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Hitchin–Thorpe inequality for noncompact Einstein 4-manifolds

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Abstract

We prove a Hitchin–Thorpe inequality for noncompact Einstein 4-manifolds with specified asymptotic geometry at infinity. The asymptotic geometry at infinity is either a cusp bundle over a compact space (the fibered cusps) or a fiber bundle over a cone with a compact fiber (the fibered boundary). Many noncompact Einstein manifolds come with such a geometry at infinity.

Keywords: Einstein manifold; Asymptotic geometry; Hitchin–Thorpe inequality

1. Introduction

Einstein manifolds are important both in mathematics and physics. They are good candidates for canonical metrics on general Riemannian manifolds and they are the vacuum solutions of Einstein's field equation (with cosmological constant) in general relativity. As a result, they are extensively studied (cf. [7,24]).

Besides space forms and irreducible symmetric spaces, a large class of compact Einstein manifolds is given by the solution of Calabi conjecture. Namely, a compact Kähler manifold with a non-positive first Chern class admits a Kähler–Einstein metric [5,36]. In the case of positive

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first Chern class, the work of [28,29] says that $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ admits a Kähler–Einstein metric if $3 \leqslant k \leqslant 8$. Other examples includes the so-called Page metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ (only Einstein) and certain principal torus bundles over Kähler–Einstein manifolds [34], see also the recent articles [2,9,10] for Sasakian Einstein metrics, compact homogeneous Einstein manifolds, and Dehn surgery construction.

On the other hand, regarding the question of topological obstructions, the obvious ones will be coming from that for the Ricci curvature. Thus, if the Einstein constant is positive, the manifold must be compact and the fundamental group is finite. If the Einstein constant is zero, there are also obstructions coming from Cheeger–Gromoll's splitting theorem [12]. Further, for noncompact manifolds, the volume growth is at least linear [35] (see also [13]).

In the case of compact Einstein 4-manifolds, there are more topological obstructions. Berger [6] observed that a compact Einstein 4-manifold must have non-negative Euler number. Moreover, the Euler number is zero if and only if the manifold is flat. This implies that, for example, $T^4 \# T^4$ and $S^1 \times S^3$ are not Einstein.

Berger's observation is considerably strengthened in the Hitchin–Thorpe inequality [20], that for any compact oriented Einstein 4-manifold M^4

$$\chi(M) \geqslant \frac{3}{2}\big|\tau(M)\big|, \tag{1.1}$$

where $\chi(M)$ denotes the Euler number of M, $\tau(M)$ the signature. Furthermore, the equality holds if and only if either M is flat or the universal cover $\tilde{M}$ is $K3$. The Hitchin–Thorpe inequality implies in particular that $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ cannot be Einstein for $k \geqslant 9$, complementing very well the result of [28,29].

There are various extensions of the Hitchin–Thorpe inequality, see [19,22,23,27] among others. The extensions can be summarized in the following generalized Hitchin–Thorpe inequality due to Kotschick [23], namely, for any compact oriented Einstein 4-manifold M^4

$$\chi(M) \geqslant \frac{3}{2}\big|\tau(M)\big| + \frac{1}{108\pi^2}\big(\lambda(M)\big)^4, \tag{1.2}$$

where $\lambda(M)$ is the volume entropy. And equality occurs if and only if either M is flat, or the universal cover $\tilde{M}$ is $K3$ or hyperbolic.

In the case of noncompact manifolds, there are results of Tian–Yau [30–32] for the existence of Kähler–Einstein metrics on the complements of a normal crossing divisor. There are also many examples from general relativity. These are all of finite topological type and moreover, most of them come with a special structure at infinity: a fibration structure and an asymptotic geometry adapted to the fibration. It should be pointed out however, that there exist Ricci flat Kähler manifolds of infinite topological type [3].

In this note we prove a Hitchin–Thorpe inequality for noncompact Einstein 4-manifolds with specified asymptotic geometry at infinity adapted to a fibration. Let (M^n, g) be a noncompact complete Riemannian manifold with finite topological type and $\bar{M} = M \cup \partial\bar{M}$ its compactification. The metric g is said to be asymptotic to a fibered cusp if there is a defining function $x \in C^\infty(\bar{M})$ of $\partial\bar{M}$ and a fibration

$$F \to \partial\bar{M} \xrightarrow{\pi} B \tag{1.3}$$

of closed manifolds such that

$$g \sim \frac{dx^2}{x^2} + \pi^* g_B + x^2 g_F. \tag{1.4}$$

Here g_B is a metric on the base manifold B and g_F is a family of metrics along the fibers. (The precise meaning of asymptotic in (1.4) and (1.5) below will be discussed in Section 3.) The coordinate change $x = e^{-r}$ transforms the metric into the more standard looking

$$g \sim dr^2 + \pi^* g_B + e^{-2r} g_F.$$

Thus, the geometry at infinity is asymptotic to a fibration over the base B with fibers given by cusps over the original fiber F, hence the name 'fibered cusps.' Clearly the volume is finite (assuming the dimension of the fiber is positive) in this case, so if the metric is also Einstein, the Einstein constant must be negative. Examples from [30] have fibered cusp geometry at infinity.

The other asymptotic geometry we will consider is the so-called fibered boundary metric:

$$g \sim \frac{dx^2}{x^4} + \frac{\pi^* g_B}{x^2} + g_F. \tag{1.5}$$

Here one can use the coordinate change $x = \frac{1}{r}$ in which the metric becomes

$$g \sim dr^2 + r^2 \pi^* g_B + g_F.$$

Hence the geometry at infinity is asymptotic to a fibration with the original fibers F, but now the base is the infinite end of the cone over the original base B. In this case the volume is infinite (assuming the dimension of the base is positive) and thus the Einstein constant could be zero or negative. The examples from general relativity, like the Euclidean Schwarzschild metric on $\mathbb{R}^2 \times S^2$, the Taub-NUT metric on $\mathbb{R}^4$, or the general Gibbons–Hawking multi-center metrics, all have fibered boundary metric with base S^2 and fiber S^1. The examples from [31,32] have fibration structure (S^1 over a smooth divisor) but the metric is not precisely of the type we consider here.

