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Mass Under the Ricci Flow

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Abstract: In this paper, we study the change of the ADM mass of an ALE space along the Ricci flow. Thus we first show that the ALE property is preserved under the Ricci flow. Then, we show that the mass is invariant under the flow in dimension three (similar results hold in higher dimension with more assumptions). A consequence of this result is the following. Let (M, g) be an ALE manifold of dimension $n = 3$. If $m(g) \neq 0$, then the Ricci flow starting at g can not have Euclidean space as its (uniform) limit.

1. Introduction

Ricci flow is an important geometric evolution equation in Riemannian Geometry. It was introduced by R. Hamilton in 1982 (see [8]) and used extensively by him to prove some outstanding results on 3-manifolds and 4-manifolds. Recently it has been used spectacularly by G. Perelman [14] to study the geometrization conjecture on 3-manifold. The flow has also been very useful in the study of pinching results and the metric smoothing process. As a natural geometric tool, Ricci flow should be used to study properties of physically meaningful objects such as mass, entropy, etc. In this paper we would like to understand the behavior of mass under the Ricci flow.

In general relativity, isolated gravitational systems are modeled by spacetimes that asymptotically approach Minkowski spacetime at infinity. The spatial slices of such spacetimes are then the so-called asymptotically flat or asymptotically Euclidean (AE in short) manifolds. That is, Riemannian manifolds (M^n, g) such that $M = M_0 \cup M_\infty$ (for simplicity we deal only with the case of one end; the case of multiple ends can be dealt with similarly) with M_0 compact and $M_\infty \simeq \mathbb{R}^n - B_R(0)$ for some $R > 0$ so that in the induced Euclidean coordinates the metric satisfies the asymptotic conditions

$$g_{ij} = \delta_{ij} + O(r^{-\tau}), \quad \partial_k g_{ij} = O(r^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = O(r^{-\tau-2}). \tag{1.1}$$

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Here $\tau > 0$ is the asymptotic order and r is the Euclidean distance to a base point. The total mass (the ADM mass) of the gravitational system can then be defined via a flux integral [1, 11],

$$m(g) = \lim_{R\to\infty} \frac{1}{4\omega_n} \int_{S_R} (\partial_i g_{ij} - \partial_j g_{ii}) * dx_j. \tag{1.2}$$

Here ω_n denotes the volume of the $(n-1)$-sphere and S_R the Euclidean sphere with radius R centered at the base point. By [3], when the scalar curvature is integrable and $\tau > \frac{n-2}{2}$, the mass $m(g)$ is well defined and independent of the coordinates at infinity, and therefore is a metric invariant. The famous Positive Mass Theorem, proved by Schoen-Yau [19] (later Witten gave an elegant spinor proof [20]), says that the mass $m(g) \geq 0$ if the scalar curvature is nonnegative (and the manifold is spin). Moreover, $m(g) = 0$ if and only if M is the Euclidean space.

There is also the notion of asymptotically *locally* Euclidean, or ALE, manifolds. For our purpose we will use the following characterization of the ALE property of a complete non-compact Riemannian manifold (M, g). Namely we use the *curvature decay condition*

$$|Rm|(x) = O(d(x)^{-(2+\tau)}) \tag{1.3}$$

for some $\tau > 0$ as $d(x) \to \infty$, and the *volume growth condition*

$$Vol(B_r, g) \geq Vr^n \tag{1.4}$$

for some constant $V > 0$. Here Rm is the Riemannian curvature tensor of the metric g, and $d(x)$ is the distance function from the base point.

According to [3, Theorem (1.1)], (1.3), (1.4) imply that (M, g) is asymptotically *locally* Euclidean. Namely, $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq \left(\mathbb{R}^n - B_R(0)\right)/\Gamma$, where $\Gamma \subset O(n)$ is a finite group acting freely on $\mathbb{R}^n - B_R(0)$, so that the asymptotic conditions (1.1) hold. For an ALE manifold (M, g), the mass $m(g)$ can be defined by (1.2) again, except that S_R should be taken as the distance sphere or, equivalently, the quotient of the Euclidean sphere by Γ.

An ALE manifold (M, g) is actually AE if the asymptotic volume ratio $\mu = 1$. Here

$$\mu = \lim_{r\to\infty} V(B_r, g)/\Omega_n r^n, \tag{1.5}$$

where Ω_n is the volume of the unit ball in the standard n-dimensional Euclidean space $\mathbb{R}^n$.

