(N, F)$ , and  $\psi \in \Omega^{i-1}(N, F)$ , one has (see [7, (6.2)])

$$\begin{aligned} \Delta\beta &= (-g'' - (m - 2i)r^{-1}g')\phi + r^{-2}g\widetilde{\Delta}\phi - 2r^{-3}g\,dr \wedge \widetilde{\delta}^F\phi \\ &\quad + (-f'' - (m - 2i + 2)r^{-1}f' + (m - 2i + 2)r^{-2}f'')\,dr \wedge \psi \\ &\quad + r^{-2}f\,dr\widetilde{\Delta}\psi - 2r^{-1}f\,\widetilde{d}^F\psi. \end{aligned} \tag{3.4}$$

For  $0 \leq i \leq m$ , we choose an orthonormal basis  $\{\phi_j^i\}_{j=1}^\infty$  of the space  $\ker \widetilde{\delta}^F \cap \Omega^i(N, F)$ , such that

$$\widetilde{\Delta}\phi_j^i = \mu_j(i)\phi_j^i, \quad \text{with } 0 \leq \mu_1(i) \leq \mu_2(i) \leq \dots \tag{3.5}$$

We use the same simplified notations as in [10, (3.1–3.3)],

$$\alpha(i) = \frac{1+2i-m}{2}, \quad \nu_j(i) = \sqrt{\mu_j(i) + \alpha^2(i)}, \quad a_j^\pm(i) = \alpha(i) \pm \nu_j(i). \tag{3.6}$$

We now proceed as in [7, pp. 289–291] to construct the Green operators in various cases.

(I) For  $\nu_j(i) > 0$ , set (cf. [7, (6.6)])

$$h_{\mu_j(i)}(r_1, r_2) = \frac{1}{2\nu_j(i)} \cdot \begin{cases} a_j^+(i) r_1 & a_j^-(i) r_2, & r_1 \leq r_2, \\ a_j^-(i) r_1 & a_j^+(i) r_2, & r_2 \leq r_1. \end{cases} \tag{3.7}$$

(II) For  $\nu_j(i) = 0$ , which means  $m = 1, i = 0, j = 1$ , set (cf. [7, (6.7)])

$$h_{\mu_1(0)}(r_1, r_2) = \begin{cases} -\log r_2, & r_1 \leq r_2, \\ -\log r_1, & r_2 \leq r_1. \end{cases} \tag{3.8}$$

Then the Green operator  $\mathcal{G}_i^F(r_1, y_1, r_2, y_2)$  for co-closed  $i$ -forms, compactly supported on  $C(N)$  and of the type  $g(r)\phi(y)$  with  $\phi(y) \in \ker \widetilde{\delta}^F \cap \Omega^i(N, F)$ , is given by (cf. [7, (6.10)])

$$\mathcal{G}_i^F(r_1, y_1, r_2, y_2) = \sum_{j=1}^{\infty} h_{\mu_j(i)}(r_1, r_2)\phi_j^i(y_1) \otimes \phi_j^i(y_2). \tag{3.9}$$

In order to obtain the Green operator for forms on  $C_{(0,1]}(N)$  which satisfy either absolute boundary condition at  $\{1\} \times N$  or relative boundary condition at  $\{1\} \times N$ , one must modify  $\mathcal{G}_i^F(r_1, y_1, r_2, y_2)$  as in [7, pp. 290–291].

We first consider the case of absolute boundary condition.