Theorem 1.1. *Let (M^4, g) be a noncompact complete Einstein manifold which is asymptotic to a fibered cusp or a fibered boundary at infinity. In the fibered boundary case, we also assume that* $\dim F > 0$ (*that is, we exclude the case when F is a point*; *see below for a separate discussion*). *Then*

$$\chi(M) \geqslant \frac{3}{2}\left|\tau(M) + \frac{1}{2} a\text{-}\lim \eta\right|,$$

where a-$\lim \eta$ is the adiabatic limit of the eta invariant of $\partial \bar{M}$ (*for the signature operator*). *Moreover, the equality holds iff (M, g) is a complete Calabi–Yau manifold.*

Remark. One can also state an inequality with a volume entropy term. However, unlike the compact case, it is unclear if the volume entropy here is a topological invariant.

The adiabatic limit of the eta invariant of $\partial\bar{M}$ (see, e.g., [15]) encodes geometric and topological information of the boundary fibration (at infinity). In the case when the fibration is a circle bundle over a surface, it is given in terms of the Euler number of the circle bundle. The case of surface bundle over a circle is more complicated. For a torus bundle over a circle (solvmanifold) the adiabatic limit is given by certain L-function [11].

Corollary 1.2. *Let (M^4, g) be a noncompact complete Einstein manifold which is asymptotic to a fibered cusp/boundary at infinity, with the fibration given by a circle bundle over a surface. Then*

$$\chi(M) \geqslant \frac{3}{2}\left|\tau(M) - \frac{1}{3}e + \operatorname{sign} e\right|,$$

where e is the Euler number of the circle bundle. Moreover, the equality holds iff (M, g) is a complete Calabi–Yau manifold.

In particular, if M^4 is the Taub-NUT manifold, then $M \# (S^1 \times N)$, for any closed 3-manifold N, does not admit Einstein metric with the same asymptotic geometry. Similarly, if M^4 is the Taub-NUT manifold or one of the Kähler–Einstein manifolds constructed in [30], the blowups $M \# k\overline{\mathbb{CP}}^2$ does not admit Einstein metric with the same asymptotic geometry for k sufficiently large.

We now look at the case of fibered boundary metrics when the fiber is a single point. In this case $B = \partial\bar{M}$ and the geometry at infinity is asymptotically conical. That is

$$g \sim dr^2 + r^2 g_{\partial\bar{M}},$$

where r can be thought as the distance from a base point. Since r can only change by adding a constant, $g_{\partial\bar{M}}$ is uniquely determined.

Theorem 1.3. *Let (M^4, g) be a complete Einstein four manifold which is asymptotic to a cone over $(\partial\bar{M}, g_{\partial\bar{M}})$. Then*

$$\chi(M) \geqslant \frac{1}{2\pi^2}\operatorname{vol}(\partial\bar{M}) + \frac{3}{2}\left|\tau(M) + \frac{1}{2}\eta(\partial\bar{M})\right| + \alpha(\partial\bar{M}),$$

where $\eta(\partial\bar{M})$ is the eta invariant of $(\partial\bar{M}, g_{\partial\bar{M}})$ and $\alpha(\partial\bar{M})$ a geometric invariant defined by

$$\alpha(\partial\bar{M}) = \frac{1}{8\pi^2}\int\limits_{\partial\bar{M}} \epsilon_{abc}\omega^a \wedge \left[\Omega_c^b - \omega^b \wedge \omega^c\right] = \frac{1}{8\pi^2}\int\limits_{\partial\bar{M}} \epsilon_{abc}\omega^a \wedge \Omega_c^b - \frac{3}{4\pi^2}\operatorname{vol}(\partial\bar{M})$$

with ω^a denoting the dual 1-forms of an orthonormal basis for $\partial\bar{M}$ and Ω_c^b the 2-form components of the curvature of $\partial\bar{M}$ with respect to the orthonormal basis. Moreover, the equality holds if and only if M is an asymptotically conical Calabi–Yau manifold.

Note that $\alpha(S^3/\Gamma) = 0$. This generalizes the previous work for ALE spaces [26]. See also the discussion below.

One can roughly classify noncompact Einstein manifolds by their volume growth. There are previous work concerning big volume growth. For asymptotic locally Euclidean (hence with Euclidean volume growth) Ricci flat 4-manifolds with end S^3/Γ, it is proved in [26] that

$$\chi(M) \geqslant \frac{1}{|\Gamma|} + \frac{3}{2}\bigl|\tau(M) + \eta_S\bigl(S^3/\Gamma\bigr)\bigr|,$$

where $\eta_S(S^3/\Gamma)$ is the eta invariant of S^3/Γ. Note that this class corresponds to our situation of fibered boundary case, with the trivial fiber F a single point and $B = S^3/\Gamma$.

For negative Einstein constant there are works [1,21] on conformally compact Einstein 4-manifolds (hence with exponential volume growth). In this case, Anderson shows that

$$\chi(M) - \frac{3}{4\pi^2}V \geqslant \frac{3}{2}\bigl|\tau(M) - \eta\bigr|,$$

where V is the so-called renormalized volume (cf. [18]) and η denotes the eta invariant of the conformal infinity.

Theorem 1.1 corresponds to the finite volume or sub-Euclidean volume growth, while Theorem 1.3 corresponds to the Euclidean volume growth.