As we mentioned, we would like to investigate the behavior of the mass $m(g)$ under the Ricci flow. Recall that the Ricci flow is a family of evolving metrics $g(t)$ such that

$$\frac{\partial}{\partial t} g(t) = -2Rc(g(t)), \tag{1.6}$$

on M with $g(0) = g$, where $Rc(g(t))$ is the Ricci tensor of the metric $g(t)$. To make sure that the mass is well defined under this evolving flow $g(t)$, we first need to show that the ALE property is preserved along the flow.

Our main result is the following

Theorem 1. *Let $g(t)$, $0 \leq t \leq T < +\infty$, be a Ricci flow on M with $(M, g(0))$ being an ALE (AE resp.) manifold of dimension n. Assume that $g(t)$ has uniformly bounded sectional curvature. Then*

A) The ALE (AE resp.) property is preserved along the flow.
B) The integrability condition $R \in L^1$ is also preserved along the flow provided (4.25) and (4.26) hold, which is the case if either $R = O(r^{-q})$, $q > n$ and $\tau > \frac{n-4}{2}$ or $\tau > \max\{n-4, \frac{n-2}{2}\}$.
C) Under the conditions above, the mass $m(t) = m(g(t))$ is well defined, and

$$m'(t) = \int_{S_{r\to\infty}} R_i dS^i, \tag{1.7}$$

where $dS^i = \frac{1}{4\omega_n} * dx_i$ and R_i denotes the covariant derivative of R. Furthermore, if $R = O(r^{-q})$ for $q > n$ and $\tau > \frac{n-2}{2}$, or $\tau > \max\{n-3, \frac{n-2}{2}\}$, the mass is invariant under the Ricci flow. In particular, the mass is invariant in dimension 3 ($\tau > \frac{1}{2}$).
D) Assume that the initial metric $g(0) = g$ satisfies the additional decay condition

$$\partial_k \partial_l \partial_p g_{ij} = O(r^{-\tau-3}), \tag{1.8}$$

then the mass is invariant under the Ricci flow if $n \le 6$ (and $\tau > \frac{n-2}{2}$) or if $\tau > \max\{n-4, \frac{n-2}{2}\}$.

That the ALE property is preserved will be obtained by using the maximum principles of Ecker-Huisken [5] and W.X.Shi [15], Cf. also P.Li and S.T.Yau [12]. To compute the changing rate of the mass of the evolving metric $g(t)$, the key part is to get a decay estimate of the spatial derivative of the scalar curvature function $R(x,t)$ of the metric $g(t)$ at infinity, which is furnished by Shi's local gradient estimate [15].

Theorem 2. *Let $g(t)$, $0 \le t < +\infty$, be a Ricci flow on M with uniformly bounded sectional curvature. Assume further that each $g(t)$ is ALE and $g(t)$ converges uniformly to an ALE metric g_∞ as t goes to infinity. Then*

$$\lim_{t\to\infty} m(g(t)) = m(g_\infty).$$

The notion of uniform convergence is introduced by using the space $\mathcal{M}_\tau$ of [11], see Definition 13. A direct consequence of Theorems 1 and 2 is the following

Corollary 3. *Let (M, g) be an ALE manifold of dimension n and of asymptotic order $\tau > \max\{n-3, \frac{n-2}{2}\}$. If $m(g) \neq 0$, then the Ricci flow starting at g can not converge uniformly to a Euclidean space.*

Note that the Ricci flow preserves nonnegative scalar curvature [8]. This raises the prospect of proving the positive mass theorem of Schoen and Yau via Ricci flow, which we intend to investigate elsewhere. On the other hand, one can also look for applications of our results by combining with the positive mass theorem. For example, one sees that there are no complete non-compact Riemannian manifolds satisfying the hypothesis of the Main Theorem in [15] in dimension 3 by using the long time convergence result of [15], Theorems 1 and 2, and the positive mass theorem, see also [4, 7].

Let us explain why Theorem 1 comes so natural. We begin by recalling some basic facts about Ricci flow on complete non-compact Riemannian manifold (M, g) with bounded sectional curvature K_0.