(I) For  $\nu_j(i) > 0$ ,

(i) If  $\mu_j(i) > 0$ , set (compare with [7, (6.11)])

$${}_a h_{\mu_j(i)}(r_1, r_2) = h_{\mu_j(i)}(r_1, r_2) - \frac{a_j^-(i)}{2\nu_j(i)a_j^+(i)}(r_1 r_2)^{a_j^+(i)}. \tag{3.10}$$

(ii) If  $\mu_j(i) = 0$ , then  $i = 0, j = 1$ , set (compare with [7, (6.17)])

$${}_a h_{\mu_1(0)}(r_1, r_2) = h_{\mu_1(0)}(r_1, r_2) + \frac{1}{2}(r_1^2 + r_2^2) + \frac{(1+m)^2}{(1-m)(3+m)}. \tag{3.11}$$

(II) For  $\nu_j(i) = 0$ , set (compare with [7, (6.20)])

$${}_a h_{\mu_1(0)}(r_1, r_2) = h_{\mu_1(0)}(r_1, r_2) + \frac{1}{2}(r_1^2 + r_2^2) - \frac{3}{4}. \tag{3.12}$$

Then the Green operator  $\mathcal{G}_{a,i}^F(r_1, y_1, r_2, y_2)$  for co-closed  $i$ -forms on  $C_{(0,1]}(N)$  with absolute boundary condition and of the type  $g(r)\phi(y)$  is given by

$$\mathcal{G}_{a,i}^F(r_1, y_1, r_2, y_2) = \sum_{j=1}^{\infty} {}_a h_{\mu_j(i)}(r_1, r_2)\phi_j^i(y_1) \otimes \phi_j^i(y_2). \tag{3.13}$$

We now consider the case of relative boundary condition.

(I) For  $\nu_j(i) > 0$ , set (compare with [7, (6.12)])

$${}_r h_{\mu_j(i)}(r_1, r_2) = h_{\mu_j(i)}(r_1, r_2) - \frac{1}{2\nu_j(i)}(r_1 r_2)^{a_j^+(i)}. \tag{3.14}$$

(II) For  $\nu_j(i) = 0$ , set (compare with [7, (6.22)])

$${}_r h_{\mu_1(0)}(r_1, r_2) = h_{\mu_1(0)}(r_1, r_2). \tag{3.15}$$

Then the Green operator  $\mathcal{G}_{r,i}^F(r_1, y_1, r_2, y_2)$  for co-closed  $i$ -forms on  $C_{(0,1]}(N)$  with relative boundary condition and of the type  $g(r)\phi(y)$  is given by

$$\mathcal{G}_{r,i}^F(r_1, y_1, r_2, y_2) = \sum_{j=1}^{\infty} {}_r h_{\mu_j(i)}(r_1, r_2)\phi_j^i(y_1) \otimes \phi_j^i(y_2). \tag{3.16}$$

Now let  $d(x_1, x_2)$  denote the distance from  $x_1$  to  $x_2$  on  $C_{(0,1]}(N)$ . Let  $G_{a,i}^F(r_1, y_1, r_2, y_2)$  denote the full Green operator on  $\Omega^i(C_{(0,1]}(N), F)$  with absolute boundary condition.



**Theorem 3.1** (Compare with [7, Theorems 6.24, 6.43]) *There exists a constant  $C > 0$  such that for  $x_1 = (r_1, y_1), x_2 = (r_2, y_2) \in C_{(0,1]}(N)$ ,*

$$\|G_{a,i}^F(x_1, x_2)\| \leq C \cdot \begin{cases} 1 + |\log d(x_1, x_2)|, & m = 1, \\ d^{1-m}(x_1, x_2), & m \geq 2, \end{cases} \tag{3.17}$$

where by  $\|\cdot\|$  we mean the pointwise norm on  $C_{(0,1]}(N)$ . We will use this notation in the remaining part without further notice.

*Proof* Without loss of generality, we assume  $r_1 \leq r_2$ .