In the process of writing this paper we learned that in the finite volume case Yugang Zhang [37] proved a similar result when the boundary admits an injective F-structure and the total space has bounded covering geometry. While there are overlaps between the finite volume case in Corollary 1.2 and his result, as any S^1 bundles over a surface have an injective F-structure iff the fundamental group of the total space is infinite, our result does cover the case of finite fundamental group. Our result in the infinite volume case is completely different from the corresponding case of [37].

The essential part of our proof is to extend the Gauss–Bonnet–Chern and Hirzebruch signature formulas to complete manifolds with fibered geometry at infinity. The index formulas we prove (Theorems 3.5 and 4.3) hold in any dimension and should be of independent interest. Our approach is based on application of Atiyah–Patodi–Singer index formula [4]. We use the asymptotic structure to approximate M by compact manifolds with boundary. The boundary will in general not be totally geodesic. Therefore, there are Chern–Simons correction terms coming from the boundary, and analyzing these Chern–Simons correction terms consists of the main part of the proof.

The paper is organized as follows. In Section 2 we review the Hitchin–Thorpe inequality for closed manifolds and the Chern–Simons correction terms from the boundary. In Section 3, we analyze the Chern–Simons correction term in the fibered cusp case, and fibered boundary case and show that they limit to zero. We found out that the language of rescaled tangent bundle introduced by Melrose [25] (see also [33]) is very useful in this analysis. We devote Section 4 to the analysis of the Chern–Simons term in the fibered boundary case without the dimensional restriction. Section 5 reviews the results for adiabatic limit of the eta invariant.

2. Chern–Simons correction term to APS

The original Hitchin–Thorpe inequality is a beautiful application of the Gauss–Bonnet–Chern formula and Hirzebruch's signature formula, two special cases of the Atiyah–Singer index theorem. For a closed oriented manifold M of even dimension n, the Gauss–Bonnet–Chern formula says that

$$\chi(M) = (-1)^{n/2} \int_M \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right),$$

where Ω is the curvature form of a Riemannian metric and Pf denotes the Pfaffian. For $n = 4$, this gives the following explicit formula:

$$\begin{aligned}\chi(M) &= \frac{1}{32\pi^2} \int_M \epsilon_{abcd} \Omega_{ab} \wedge \Omega_{cd} \\ &= \frac{1}{8\pi^2} \int_M \left(|W|^2 - |Z|^2 + \frac{1}{24} S^2 \right) d\mathrm{vol}.\end{aligned}$$

Here W is the Weyl curvature, Z the traceless Ricci, S the scalar curvature, and ϵ_{abcd} denotes the totally anti-symmetric tensor with $\epsilon_{1234} = 1$ (in other words, ϵ_{abcd} is the sign of the permutation σ where $\sigma(1) = a, \ldots, \sigma(4) = d$).

Similarly, the Hirzebruch signature formula gives

$$\tau(M) = \int_M L\left(\frac{\Omega}{2\pi}\right),$$

where L denotes the L-polynomial. Again, in dimension 4, the formula simplifies to

$$\begin{aligned}\tau(M) &= -\frac{1}{24\pi^2} \int_M \mathrm{Tr}(\Omega \wedge \Omega) \\ &= \frac{1}{12\pi^2} \int_M \left(|W^+|^2 - |W^-|^2\right) d\mathrm{vol}.\end{aligned}$$

Since $|W|^2 = |W^+|^2 + |W^-|^2$ and $Z = 0$ for Einstein manifolds, the Hitchin–Thorpe inequality follows. Furthermore, it follows that in the case of equality we must have $S = 0$, and either $W^+ = 0$ or $W^- = 0$. That is, these must be Ricci flat manifolds with either vanishing self-dual or anti-self-dual Weyl curvature. (They are shown by Hitchin [20] to be either flat or covered by $K3$.)

Assume now that (M, g) is a complete noncompact manifold with fibered geometry at infinity as defined in the previous section. We now look at the index formula for the Euler number and signature of such manifolds. By their topological nature, we have

$$\chi(M) = \chi(M_\epsilon), \qquad \tau(M) = \tau(M_\epsilon), \tag{2.1}$$

for $\epsilon > 0$ sufficiently small, where $M_\epsilon = \{x \geqslant \epsilon\}$. We are now in a position to apply the Atiyah–Patodi–Singer index formula [4].

If N^n is an even-dimensional compact oriented Riemannian manifold with boundary ∂N, whose metric is the product type near the boundary, then

$$\chi(N) = (-1)^{n/2} \int_N \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right),$$

and

$$\tau(N) = \int_N L\left(\frac{\Omega}{2\pi}\right) - \frac{1}{2}\eta(\partial N),$$

with $\eta(\partial N)$ denoting the eta invariant of the signature operator A on the boundary with respect to the induced metric. However, M_ϵ does not have product metric near its boundary. Hence there will be Chern–Simons terms coming out as well.

Let P be an invariant polynomial of a Lie group G, of degree k. By the Chern–Weil theory, for any G-connection ω with curvature Ω,

$$P(\Omega)$$

defines a characteristic form. If ω' is another G-connection whose curvature form is denoted by Ω', then their corresponding characteristic forms differ by an exact form:

$$P(\Omega') - P(\Omega) = dQ, \tag{2.2}$$

where

$$Q(\omega', \omega) = k \int_0^1 P(\omega' - \omega, \Omega_t, \ldots, \Omega_t)\, dt. \tag{2.3}$$

Here we have denoted by Ω_t the curvature form of the connection $\omega_t = t\omega' + (1-t)\omega$ interpolating between the two connections.