Let $g(t)$ be a family of the metrics evolving under the Ricci flow on M with initial data g, $0 \le t \le T < +\infty$. We shall write by $\nabla_{g(t)}$ and $R_{ijkl}(t)$ the Riemannian connection and Riemannian curvature tensor of $g(t)$ respectively. Hamilton proved in [8] that the asymptotic volume ratio $\mu(t) = \mu(g(t))$ of (1.5) is a constant under the Ricci flow with bounded curvature and nonnegative Ricci curvature, where $|Rm| \to 0$ at infinity on the complete non-compact Riemannian manifold. This result tells us that if $\mu \neq 1$ at $t = 0$, then the Ricci flow can not have Euclidean space as its limit.

It is well known that Ricci flow smoothes out the metric. W.X.Shi [16] showed that there exists a positive constant $T > 0$ such that for any integer $\alpha \geq 0$ and any $0 < t \leq T$, there exist constants $c(n, K_0)$, $c(n, K_0, T)$ and $c(n, , K_0, \alpha, t)$ such that

$$e^{-c(n,K_0)t} g \leq g(t) \leq e^{c(n,K_0)t} g, \quad |\nabla_g - \nabla_{g(t)}| \leq c(n, K_0)t,$$
$$|R_{ijkl}| \leq c(n, K_0, T),$$

and

$$|\nabla^{\alpha}_{g(t)} R_{ijkl}(t)| \leq c(n, K_0, \alpha, t).$$

All these facts will be implicitly used in this paper. It is clear that the volume growth condition (1.4) is preserved along the Ricci flow. In Sect. 18 in [8], Hamilton further proved that if the curvature $Rm \to 0$ as $s \to +\infty$ for the initial metric, where s is the distance function to a fixed point of the metric g, the same is true for each $g(t)$. So it is very natural for one to expect that if the curvature of the initial metric has decay at infinity, then the same is true for the evolving metric $g(t)$. With this understanding, one would like to know the change of the mass, a natural invariant of the metric, under the flow.

Finally, we refer the reader to [6] for related discussions under the (worldsheet) RG flow. Also, in a recent preprint [13], a different approach is used to study these questions that we address here.

Throughout this paper we will denote by C, c various constants depending only on dimension.

2. Preliminaries

In this section we briefly introduce some facts on Ricci flow. We shall use notations from [9]. Let M be a manifold of dimension n, $g(t)$ a family of metrics evolving by Ricci flow (1.6). Then the curvature tensor evolves by the equation

$$\frac{\partial}{\partial t} Rm = \Delta Rm + Rm * Rm, \tag{2.9}$$

where $Rm * Rm$ denotes a quadratic expression of the curvature tensor. It follows then

$$\frac{\partial}{\partial t}|Rm|^2 = \Delta|Rm|^2 - 2|\nabla Rm|^2 + Rm * Rm * Rm,$$

which yields

$$\frac{\partial}{\partial t}|Rm|^2 \leq \Delta|Rm|^2 + C|Rm|^3. \tag{2.10}$$

The evolution equation for the scalar curvature is much simpler, and one has

$$\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2. \tag{2.11}$$

Now let X be a point in M. Let $Y = \{Y_a\}, 1 \leq a \leq n$ be a frame at X. In local coordinates $X = \{x^i\}$, we have

$$Y_a = y_a^i \partial/\partial x^i.$$

Let

$$g_{ab} = g(Y_a, Y_b),$$

and let

$$\nabla_b^a = y_b^i \partial/\partial y_a^i$$

be the vector fields tangent to the fibers of the frame bundle. Write by D_a the vector field on the frame bundle $F(M)$ which is the lift of the vector Y_a at $Y \in F(M)$. Then we have

$$D_a = y_a^i[\partial x^i - \Gamma_{ij}^k y_b^j \partial/\partial y_b^k],$$

where Γ_{ij}^k's are the Christoffel symbols of the connection. Under the Ricci flow, we can define the evolving orthonormal frame on M such that

$$\partial_t F_a^i = g^{ij} R_{jk} F_a^k,$$

where (g^{ij}) is the inverse matrix of (g_{ij}). Then we set

$$D_t = \partial_t + R_{ab} g^{bc} \nabla_c^b.$$

Note that

$$D_t g_{ab} = 0.$$

This says that D_t is the unique tangent vector field to the orthonormal bundle. Choose a metric on $F(M)$ such that D_a, ∇_c^b are an orthonormal basis. Then we can see that $D_t - \partial_t$ is a space-like vector orthonormal to the orthonormal frame bundle. A useful fact for us is that for a smooth function u on $M \times (0, T)$, we have