We rewrite the co-exact part of (3.9) as follows (cf. [7, (6.25)]),

$$\begin{aligned} \underline{\mathcal{G}}_i^F(x_1, x_2) &:= \frac{1}{2}(r_1 r_2)^{\alpha(i)} \sum_{\mu_j(i) > 0} \frac{(r_1/r_2)^{v_j(i)}}{v_j(i)} \phi_j^i(y_1) \otimes \phi_j^i(y_2) \\ &= \frac{1}{2}(r_1 r_2)^{\alpha(i)} P_{\tilde{c}\tilde{e},i} \frac{e^{\log(r_1/r_2) \cdot (\alpha^2(i) + \tilde{\Delta})^{1/2}}}{(\alpha^2(i) + \tilde{\Delta})^{1/2}}, \end{aligned} \tag{3.18}$$

where by  $P_{\tilde{c}\tilde{e},i}$  we mean the orthogonal projection on  $\widetilde{\delta^F \Omega}^{i+1}(N, F)$ .

Observe that the pointwise norm of (3.18), viewed as a kernel on  $C_{(0,1]}(N)$ , is equal to the pointwise norm of (3.18) viewed as a kernel on  $N$ , multiplied by  $(r_1 r_2)^{-i}$ .

For  $\frac{r_1}{r_2} \geq \frac{1}{2}$ , using [7, Theorem 6.23(1)], one sees that there exists a constant  $C > 0$  such that

$$\begin{aligned} \|\underline{\mathcal{G}}_i^F(x_1, x_2)\| &\leq C(r_1 r_2)^{\frac{1-m}{2}} \\ &\cdot \begin{cases} 1 + |\log(\log^2(r_1/r_2) + d^2(y_1, y_2))|, & m = 1, \\ 1 + (\log^2(r_1/r_2) + d^2(y_1, y_2))^{\frac{1-m}{2}}, & m \geq 2, \end{cases} \end{aligned} \tag{3.19}$$

where  $d(y_1, y_2)$  denote the distance from  $y_1$  to  $y_2$  on  $(N, g^{TN})$ .

As in [7, (6.28–6.32)], one has

$$(r_1 r_2)^{\frac{1-m}{2}} \leq 2^{3(1-m)/2} d^{1-m}(x_1, x_2), \tag{3.20}$$

$$r_1 r_2 (\log^2(r_1/r_2) + d^2(y_1, y_2)) \geq \frac{r_1}{r_2} d^2(x_1, x_2), \tag{3.21}$$

$$\log^2(r_1/r_2) + d^2(y_1, y_2) \geq \frac{1}{r_2^2} d^2(x_1, x_2). \tag{3.22}$$

From (3.19)–(3.22), one deduces that when  $\frac{r_1}{r_2} \geq \frac{1}{2}$ ,

$$\|\underline{\mathcal{G}}_i^F(x_1, x_2)\| \leq C \cdot \begin{cases} 1 + |\log d(x_1, x_2)|, & m = 1, \\ d^{1-m}(x_1, x_2), & m \geq 2. \end{cases} \tag{3.23}$$

For  $\frac{r_1}{r_2} \leq \frac{1}{2}$ , by the Sobolev inequality, the series (3.18), viewed as a kernel on  $N$ , converges uniformly. Therefore,

$$\|\underline{\mathcal{G}}_i^F(x_1, x_2)\| \leq C(r_1 r_2)^{\frac{1-m}{2}} (r_1/r_2)^{\widehat{v}(i)}, \quad \text{with } \widehat{v}(i) = \min_{\mu_j(i) > 0} \{v_j(i)\}. \tag{3.24}$$

From the fact that (cf. [7, (6.34)])

$$\min_{\mu_j(i) > 0} \{\mu_j(i)\} \geq (m - i)(i + 1), \tag{3.25}$$

one obtains  $\widehat{v}(i) \geq \frac{m+1}{2}$ . Thus,

$$\|\underline{\mathcal{G}}_i^F(x_1, x_2)\| \leq Cr_1r_2^{-m} \leq \frac{1}{2}Cr_2^{1-m} \leq 2^{m-2}C d^{1-m}(x_1, x_2). \tag{3.26}$$