Now, suppose N is an compact oriented manifold with boundary whose metric g may not be product near the boundary. Then, near the boundary ∂N,

$$g = dr^2 + h(r),$$

where r is the geodesic distance from the boundary and $h(r)$ is the restriction of g on the constant r hypersurface which is diffeomorphic to ∂N, for r sufficiently small. Let g_0 be a metric on N which is equal to g except near the boundary, and is a product sufficiently close to the boundary:

$$g_0 = dr^2 + h(0).$$

Denote by ω and ω_0 the connection 1-forms of the Levi-Civita connections of g and g_0, respectively. Then, by (2.2),

$$\int_N P(\Omega) - \int_N P(\Omega_0) = \int_{\partial N} Q(\omega, \omega_0),$$

where

$$Q(\omega, \omega_0) = k \int_0^1 P(\theta, \Omega_t, \dots, \Omega_t)\, dt, \tag{2.4}$$

and

$$\theta = \omega - \omega_0$$

is the second fundamental form at the boundary. This is the general form of the Chern–Simons correction to the Atiyah–Patodi–Singer index formula for a non-product type metric. Namely,

$$\chi(N) = (-1)^{n/2} \int_N \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right) - \int_{\partial N} Q(\omega, \omega_0),$$

and

$$\tau(N) = \int_N L\left(\frac{\Omega}{2\pi}\right) - \int_{\partial N} Q(\omega, \omega_0) - \frac{1}{2}\eta(\partial N).$$

Here Q is associated to the Pfaffian and the L-polynomial, respectively. These formula are obtained by applying the APS index theorem to g_0 and then replacing the characteristic integral of g_0 by that of g.

In dimension 4, the Chern–Simons correction terms can be made more explicit [14,17]. When P is the Pfaffian, one has

$$\int_{\partial N} Q(\omega, \omega_0) = \frac{1}{32\pi^2} \int_{\partial N} \epsilon_{abcd}\left(2\theta_b^a \wedge \Omega_d^c - \frac{4}{3}\theta_b^a \wedge \theta_e^c \wedge \theta_d^e\right). \tag{2.5}$$

For $P = \frac{1}{3}p_1$, it is given by

$$-\frac{1}{24\pi^2} \int_{\partial N} \mathrm{Tr}(\theta \wedge \Omega) = -\frac{1}{12\pi^2} \int_{\partial N} \theta_i^0 \wedge \Omega_0^i. \tag{2.6}$$

In the following section we study these Chern–Simons correction terms for manifolds with fibered geometry at infinity.

3. Fibered geometry at infinity

Now let (M^n, g) be a noncompact complete Riemannian manifold with finite topological type and $\bar{M} = M \cup \partial\bar{M}$ its compactification. Moreover, there is a fibration structure on the boundary (at infinity)

$$F \to \partial\bar{M} \xrightarrow{\pi} B$$

with B, F closed manifolds, as in (1.3). Let x be a boundary defining function, i.e., $x \in C^\infty(\bar{M})$, $x > 0$ in M and $x = 0$ on $\partial \bar{M}$; in addition dx is nowhere vanishing on $\partial \bar{M}$. Associated to the compactification $\bar{M}$ of the manifold M with fibered structure at infinity (and the defining function), there is a Lie algebra of vector fields

$$^\phi\mathcal{V}(\bar{M}) = \{\text{vector field } X \text{ on } \bar{M} \text{ tangent to the fibers at the boundary, and } X(x) = O(x^2)\}.$$

It defines a vector bundle $^\phi T\bar{M}$, the rescaled tangent bundle, on $\bar{M}$ via

$$^\phi\mathcal{V}(\bar{M}) = \Gamma({}^\phi T\bar{M}).$$

If y, z are local coordinates for the base B and fiber F, respectively, a local frame near $\partial\bar{M}$ for $^\phi T\bar{M}$ is then given by $x^2\partial_x$, $x\partial_y$, ∂_z. Thus, on M, where $x > 0$, $^\phi T\bar{M}$ is (non-canonically) isomorphic to $T\bar{M}$ (or TM). In turn, this induces a non-canonical identification

$$\operatorname{End}({}^\phi T\bar{M})\big|_M \cong \operatorname{End}(TM)$$

where different identifications differ by the adjoint action, i.e., by conjugation. This implies that invariant polynomials are canonically identified. For example, the trace functionals are canonically identified:

$$\begin{array}{cccc} \mathrm{Tr}: & \operatorname{End}({}^\phi T\bar{M})|_M & \longrightarrow & \mathbb{R} \\ \| & \downarrow \cong & & \| \\ \mathrm{Tr}: & \operatorname{End}(TM) & \longrightarrow & \mathbb{R}. \end{array} \tag{3.1}$$

A metric g_1 is said to be a fibered boundary metric if there is a defining function $x \in C^\infty(\bar{M})$ of $\partial\bar{M}$ such that

$$g_1 = \frac{dx^2}{x^4} + \frac{\pi^* g_B}{x^2} + g_F, \tag{3.2}$$

where g_B is a metric on the base manifold B and g_F is a family of metrics along the fibers. Note that g_1 in fact defines a smooth metric on the rescaled tangent bundle $^\phi T\bar{M}$.

Definition 3.1. A metric g is asymptotic to a fibered boundary metric if

$$g = g_1 + a,$$

where g_1 is a fibered boundary metric defined by (3.2) and $a = xa_1$ where a_1 is a smooth section of $\mathcal{S}^2({}^\phi T\bar{M})$ such that $a_1(x^2\partial_x, \cdot) \equiv 0$. Here $\mathcal{S}^2$ denotes the space of symmetric two tensors.

Remark. The condition on the perturbation term in Definition 3.1 means that a contains no terms in dx. Thus, the normal direction to $\partial\bar{M}$ is still given by ∂_x. This condition, however, can be relaxed, see [33].