$$(D_t - \Delta) D_a u = D_a(\partial_t - \Delta)u. \tag{2.12}$$

We now recall Hamilton's argument [8]. Assume that a K-bounded smooth function u satisfies the heat equation

$$u_t = \Delta u, \quad \text{in} \quad M \tag{2.13}$$

with $|Du|^2 \leq \delta$ at $t = 0$. Then we have by (2.12),

$$D_t D_a u = \Delta D_a u,$$

and thus

$$\partial_t |Du|^2 = \Delta |Du|^2 - 2|D^2 u|^2.$$

By the maximum principle of Shi [15] we have

$$|Du|^2(x, t) \leq \delta. \tag{2.14}$$

Let $F = t|D^2u|^2 + |Du|^2$. Then by a direct computation, we have

$$\partial_t F \leq \Delta F - (1 - cKt)|D^2u|^2.$$

Hence, by the maximum principle of Shi [15] again we get for $t \leq 1/cK$,

$$F(x,t) \leq \delta^2,$$

which implies that

$$|D^2u| \leq \delta/\sqrt{t}.$$

Note that $|\Delta u|^2 \leq n|D^2u|^2$. Using the heat equation we obtain that for $t \leq 1/cK$,

$$|u_t| \leq \sqrt{n}\delta/\sqrt{t}.$$

Therefore, we have

$$|u(x,t) - u(x,0)| \leq 2\sqrt{n}\delta\sqrt{t}.$$

Take $\delta \leq \sqrt{K}\epsilon^2$ and assume that

$$\lim_{x\to\infty} u(x,0) = 0, \quad \text{in} \quad M$$

uniformly. Then we can conclude using an iteration argument that for any $t \in [0,T]$,

$$\lim_{x\to\infty} u(x,t) = 0, \quad \text{in} \quad M$$

uniformly. In fact, Hamilton [8] showed that for any $\delta > 0$ and for any bounded smooth function $u_0 \in C^1(M)$ with $\lim_{x\to\infty} u_0(x) = 0$, one can find a bounded smooth solution $u(x,t)$ to the heat equation such that $u_0(x) \leq u(x,0)$ and $|Du|^2(x,t) \leq \delta$ on $M \times [0,T]$.

3. ALE is Preserved

In this section we study the ALE property under the Ricci flow and show that it is preserved. The question can be reduced to the study of the non-negative solutions to the heat equation

$$u_t = \Delta u, \quad \text{in} \quad M \tag{3.15}$$

with initial data $u(0) = u_0$, where $\Delta = \Delta_{g(t)}$ is the Laplacian operator of the family of metrics $g(t)$. We assume that u_0 has a decay $O(d(x)^{-\sigma})$ for some $\sigma > 0$.

Theorem 4. *Let $g(t)$ be the solution of the Ricci flow (1.6) over* $[0,T]$. *Assume that $g(t)$ has uniform curvature bound $|Rm(g(t))| \leq K$. Then non-negative solutions to (3.15) have the same decay rate as the initial data u_0.*

The main tools here are the maximum principles, especially the maximum principle of [5, Theorem 4.3]. For the reader's convenience, we quote the result here (the superscript 't' is put in here to emphasize the t-dependence from the metric $g(t)$).

Theorem 5 (Ecker-Huisken). *Suppose that the complete non-compact manifold M^n with Riemannian metric $g(t)$ satisfies the uniform volume growth condition*

$$\mathrm{vol}^t(B_r^t(p)) \leq \exp\left(k(1+r^2)\right) \tag{3.16}$$

for some point $p \in M$ and a uniform constant $k > 0$ for all $t \in [0, T]$. Let w be a function on $M \times [0, T]$ which is smooth on $M \times (0, T]$ and continuous on $M \times [0, T]$. Assume that w and $g(t)$ satisfy

i) the differential inequality

$$\frac{\partial}{\partial t} w - \Delta^t w \leq \mathbf{a} \cdot \nabla w + bw, \tag{3.17}$$

where the vector field $\mathbf{a}$ and the function b are uniformly bounded

$$\sup_{M\times[0,T]} |\mathbf{a}| \leq \alpha_1, \quad \sup_{M\times[0,T]} |b| \leq \alpha_2 \tag{3.18}$$