Now from (3.13), one sees

$$\begin{aligned} \mathcal{G}_{a,i}^F(x_1, x_2) &= \underline{\mathcal{G}}_i^F(x_1, x_2) + {}_a h_{\mu_1(0)}(r_1, r_2)\phi_1^0(y_1) \otimes \phi_1^0(y_2) \\ &\quad - \sum_{\mu_j(i)>0} \frac{a_j^-(i)}{2v_j(i) a_j^+(i)} (r_1r_2)^{a_j^+(i)} \phi_j^i(y_1) \otimes \phi_j^i(y_2). \end{aligned} \tag{3.27}$$

The last term in (3.27) is the same as the term in [7, (6.38)], so it satisfies the required estimates. On the other hand, it is straightforward to show the second term in (3.27) implies the required estimates. Thus, one has

$$\|\mathcal{G}_{a,i}^F(x_1, x_2)\| \leq C \cdot \begin{cases} 1 + |\log d(x_1, x_2)|, & m = 1, \\ d^{1-m}(x_1, x_2), & m \geq 2. \end{cases} \tag{3.28}$$

A similar argument shows that  $\mathcal{G}_{r,i}^{F*}$  satisfies the estimates in (3.28). Moreover, with the help of [7, Theorem 6.23(2)], in the same way one can show that  $d_{x_1}^F \mathcal{G}_a^F(x_1, x_2)$ ,  $d_{x_2}^F \mathcal{G}_a^F(x_1, x_2)$ ,  $d_{x_1}^{F*} \mathcal{G}_r^{F*}(x_1, x_2)$ ,  $d_{x_2}^{F*} \mathcal{G}_r^{F*}(x_1, x_2)$  are bounded by  $d^{-m}(x_1, x_2)$ . Since

$$\begin{aligned} \mathcal{G}_{a,i}^F &= \mathcal{G}_{a,i}^F + d_1^F \mathcal{G}_{a,i-1}^F \circ d_2^F \mathcal{G}_{a,i-1}^F + \left(*_1^F\right)^{-1} \mathcal{G}_{r,m+1-i}^{F*} *_2^F \\ &\quad + \left(*_1^F\right)^{-1} d_1^{F*} \mathcal{G}_{r,m-i}^{F*} \circ d_2^{F*} \mathcal{G}_{r,m-i}^{F*} *_2^F, \end{aligned} \tag{3.29}$$

applying [7, Lemma 5.6], we complete the proof. □

The main result of this subsection is the Sobolev estimates on  $C_{(0,1]}(N)$  as follows.

**Theorem 3.2** *For  $n > \frac{m+1}{4}$ , there exists a constant  $C(n) > 0$  with the following property. Let  $\beta \in \Omega^i(C_{(0,1]}(N), F)$  such that  $\Delta^j \beta$  satisfies absolute boundary condition for  $0 \leq j \leq n$ . Then,*

(i) *If  $i \neq 0$ ,*

$$\|\beta\| \leq C(n) \|\Delta^n \beta\|_{L^2(C_{(0,1]}(N))}. \tag{3.30}$$

(ii) *If  $i = 0$ ,*

$$\|\beta\| \leq C(n) \left( \|\beta\|_{L^2(C_{(0,1]}(N))} + \|\Delta^n \beta\|_{L^2(C_{(0,1]}(N))} \right). \tag{3.31}$$

Here by  $\|\cdot\|_{L^2(C_{(0,1]}(N_p))}$  we mean the  $L^2$  norm on the cone  $C_{(0,1]}(N_p)$ .

*Proof* If  $i \neq 0$ ,

$$\beta(x_1) = \left(G_{a,i}^F\right)^n(x_1, x_2) \Delta^n \beta(x_2). \tag{3.32}$$

By [7, Lemma 5.6] and Theorem 3.1, the norm of  $(G_{a,i}^F)^n$  as a function of  $x_2$  is finite provided  $n > \frac{m+1}{4}$ . Thus, applying the Schwartz inequality, one gets (3.30).

If  $i = 0$ , the argument is the same except for the fact that the harmonic function must be split off and treated separately. We complete the proof.