A special example of asymptotically fibered boundary metric is a metric of the form

$$g = \frac{dx^2}{x^4} + \frac{\pi^* g_B(x)}{x^2} + g_F(x),$$

where $g_B(x)$ is a family of metrics on the base manifold B depending smoothly on x and $g_F(x)$ is a family of metrics along the fibers, also depending smoothly on x. Many examples appear in this form. For example, the Euclidean Schwarzschild metric on $\mathbb{R}^2 \times S^2$, the Taub-NUT metric on $\mathbb{R}^4$, or the general Gibbons–Hawking multi-center metrics are in this form with fiber S^1.

The vector bundle ${}^{\phi}T\bar{M}$ captures geometric information about fibered boundary metric. The following is proved in [33].

Proposition 3.2. *The Levi-Civita connection for a metric asymptotic to the fibered boundary metric is a true connection, i.e.,*

$$\nabla^{\phi} : \Gamma\big({}^{\phi}T\bar{M}\big) \to \Gamma\big(T^*\bar{M} \otimes {}^{\phi}T\bar{M}\big).$$

Moreover,

$$R^{\phi} \in \Gamma\big(\Lambda^2 T^*\bar{M} \otimes \mathrm{End}\big({}^{\phi}T\bar{M}\big)\big).$$

The asymptotic fibered cusp metric g_d and asymptotic fibered boundary metric g_{ϕ} are related by a conformal rescaling:

$$g_d = x^2 g_{\phi},$$

and we will use this as the definition of asymptotic fibered cusp metric. Let ${}^{d}T\bar{M} = x^{-1}\,{}^{\phi}T\bar{M}$, i.e., a local frame near the boundary for ${}^{d}T\bar{M}$ will be $x\partial_x$, ∂_y, $x^{-1}\partial_z$. Then one also has canonical identification of the invariant polynomials such as the trace functionals

$$\begin{array}{cccc} \mathrm{Tr}: & \mathrm{End}({}^{d}T\bar{M}) & \longrightarrow & \mathbb{R} \\ \Vert & \downarrow \cong & & \Vert \\ \mathrm{Tr}: & \mathrm{End}(T\bar{M}) & \longrightarrow & \mathbb{R}. \end{array} \tag{3.3}$$

Furthermore, one has similarly [33],

Proposition 3.3. *The Levi-Civita connection for a metric asymptotic to the fibered cusp metric is a true connection, i.e.,*

$$\nabla^{d} : \Gamma\big({}^{d}T\bar{M}\big) \to \Gamma\big(T^*\bar{M} \otimes {}^{d}T\bar{M}\big).$$

Moreover,

$$R^{d} \in \Gamma\big(\Lambda^2 T^*\bar{M} \otimes \mathrm{End}\big({}^{d}T\bar{M}\big)\big).$$

Important to our consideration is the following lemma from [33] regarding the second fundamental form of asymptotic fibered cusp metric.

Lemma 3.4. *For any $T \in \Gamma({}^bT\bar{M})$, and $A \in \Gamma({}^dT\bar{M})$,*

$$\nabla^d_T \frac{dx}{x}(A)\bigg|_{\partial\bar{M}} = 0.$$

We are now in a position to prove the following

Theorem 3.5. *Let (M, g) be an even-dimensional complete manifold which is asymptotic to a fibered cusp metric at infinity. Then*

$$\chi(M) = (-1)^{n/2} \int_M \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right),$$

and

$$\tau(M) = \int_M L\left(\frac{\Omega}{2\pi}\right) - \frac{1}{2} a\text{-}\lim \eta,$$

where $a\text{-}\lim \eta = \lim_{\epsilon \to 0} \eta(\partial M_\epsilon)$ denotes the adiabatic limit of the eta invariant.

Proof. Since the proofs of both formula are similar, we do it for the signature formula here. Applying the Atiyah–Patodi–Singer formula with the Chern–Simons correction term to $M_\epsilon = \{x \geqslant \epsilon\}$ (and ϵ sufficiently small), we have

$$\tau(M_\epsilon) = \int_{M_\epsilon} L\left(\frac{\Omega}{2\pi}\right) - \int_{\partial M_\epsilon} Q - \frac{1}{2}\eta(\partial M_\epsilon),$$

where Q is the Chern–Simons terms involving the second fundamental form of ∂M_ϵ. By Proposition 3.3 and the discussion preceding it, we can take $\Omega \in \Gamma(\Lambda^2 T^*\bar{M} \otimes \mathrm{End}({}^dT\bar{M}))$. It follows that the first term on the right-hand side of the APS index formula has a finite limit as ϵ goes to zero. The metric on ∂M_ϵ is approaching

$$\pi^* g_B + \epsilon^2 g_F = \epsilon^2\left(\epsilon^{-2}\pi^* g_B + g_F\right).$$

By the scale invariance of the eta invariance,

$$\lim_{\epsilon \to 0} \eta(\partial M_\epsilon) = a\text{-}\lim \eta$$

is the adiabatic limit. On the other hand, the limit as ϵ goes to zero of the Chern–Simons term is zero, since the limit of the second fundamental form is zero as follows from Lemma 3.4. Our result follows. □