for some constants $\alpha_1, \alpha_2 < \infty$;

ii) the initial data

$$w(p, 0) \leq 0 \tag{3.19}$$

for all $p \in M$;

iii) the growth condition

$$\int_0^T \left(\int_M \exp\left[-\alpha_3 d^t(p, y)^2\right] |\nabla w|^2(y) d\mu_t \right) dt < \infty \tag{3.20}$$

for some constant $\alpha_3 > 0$;

iv) bounded variation condition in metrics

$$\sup_{M\times[0,T]} \left| \frac{\partial}{\partial t} g(t) \right| \leq \alpha_4 \tag{3.21}$$

for some constant $\alpha_4 < \infty$.

Then, we have

$$w \leq 0 \tag{3.22}$$

on $M \times [0, T]$.

In our situation, with the metric $g(t)$ coming from the Ricci flow (1.6), the condition (3.21) is clearly satisfied by the uniform curvature bound. The uniform volume growth condition (3.16) also follows immediately from the volume comparison theorem via the curvature bound. The differential inequality will be coming from a modification of the solution of the heat equation (3.15). To see that the coefficients are uniformly bounded per (3.18) requires the following lemma.

Lemma 6. *Let $g(t)$ be the solution of the Ricci flow (1.6) over $[0, T]$ with $g(0) = g$ being an ALE. Assume that $g(t)$ has uniform curvature bound $|Rm(g(t))| \le K$. Then, for sufficiently large R, there is a smooth positive function f on M such that*

$$\begin{aligned} &f(x) = C_0 >> 1, \quad \textit{for } x \in B_R; \\ &c\, d^t(x) \le f(x) \le C d^t(x) \quad \textit{for } x \in M - B_R. \end{aligned}$$

Moreover

$$f \ge C_0, \quad |\nabla^t f| \le C_1, \quad |\Delta^t f| \le C_2.$$

Proof. Since (M, g) is ALE, we have coordinates at infinity, which we denote by x. Let $|x|$ be the Euclidean distance function.

Choose a smooth increasing function $\phi(s)$ on $\mathbb{R}$ such that

$$\begin{aligned} &\phi(s) = C_0 = R - 1, \quad \text{if } s \le R - 1; \\ &\phi(s) = s, \quad \text{if } s \ge R, \end{aligned}$$

and

$$|\phi'| \le 1, \qquad |\phi''| \le 2.$$

We define our function f to be $f(x) = \phi(|x|)$. Then clearly $f \ge C_0$. Since the metrics $g(t)$ are all equivalent and $g(0) = g$ is ALE, we can use the Euclidean norm in estimating $|\nabla^t f|$ and $|\Delta^t f|$. Then

$$|\nabla^t f| = |\phi'||\nabla^t |x|| \le C_2.$$

Similarly the estimate

$$|\Delta^t f| \le C_2$$

follows from the coordinate expression of the Laplacian

$$\Delta^t = \frac{1}{\sqrt{\det g(t)}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g(t)} g^{ij}(t) \frac{\partial}{\partial x_j} \right)$$

and the known estimate for $g(t)$. $\square$

We now suppress the superscript 't' with the understanding that all covariant derivatives and the Laplacian are taken with respect to $g(t)$.

In our application of Theorem 5 to the proof of Theorem 4, we will let $w = f^\sigma u$, where $\sigma > 0$. Then the growth condition (3.20) follows from the gradient bound (2.14), which implies that

$$|\nabla w| \le C(T) f^{\sigma+1}.$$

We now turn to the proof of Theorem 4.

Proof. For simplicity, we assume that (M, g_0) is an ALE with one end. Let $u_0(x) = O(d(x)^{-\sigma})$ as $d(x) \to \infty$, where the distance function is with respect to a fixed point o in M. Choose a global smooth positive function $f(x)$ on M as in Lemma 6 and let

$$h(x) = f(x)^\sigma.$$

Set $w(x,t) = h(x)u(x,t)$. Then, by a direct computation we have that

$$w_t = hu_t,$$
$$w_i = h_i u + hu_i,$$

and

$$\Delta w = \Delta h u + 2\nabla h \nabla u + h\Delta u.$$

Hence,

$$(\partial_t - \Delta)w = Bw - 2\nabla \log h \nabla w,$$

where $B(x,t) = \frac{2|\nabla h|^2 - h\Delta h}{h^2}$. Note that the coefficients B and $\nabla \log h$ are uniformly bounded by Lemma 6. In particular, $|B| \le b$. Since