### 3.2 Proof of the main result

Let  $(X, g^{TX})$  be a Riemannian manifold of dimension  $m + 1$  ( $m$  odd) with isolated conical singularities  $\Sigma$ , which carries an orbifold structure. As pointed out in [1, p. 2], any orbifold has an atlas consisting of linear charts. For  $p \in \Sigma$ , let  $B^X(p, \varepsilon)$  be the open ball in  $X$  of center  $p$  and radius  $\varepsilon > 0$  small enough. In a neighborhood of  $p$ , we take the following chart of the form,

$$\left(G_p \subset O(m + 1), B^X(p, \varepsilon), B(2)\right),$$

where  $B(2) \subset \mathbb{R}^{m+1}$  denotes the open ball of center 0 and radius 2, which carries an orthogonal action of the finite group  $G_p$  with the only fixed point  $p$ . Let  $g^{B(2)}$  denote the lift of  $g^{TX}|_{B^X(p, \varepsilon)}$  to  $B(2)$ . Since  $X$  is both conical and orbifold near  $p$ ,  $g^{B(2)}$  is the Euclidean metric.

Let  $F$  be a flat real honest orbifold vector bundle over  $X$  with the flat connection  $\nabla^F$  and an Euclidean metric  $g^F$ .

We make the following assumption.

**Assumption 3.3**  $g^F|_{B^X(p, \varepsilon)}$  is flat.

Let  $S^m \subset B(2)$  be the sphere of center 0 and radius 1, which carries the induced free  $G_p$  action and the Euclidean metric induced by  $g^{B(2)}$ . Then we have the isometric identification

$$B^X(p, \varepsilon) = C_{(0,2)}^*(N_p), \quad \text{with } N_p = S^m / G_p. \tag{3.33}$$

The following theorem compares the  $L^2$ -cohomology with the singular cohomology.

**Theorem 3.4** *We have the isomorphism as follows,*

$$H_{(2)}^i(X; F) \simeq H^i(X; F), \quad 0 \leq i \leq m + 1. \tag{3.34}$$

Moreover, we have the canonical isometry,

$$\ker \Delta_{c,i} \simeq \ker \Delta_{o,i}, \quad 0 \leq i \leq m + 1. \tag{3.35}$$

*Proof* Since  $X$  has a good cover, using Mayer-Vietoris sequence, it is sufficient to show for  $p \in \Sigma$  and  $u > 0$  small enough,

$$H_{(2)}^i(C_{(0,u]}^*(N_p); F) \simeq H^i(C_{(0,u]}^*(N_p); F), \quad 0 \leq i \leq m + 1. \tag{3.36}$$

We have the following fact from [9],

$$H_{(2)}^i(C_{(0,u]}^*(N_p); F) \simeq \begin{cases} H^i(N_p; F), & i \leq \frac{m}{2}, \\ 0, & i \geq \frac{m+1}{2}. \end{cases} \tag{3.37}$$

On the other hand,

$$H^i(N_p; F) \simeq (H^i(S^m; \tilde{F}_p))^{G_p} \simeq \begin{cases} F|_{\{p\}}, & i = 0, \quad \text{or } i = m, \\ 0, & i \neq 0, \quad \text{and } i \neq m. \end{cases} \tag{3.38}$$

Since  $C_{(0,u]}^*(N_p)$  is contractible, (3.36) follows from (3.37) and (3.38).

Observe that for any element  $s \in \ker \Delta_{o,i}$ ,  $s$  and  $d^F s$  are both  $L^2$  integrable, hence  $s$  determines an element in  $\ker \Delta_{c,i}$ . From (2.10), (2.18) and (3.34), one sees the injection from  $\ker \Delta_{o,i}$  to  $\ker \Delta_{c,i}$  gives the isomorphism in (3.35) which is indeed an isometry.  $\square$

*Remark 3.5* One can verify (3.35) directly. In fact, using the explicit expression of the harmonic forms on the cone in [7, 10], one finds the harmonic element in  $\ker \Delta_c$  has removable singularities, so it lies in  $\ker \Delta_o$  by elliptic regularity.