For fibered boundary geometry at infinity, the analysis of the Chern–Simons term is more complicated. We will restrict ourself to dimension 4 in this section and leave the general discussion to the next section. As we see from (2.5) and (2.6), this involves computing the second fundamental form θ and the curvature form Ω. Taking a cue from our treatment in the fibered cusp case, we express both θ, Ω as matrices with respect to an orthonormal basis, but with entries differential forms that are smooth up to the boundary at infinity $x = 0$. First, assume that $g = g_1$ is a fibered boundary metric as defined by (3.2). Fix a local orthonormal frame e_a, e_i of $\partial\bar{M}$ compactible with the submersion metric $\pi^*(g_B) + g_F$ and let θ^a, θ^i be the dual 1-forms, where a ranges over the coordinates of B and i that of F. Then near infinity,

$$x^2\partial_x,\ x\,e_a,\ e_i$$

form an orthonormal basis for the metric g. Computing with respect to this basis, we find at $x = 0$

$$\theta_a^0 = \theta^a, \qquad \theta_i^0 = 0.$$

Similarly, we find

$$\Omega_0^a = f_{bc}^a(x)\theta^b \wedge \theta^c + O\big(x^2\big),$$

where $f_{bc}^a(x) = O(1)$ as $x \to 0$. This shows that, for fibered boundary geometry at infinity where the fibration has positive dimensional fiber, the Chern–Simons term (cf. (2.6)) for the signature vanishes in dimension 4:

$$\theta_a^0 \wedge \Omega_0^a = O\big(x^2\big).$$

For the Chern–Simons terms for the Euler number (2.5), one term involves only the second fundamental form:

$$\epsilon_{abcd}\theta_b^a \wedge \theta_e^c \wedge \theta_d^e.$$

Since θ_b^a is zero unless one of the indices is 0, this term reduces to a multiple of

$$\theta_1^0 \wedge \theta_2^0 \wedge \theta_3^0$$

which vanishes by the explicit form of θ_b^a computed above (i.e. $\theta_3^0 = 0$ as 3 is the index for the fiber coordinate here). The other term involved is (up to a constant multiple)

$$\epsilon_{abcd}\theta_b^a \wedge \Omega_d^c$$

which reduces to a multiple of

$$\theta_1^0 \wedge \Omega_3^2 - \theta_2^0 \wedge \Omega_3^1.$$

Again, explicit computation gives

$$\Omega_i^a = f_b^a(x)\theta^b \wedge \theta^3 + f_{bc}^a(x)\theta^b \wedge \theta^c + g_b^a(x)\theta^b \wedge dx + g(x)\theta^3 \wedge dx,$$

where $f_b^a(x) = O(x)$. It follows then that

$$\lim_{\epsilon\to 0} \int\limits_{\partial M_\epsilon} \theta_1^0 \wedge \Omega_3^2 - \theta_2^0 \wedge \Omega_3^1 = 0.$$

Thus, we have proved the following theorem in the special case when $g = g_1$ is a fibered boundary metric.

Theorem 3.6. *Let (M, g) be a complete manifold of dimension 4 which is asymptotic to a fibered boundary metric at infinity and the fiber has positive dimension. Then*

$$\chi(M) = (-1)^{n/2} \int\limits_M \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right),$$

and

$$\tau(M) = \int\limits_M L\left(\frac{\Omega}{2\pi}\right) - \frac{1}{2} a\text{-}\lim \eta.$$

In order to prove the theorem in general, we now consider the effect of the perturbation term. This part of discussion is not restricted to dimension four. Thus let $g = g_1 + a$ and denote by ∇, ∇^1, the Levi-Civita connection of g, g_1, respectively.

Lemma 3.7. *Let Q, Q^1 denote the Chern–Simons correction terms with respect to the metrics g, g_1, respectively. Then, for perturbation a satisfying the condition in Definition 3.1, we have*

$$\lim_{\epsilon\to 0} \int\limits_{\partial M_\epsilon} Q = \lim_{\epsilon\to 0} \int\limits_{\partial M_\epsilon} Q^1.$$

Proof. Let $S = \nabla - \nabla^1$ be the difference tensor. An easy calculation using Koszul's formula yields

$$\begin{aligned} g\big(S(X)Y, Z\big) + a\big(\nabla^1_X Y, Z\big) = \frac{1}{2}\big[&X\big(a(Y,Z)\big) + Y\big(a(X,Z)\big) - Z\big(a(X,Y)\big) \\ &- a\big(X, [Y,Z]\big) - a\big(Y, [X,Z]\big) + a\big(Z, [X,Y]\big)\big], \end{aligned} \tag{3.4}$$

for vector fields X, Y, Z.

By Proposition 3.2, S is a (regular) 1-form valued endomorphism of ${}^{\phi}T\bar{M}$. Effectively, this means that in (3.4) we let X be a usual vector field while letting Y, Z be smooth sections of ${}^{\phi}T\bar{M}$, i.e., rescaled vector fields. It follows from the assumption on the perturbation a that

$$S = xS_1 + dx \otimes S',$$

where S_1 is a 1-form valued endomorphism of ${}^{\phi}T\bar{M}$, and S' an endomorphism of ${}^{\phi}T\bar{M}$. The crucial point here is that the precise form of S' is not important when we restrict to $\partial M_\epsilon = \{x = \epsilon\}$.

Now the curvature of g is related to that of g_1 via

$$\Omega = \Omega_1 + [\nabla^1, S] + S^2.$$

Hence,

$$\Omega = \Omega_1 + x\Omega' + dx \wedge \Omega'',$$

where Ω' is a (regular) 2-form valued endomorphism of ${}^{\phi}T\bar{M}$ and Ω'' a (regular) 1-form valued endomorphism of ${}^{\phi}T\bar{M}$.

Similarly we find that the second fundamental forms of ∂M_ϵ with respect to the metrics g and g_1, respectively, differ by a term vanishing to first order of ϵ:

$$\theta = \theta_1 + \epsilon\theta',$$

where θ' is a (regular) 1-form valued endomorphism of ${}^{\phi}T\bar{M}$. Our lemma follows. □

4. Chern–Simons term for fibered boundary geometry

It turns out that Theorem 3.6 holds in any dimension. In order to see this, we now discuss briefly some elementary geometry of a fibration following [8]. Thus let $F \to N \xrightarrow{\pi} B$ be a fibration of smooth manifolds. It gives rise to a subbundle of TN, the vertical bundle T^VN, whose section consists of vector fields of N tangent to the fibers. This leads to the exact sequence of vector bundles

$$0 \to T^VN \to TN \to \pi^*TB \to 0.$$

A connection for the fibration is a splitting

$$TN = T^HN \oplus T^VN, \tag{4.1}$$

where $T^HN \cong \pi^*TB$ is the horizontal bundle. For example, a Riemannian metric g on N determines such a splitting, where T^HN is the orthogonal complement of T^VN.