$$w(x,0) = d(x)^\sigma u_0(x) \le D < +\infty,$$

and

$$(\partial_t - \Delta)(w - De^{tb}) \le B(w - De^{tb}) - 2\nabla \log h \nabla (w - De^{tb}),$$

we have by the maximum principle of Ecker-Huisken, Theorem 5, (see also the proofs of Theorem 18.2 [8] and Theorem 4.3 in [5], see also [15]) that there exists a uniform constant $C_1 > 0$ such that

$$max_{M_T} w \le C_1,$$

where $M_T = M \times [0,T]$. This implies the desired decay for $u(x,t)$. □

We are now in a position to prove the first part of Theorem 1. In fact, since $|Rm(g_0)| \le C_0 d(x)^{-\sigma}$ for $d(x) >> 1$, where $\sigma = 2 + \tau$, we can choose a bounded smooth function u_0, which dominates the function $|Rm(g_0)|^2$ such that it is $C_0 d(x)^{-2\sigma}$ for $d(x) >> 1$ and has bounded gradient. Let u be the solution of the heat equation as above. Then under Ricci flow, we have from (2.10) and the uniform curvature bound,

$$\partial_t |Rm|^2 \le \Delta |Rm|^2 + CK|Rm|^2$$

while

$$\partial_t (e^{CKt}u) = \Delta(e^{CKt}u) + CKe^{CKt}u.$$

Therefore, we have by the maximum principle of Shi [15] and Theorem 4 that

$$|Rm|^2 \le e^{CKt}u \le e^{CKt}d(x)^{-2\sigma}, \quad \text{on} \quad M.$$

Thus, under the Ricci flow, the ALE property is preserved.

Remark 7. That the ALE property is preserved does not follow from [10, Remark 0.11], as was claimed there.

The same is true for AE, as we have the following analog of a theorem of Hamilton [8].

Corollary 8. *Let $g(t)$, $0 \le t \le T$, be a Ricci flow on M with uniformly bounded sectional curvature. Assume further that $g(0)$ is ALE. Then the asymptotic volume ratio $\mu(t) = \mu(g(t))$ is constant along the Ricci flow.*

Proof. If (M, g) is ALE, it follows from the characterization in [3] that $M = M_0 \cup M_\infty$ with M_0 compact and $M_\infty \simeq \left(\mathbb{R}^n - B_R(0)\right) / \Gamma$, where $\Gamma \subset O(n)$ is a finite group acting freely on $\mathbb{R}^n - B_R(0)$, so that the asymptotic conditions (1.1) hold, where the asymptotic coordinates come from the projection of the Euclidean coordinates under $\mathbb{R}^n - B_R(0) \to \left(\mathbb{R}^n - B_R(0)\right) / \Gamma$. Therefore, (M, g) has the asymptotic volume ratio

$$\mu = \frac{1}{|\Gamma|}.$$

Since ALE is preserved along Ricci flow by Theorem 1, we deduce that $\mu(t)$ is a constant by the continuity of $\mu(t)$, as shown in the following lemma. □

Lemma 9. *Let $g(t)$, $0 \le t \le T$, be a Ricci flow on M with uniformly bounded scalar curvature. Assume that the asymptotic volume ratio $\mu(t) = \mu(g(t))$ are well defined. Then $\mu(t)$ is a continuous function of t.*

Proof. The volume of a ball changes according to the formula

$$\frac{d}{dt} V(B_r, g(t)) = -\int_{B_r} R(g(t)) dv_{g(t)}. \tag{3.23}$$

Hence

$$|\frac{d}{dt} V(B_r, g(t))| \le K V(B_r, g(t)),$$

where K denotes the uniform bound on the scalar curvature. Therefore,

$$e^{-K(t-t_0)} V(B_r, g(t_0)) \le V(B_r, g(t)) \le e^{K(t-t_0)} V(B_r, g(t_0)),$$

from which the continuity follows. □

4. The Changing Rate of Mass

For the mass to be well defined, one needs the integrability condition $R \in L^1$ in addition to the requirement that the asymptotic order $\tau > \frac{n-2}{2}$ [2, 17]. We have seen that the ALE property is preserved along the Ricci flow. We now examine the integrability condition.