Recall that  $X_u = X \setminus (\sqcup_{p \in \Sigma} C_{(0,u)}^*(N_p))$ ,  $0 < u \leq 2$ .

**Theorem 3.6** *Given  $T > 0$ ,  $0 < u_0 < 1/5$ ,  $n \in \mathbb{N}$ , and  $n > \frac{m+1}{4}$ , there exists a constant  $C(T, u_0, n) > 0$  such that for  $0 < t \leq T$ ,  $x \in X \setminus X_{u_0}$  and  $y \in X_{5u_0}$ ,*

$$\|K_c(t, x, y)\| \leq C(T, u_0, n) t^n. \tag{3.39}$$

The same estimates hold for  $d_x^F K_c(t, x, y)$  and  $\delta_y^F K_c(t, x, y)$ .

*Proof* As explained in [7, p. 287], it suffices to prove the estimate holds for  $K_c(t, x, y)$ .

We first establish the estimates for the heat kernel  $\widehat{K}_p(t, x, y)$  on the bounded cone  $C_{(0,1]}(N_p)$ . Choose a parametrix  $P_n(t, x, y)$  of order  $n$  such that (cf. [7, p. 272])

$$P_n(0, \cdot, y) = \delta_y(\cdot), \quad \text{for } y \in C_{(0,1]}(N_p) \tag{3.40}$$

and

$$P_n(t, x, y) = 0, \quad \text{for } x \in C_{(0,u_0]}(N_p), y \in C_{[2u_0,1]}(N_p), \tag{3.41}$$

where  $C_{(0,u_0]}(N_p)$  is defined as in (2.1) and

$$C_{[2u_0,1]}(N_p) = \{(r, y) \in C(N) \mid 2u_0 \leq r \leq 1\}.$$

Then there exists a constant  $C'(T, u_0, n) > 0$  such that for  $j \in \mathbb{N}$ ,  $t \in (0, T]$ ,  $x \in C_{(0,u_0]}(N_p)$ ,

$$\|\Delta^j(\widehat{K}_p(t, x, y) - P_n(t, x, y))\|_{L^2(C_{(0,1]}(N_p))} \leq C'(T, u_0, n) t^n, \tag{3.42}$$

where by  $\|\cdot\|_{L^2(C_{(0,1]}(N_p))}$  we mean taking the  $L^2$  norm with respect to  $y$ .

By (3.42) and the Sobolev inequality on the bounded cone, Theorem 3.2, one sees for  $n > \frac{m+1}{4}$ , there exists a constant  $C(T, u_0, n) > 0$  such that for  $x \in C_{(0,u_0]}(N_p)$  and  $y \in C_{(0,1]}(N_p)$ ,

$$\|\widehat{K}_p(t, x, y) - P_n(t, x, y)\| \leq C(T, u_0, n) t^n. \tag{3.43}$$

In particular, from (3.41) and (3.43) we see that for  $x \in C_{(0,u_0]}(N_p)$  and  $y \in C_{[2u_0,1]}(N_p)$

$$\|\widehat{K}_p(t, x, y)\| \leq C(T, u_0, n) t^n. \tag{3.44}$$

As explained in [7, p. 286], the same estimates hold for  $\frac{\partial^\alpha}{\partial y^\alpha} \widehat{K}_p(t, x, y)$ .