If ∇^F is a family of connections on F parametrized by B, then it defines a connection (still denoted by the same notation) on T^VN by adding

$$\nabla^F_{X^H} Y = [X^H, Y],$$

where X^H is a horizontal vector field and Y vertical vector field (a section T^VN). In particular, if g^F is a family of Riemannian metrics on F (parametrized by B), the corresponding Levi-Civita connections define such a connection on the vertical bundle.

Together with a connection ∇^B (determined by a metric g^B, for example) on B, one can define a connection ∇ on TN which is diagonal with respect to the splitting (4.1):

$$\nabla = \pi^*\nabla^B \oplus \nabla^F. \tag{4.2}$$

Now let $g^N = \pi^*g^B + g^F$ be a submersion metric on N. Then the above discussion gives a diagonal connection ∇ on TN determined by the Levi-Civita connections of g^B and g^F. Let ∇^L

be the Levi-Civita connection of g^N and $S = \nabla^L - \nabla$ the difference tensor. Since Levi-Civita connections are scale invariant, the diagonal connection ∇ stays the same under the adiabatic limit $g^N_\epsilon = \epsilon^{-2}\pi^* g^B + g^F$.

Let $\nabla^{L,\epsilon}$ be the Levi-Civita connection of g^N_ϵ and $S^\epsilon = \nabla^{L,\epsilon} - \nabla$ the corresponding difference tensor. Denote by P^H, P^V the projections associated with the splitting (4.1). The following observation is from [8].

Lemma 4.1. *For any vector field X on N, $S(X)$ defines an odd endomorphism of TN with respect to the splitting* (4.1). *That is,*

$$S(X): T^H N \to T^V N, \qquad S(X): T^V N \to T^H N.$$

Moreover,

$$P^H S^\epsilon = \epsilon^2 P^H S, \qquad P^V S^\epsilon = P^V S.$$

For later purpose and also for symmetry, we paraphrase it in terms of the rescaled splitting

$$^\epsilon TN = \epsilon T^H N \oplus T^V N, \tag{4.3}$$

and think of the connection $\nabla^{L,\epsilon}$ as a connection $^\epsilon\nabla^L$ on $^\epsilon TN$. (Effectively this is computing the connection with respect to an orthonormal basis of the adiabatic metric g^N_ϵ but with the crucial difference that the directional vector field is the usual vector field.) Note that ∇ stays unchanged. Then

$$^\epsilon\nabla^L = \nabla + O(\epsilon). \tag{4.4}$$

We now consider a Riemannian manifold (M, g) which is asymptotic to a fibered boundary metric at infinity. By Lemma 3.7, we can actually assume that g is a fibered boundary metric. That is, we have a fibration $F \to \partial\bar{M} \xrightarrow{\pi} B$ and

$$g = \frac{dx^2}{x^4} + \frac{\pi^* g^B}{x^2} + g^F.$$

Thus, near $\partial\bar{M}$, we have a direct sum decomposition

$$^\phi TM = \langle x^2\partial_x\rangle \oplus xT^H(\partial\bar{M}) \oplus T^V(\partial\bar{M}). \tag{4.5}$$

Let ∇^M be the Levi-Civita connection of g. For each hypersurface $x = \epsilon$, the metric

$$g_0 = \frac{dx^2}{\epsilon^4} + \frac{\pi^* g^B}{\epsilon^2} + g^F$$

is a product metric near $x = \epsilon$ and restricts to $x = \epsilon$ to the same metric as g. Let ∇^0 be its Levi-Civita connection. The difference

$$\theta = \nabla^M - \nabla^0 \in \Omega^1\big(M, \mathrm{End}(TM)\big)$$

is a matrix with 1-form entries and, when restricted to $x=\epsilon$, has only normal components (i.e. off-diagonal with respect to the decomposition into tangential and normal part) determined by the second fundamental form of $x=\epsilon$. As before, we reinterpret θ as a 1-form taking values in $\mathrm{End}({}^{\phi}TM)$ and thus, θ is off-diagonal with respect to the decomposition (4.5). In fact, if we take the orthonormal basis $x^2\partial_x, xe_i, f_\alpha$ where e_i is (the lift of) an orthonormal basis of (B, g^B) and f_α that of g^F, then

$$\theta_i^0 = -\theta_0^i = \omega^i,$$

and all other components of θ vanish. Here ω^i are the (pullback of) dual 1-forms of e_i. The crucial observation is that θ is in block form with respect to the splitting (4.5) with nontrivial entries only in the block from $\langle x^2\partial_x\rangle \oplus x\,T^H(\partial\bar{M})$ to itself. Moreover, the nontrivial entries are (pullbacks) of forms on B.