One thing that in particular guarantees the integrability is the decay condition

$$R = O(r^{-q}), \qquad q > n. \tag{4.24}$$

We have

Theorem 10. *Let $g(t)$ be the solution of the Ricci flow (1.6) over $[0, T]$ with uniformly bounded curvature. Assume that $g(0)$ is ALE with asymptotic order $\tau > 0$ and its scalar curvature satisfy the decay condition $R(0) = O(r^{-q}), \quad q > 0$. Then the scalar curvature of $g(t)$ satisfies the decay condition*

$$R(t) = O(r^{-q'}), \quad q' = \min\{q, 2(\tau + 2)\}.$$

Proof. This is similar to our proof of the ALE property. The scalar curvature satisfies the evolution equation

$$\frac{\partial}{\partial t} R = \Delta R + 2|Rc|^2.$$

By the assumption and our result on the ALE property, we have $|Rc|^2 = O(r^{-2(\tau+2)})$. Let f be the function in Lemma 6 and $w = f^{q'} R$. Then

$$(\partial_t - \Delta)w \le Bw + 2q'\nabla \log f \nabla w + C,$$

where $f^{q'}|Rc|^2 \le C$. Hence, by the argument in the proof of ALE property, we have

$$\max_{M_T} w \le C_1.$$

□

In particular, when the order of decay of the initial scalar curvature $q > n$ and the asymptotic order $\tau > \frac{n-4}{2}$, then the order of decay of the evolving scalar curvature also satisfies $q' > n$.

In general, without assuming the ALE conditions, we show that under the natural condition

$$\int_0^T \int_M |Rc|^2 < +\infty, \tag{4.25}$$

the property $R \in L^1$ is preserved under the Ricci flow if the decay condition

$$R(t) = O(r^{-\sigma}) \tag{4.26}$$

holds uniformly for some $\sigma \ge n - 2$ and all $t \in [0, T]$. We remark that both conditions (4.25) and (4.26) are always true for $0 < T < +\infty$ if the initial metric is ALE, provided the order $\sigma = 2 + \tau \ge n - 2$ (i.e. $\tau \ge n - 4$) and $2\sigma > n$ in the curvature decay condition (1.3), as it follows from our result on ALE property and Theorem 10.

Theorem 11. *Let $g(t)$ be the solution of the Ricci flow (1.6) over $[0, T]$ with uniformly bounded curvature. Assume that the conditions (4.25) and (4.26) hold. Then the property $R \in L^1$ is preserved under the Ricci flow.*

Proof. Recall that on M,

$$\Delta R = R_t - 2|Rc|^2.$$

Let $p = 1 + \epsilon$ with small $\epsilon > 0$. Let ϕ be a non-negative cut-off function such that $0 \le \phi \le 1$ on M, $\phi = 1$ on $B_r(o)$, $\phi = 0$ outside $B_{2r}(o)$, and

$$|\nabla\phi|^2 \le 4\phi/r^2.$$

Then

$$\begin{aligned}
&\int_0^t dt \int_M \phi^2 |R|^{p-1}(R_t - 2|Rc|^2)\,\mathrm{sgn}R \\
&= \int_0^t dt \int_M \phi^2 |R|^{p-1}(\Delta R)\,\mathrm{sgn}R \\
&= -2\int_0^t dt \int_M \phi |R|^{p-1} < \nabla\phi, \nabla R > \mathrm{sgn}R \\
&\quad - (p-1)\int_0^t dt \int_M \phi^2 |R|^{p-2}|\nabla R|^2 \\
&\le \frac{2}{p-1}\int_0^t dt \int_M |\nabla\phi|^2 |R|^p - \frac{2(p-1)}{p^2}\int_0^t dt \int_M \phi^2 |\nabla(|R|^{p/2})|^2 \\
&\le \frac{2}{(p-1)r^2}\int_0^t dt \int_M \phi |R|^p - \frac{2(p-1)}{p^2}\int_0^t dt \int_M \phi^2 |\nabla(|R|^{p/2})|^2 \\
&\le \frac{2}{(p-1)r^2}\int_0^t dt \int_M \phi |R|^p \le \frac{2CTr^{n-2-p\sigma}}{p-1} \to 0
\end{aligned}$$

as $r \to \infty$. Here we have used the decay condition $R = O(r^{-\sigma})$ for some $\sigma > n-2$.