Let  $\phi(r): [0, 1] \rightarrow [0, 1]$  be a smooth function such that

$$\phi(r) = \begin{cases} 1, & 0 \leq r \leq 3u_0, \\ 0, & 4u_0 \leq r \leq 1. \end{cases} \tag{3.45}$$

Let  $r$  denote the radial coordinate of  $y$ . Then as in [7, (7.7)], one has for  $j \in \mathbb{N}$ ,  $x \in X \setminus X_{u_0}$ ,

$$\begin{aligned} \|\Delta_y^j K_c(t, x, y)\|_{L^2(X_{5u_0})} &= \left\| \Delta_y^j \left( K_c(t, x, y) - \sum_{p \in \Sigma} \phi(r) \widehat{K}_p(t, x, y) \right) \right\|_{L^2(X_{5u_0})} \\ &\leq \left\| \Delta_y^j \left( K_c(t, x, y) - \sum_{p \in \Sigma} \phi(r) \widehat{K}_p(t, x, y) \right) \right\|_{L^2(X \setminus \Sigma)} \\ &\leq \sum_{p \in \Sigma} \int_0^t \left\| \Delta_y^j (\partial_s + \Delta_y) (\phi(r) \widehat{K}_p(s, x, y)) \right\|_{L^2(C_{[2u_0,1]}(N_p))}, \end{aligned} \tag{3.46}$$

where we use  $\|\cdot\|_{L^2}$  to denote the  $L^2$  norm with respect to  $y$  as in (3.42).

In view of the estimates already established for  $\frac{\partial^\alpha}{\partial y^\alpha} \widehat{K}_p(t, x, y)$  in (3.44), the estimates for  $K_c(t, x, y)$ ,  $d_x^F K_c(t, x, y)$  and  $\delta_y^F K_c(t, x, y)$  follow from the standard Sobolev inequality (cf. [7, (5.7)]) applied with respect to  $y$ .  $\square$

*Remark 3.7* A weaker but more general estimate for the conical heat kernel is given in [19].

We are now in a position to prove our main theorem which compares the two heat kernels outside the singular points.

**Theorem 3.8** *For  $t > 0$ ,  $x, y \in X \setminus \Sigma$ , one has*

$$K_{o,i}(t, x, y) = K_{c,i}(t, x, y), \quad 0 \leq i \leq m + 1. \tag{3.47}$$

*Proof* Fix any  $T > 0$ , and we will show (3.47) holds for any  $t \in (0, T]$ .

For any  $u_0 > 0$  small enough fixed temporarily and  $x, y \in X_{u_0}$ , applying the Duhamel principle [7, (3.9)] on  $X_u$  with  $0 < u < \frac{1}{5}u_0$ , one gets

$$\begin{aligned} K_{c,i}(t, x, y) - K_{o,i}(t, x, y) &= \int_0^t \int_{\partial X_u} \langle K_{o,i}(t-s, x, z) \wedge *^F d^F K_{c,i}(s, z, y) \rangle \\ &\quad - \int_0^t \int_{\partial X_u} \langle *^F d^F K_{o,i}(t-s, x, z) \wedge K_{c,i}(s, z, y) \rangle \\ &\quad + \int_0^t \int_{\partial X_u} \langle \delta^F K_{o,i}(t-s, x, z) \wedge *^F K_{c,i}(s, z, y) \rangle \\ &\quad - \int_0^t \int_{\partial X_u} \langle *^F K_{o,i}(t-s, x, z) \wedge \delta^F K_{c,i}(s, z, y) \rangle, \end{aligned} \tag{3.48}$$

where on the right hand side all operations are applied to the variable  $z$ .

Using the uniform estimates in Theorems 2.2, 3.6, one sees as  $u \rightarrow 0$ ,

$$K_{o,i}(t, x, y) = K_{c,i}(t, x, y) \text{ holds for } x, y \in X_{u_0}. \tag{3.49}$$

Since  $u_0 > 0$  can be taken arbitrarily small, (3.49) holds on  $X \setminus \Sigma$ .

From Theorems 3.4, 3.8, one gets

**Corollary 3.9** *The two torsions are equal, i.e.,*

$$T_c(X, g^{TX}, g^F) = T_o(X, g^{TX}, g^F). \tag{3.50}$$

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