Lemma 4.2. *For a complete manifold (M, g) which is asymptotic to a fibered boundary metric at infinity and the fiber has positive dimension, and for any invariant polynomial P, the Chern–Simons term vanishes at infinity*:

$$\lim_{\epsilon\to 0} \int_{\partial M_\epsilon} Q = 0.$$

Here Q is defined in (2.4) *and ∂M_ϵ is the hypersurface $x=\epsilon$.*

Proof. By the discussion above, we have a similar block structure for Ω_t with diagonal blocks from $\langle x^2\partial_x\rangle \oplus xT^H(\partial\bar{M})$ to itself and from $T^V(\partial\bar{M})$ to itself plus an error term of $O(\epsilon)$. Moreover, the diagonal block from $\langle x^2\partial_x\rangle \oplus xT^H(\partial\bar{M})$ to itself involves only pullbacks of forms on B. It follows from the explicit block structure of θ that

$$P(\theta, \Omega_t, \ldots, \Omega_t) = \pi^*(\alpha) + O(\epsilon),$$

where α is a differential form on B. Hence

$$\lim_{\epsilon\to 0} \int_{\partial M_\epsilon} Q = k \int_0^1 \int_{\partial\bar{M}} \pi^*(\alpha) = 0. \qquad \square$$

Thus, we have

Theorem 4.3. *Let (M, g) be a complete manifold which is asymptotic to a fibered boundary metric at infinity and the fiber has positive dimension. Then*

$$\chi(M) = (-1)^{n/2} \int_M \mathrm{Pf}\left(\frac{\Omega}{2\pi}\right),$$

and

$$\tau(M) = \int_M L\left(\frac{\Omega}{2\pi}\right) - \frac{1}{2}a\text{-}\lim \eta.$$

5. The adiabatic limit of the eta invariant

There is extensive work on the adiabatic limit of the eta invariant (and other geometric invariants) (cf. [8,15] among others). In general if M is a closed oriented manifold that has a fibration structure

$$Y \to M \xrightarrow{\pi} B \tag{5.1}$$

and g_M a submersion metric,

$$g_M = \pi^* g_B + g_Y,$$

then blowing up the metric in the horizontal direction by a factor ϵ^{-2} gives us a family of metrics g_x,

$$g_\epsilon = \epsilon^{-2}\pi^* g_B + g_Y.$$

Let A_ϵ be the signature operator on M with respect to the adiabatic metric g_ϵ. A general formula for $\lim_{\epsilon\to 0}\eta(A_\epsilon)$ is given in [15], which, in fact, comes from a more general formula for Dirac operators (cf. [15]). Namely,

$$\lim_{\epsilon\to 0}\eta(A_\epsilon) = 2\int_B \mathcal{L}\left(\frac{R^B}{2\pi}\right)\wedge\tilde{\eta} + \eta(A_B\otimes \ker A_Y) + 2\tau, \tag{5.2}$$

where $\tilde{\eta}$ is the $\tilde{\eta}$-form of Bismut–Cheeger [8], R^B is the curvature tensor of g_B and A_B denotes the signature operator on B and A_Y the family of signature operators along Y. The integer τ is a topological invariant computable from the Leray spectral sequence.

In the case of circle bundles, i.e., $Y = S^1$, the terms on the right-hand side of (5.2) can be explicitly computed. For example

$$\tilde{\eta} = 2\left(\frac{1}{2\tanh\frac{e}{2}} - \frac{1}{e}\right),$$

and

$$\tau = \operatorname{sign}(B_e),$$

where B_e is the quadratic form

$$B_e : H^{2k-2}(B)\otimes H^{2k-2}(B) \to \mathbb{R},$$
$$B_e(x\otimes y) = \langle xye, [B]\rangle.$$

Here e is the Euler class of the circle bundle. This gives us the following result of [16].

Theorem 5.1. *We have*

$$\frac{1}{2}\lim_{\epsilon\to 0}\eta(A_\epsilon)=\left\langle L(TB)\left(\frac{1}{\tanh e}-\frac{1}{e}\right),[B]\right\rangle-\operatorname{sign}(B_e). \tag{5.3}$$

When $\dim B=2$, i.e., we have a circle bundle over a surface, the formula (5.3) gives

$$\frac{1}{2}\lim_{\epsilon\to 0}\eta(A_\epsilon)=\frac{1}{3}e-\operatorname{sign}e. \tag{5.4}$$

6. Proof of the theorems

We now proceed to prove Theorem 1.1. By Theorems 3.5, 3.6, formula (5.4), and the decomposition of curvature in dimension 4, we have

$$\chi(M)=\frac{1}{8\pi^2}\int_M\left(|W|^2-|Z|^2+\frac{1}{24}S^2\right)d\mathrm{vol},$$

and

$$\tau(M)+\frac{1}{3}e-\operatorname{sign}e=\frac{1}{12\pi^2}\int_M\left(|W^+|^2-|W^-|^2\right)d\mathrm{vol}.$$

The rest of the proof is the same as in the closed case.

Note that the equality holds exactly as in the closed case, namely, for Ricci flat manifolds with either vanishing self-dual or anti-self-dual Weyl curvature. Thus M must be Kähler as follows from the same argument of [20], and hence Calabi–Yau.

For Theorem 1.3, we can no longer apply Theorem 3.6. However, using Lemma 3.7, the conformal invariant of the Pontryagin forms and the scale invariance of the eta invariant, one still has

$$\tau(M)+\frac{1}{2}\eta(\partial\bar{M})=\frac{1}{12\pi^2}\int_M\left(|W^+|^2-|W^-|^2\right)d\mathrm{vol}.$$

On the other hand,

$$\chi(M)+\lim_{\epsilon\to 0}\int_{\partial M_\epsilon}Q=\frac{1}{8\pi^2}\int_M\left(|W|^2-|Z|^2+\frac{1}{24}S^2\right)d\mathrm{vol},$$

where Q is given by (2.5):

$$Q=\epsilon_{abcd}\left(2\theta^a_b\wedge\Omega^c_d-\frac{4}{3}\theta^a_b\wedge\theta^c_e\wedge\theta^e_d\right).$$

Here we emphasize that Ω_d^c denotes the two form components of the curvature of M. In this case, using Lemma 3.7, an explicit computation shows that

$$\lim_{\epsilon\to 0} \int_{\partial M_\epsilon} Q = \frac{1}{2\pi^2}\operatorname{vol}(\partial\bar{M}) + \alpha(\partial\bar{M}).$$

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