By direct computation, we have

$$\begin{aligned}
&\int_0^t dt \int_M \phi^2 |R|^{p-1}(R_t - 2|Rc|^2)\,\mathrm{sgn}R \\
&= -2\int_0^t dt \int_M \phi^2 |R|^{p-1}|Rc|^2\,\mathrm{sgn}R + \int_0^t dt \int_M \phi^2 |R|^{p-1} R_t\,\mathrm{sgn}R \\
&= -2\int_0^t dt \int_M \phi^2 |R|^{p-1}|Rc|^2\,\mathrm{sgn}R + \frac{1}{p}\int_0^t dt \frac{d}{dt}\int_M \phi^2 |R|^p \\
&\quad + \frac{1}{p}\int_0^t dt \int_M \phi^2 |R|^p R \\
&= -2\int_0^t dt \int_M \phi^2 |R|^{p-1}|Rc|^2\,\mathrm{sgn}R + \frac{1}{p}\int_M \phi^2 |R|^p(t) - \frac{1}{p}\int_M \phi^2 |R|^p(0) \\
&\quad + \frac{1}{p}\int_0^t dt \int_M \phi^2 |R|^p R.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&-2\int_0^t dt \int_M \phi^2 |R|^{p-1}|Rc|^2\,\mathrm{sgn}R + \frac{1}{p}\int_M \phi^2 |R|^p(t) \\
&\quad - \frac{1}{p}\int_M \phi^2 |R|^p(0) + \frac{1}{p}\int_0^t dt \int_M \phi^2 |R|^p R \\
&\le \frac{2CTr^{n-2-p\sigma}}{p-1}.
\end{aligned}$$

Sending $r \to +\infty$, we get

$$-2\int_0^t dt \int_M |R|^{p-1}|Rc|^2 \operatorname{sgn} R + \frac{1}{p}\int_M |R|^p(t) - \frac{1}{p}\int_M |R|^p(0) + \frac{1}{p}\int_0^t dt \int_M |R|^p R \le 0.$$

Sending $p \to 1$, we have that

$$-2\int_0^t dt \int_M |Rc|^2 \operatorname{sgn} R + \int_M |R|(t) - \int_M |R|(0) + \int_0^t dt \int_M |R|R \le 0.$$

That is,

$$\int_M |R(t)| - \int_M |R(0)| \le \int_0^T dt \int_M R^2 + 2\int_0^T dt \int_M |Rc|^2,$$

which implies that $R \in L^1$ for each $t > 0$. □

Using a similar argument, we can show that the property $|Rm| \in L^p$ $(p \geq 1)$ is preserved under the Ricci flow with bounded curvature.

We now look at the change of mass under the Ricci flow. Let S be a hypersurface in M. Without loss of generality, we can assume that M is oriented.

We can take the local frame F_a such that $F_1, ..., F_{n-1}$ are tangent to S and $F_n = \nu$ is orthogonal to S at X.

Let ω^a be a local frame dual to F_a. Then the area form on S is

$$dS = \omega^1 \wedge ... \wedge \omega^{n-1}.$$

Let

$$f^a_b = \partial/\partial_t \omega^a(F_b).$$

Then we have

$$f^a_b = -\omega^a(\partial/\partial_t F_b),$$

which is a decay term of the same order as $R_{jk} = O(r^{-\sigma})$, and

$$\partial/\partial_t \omega^a = f^a_b \omega^b.$$

Hence,

$$\partial/\partial_t dS = O(r^{-\sigma}).$$

It is also clear that

$$\partial/\partial_t g(F_n, F_a) = O(r^{-\sigma}).$$

Hence

$$\partial/\partial_t dS^i = O(r^{-\sigma}).$$

Therefore,

$$m'(t) = \int_{S_{r\to\infty}} \left(\frac{\partial}{\partial t} g_{ij,j} - \frac{\partial}{\partial t} g_{jj,i}\right) dS^i + \int_{S_{r\to\infty}} \left(g_{ij,j} - g_{jj,i}\right) \frac{\partial}{\partial t} dS^i$$

and the second term in the above equation is zero. So, under the flow, we have

$$m'(t) = -2 \int_{S_{r\to\infty}} (R_{ij,j} - R_{jj,i}) dS^i.$$

By using the contracted first Bianchi identity

$$2R_{ij,j} = R_i,$$

we have that

$$m'(t) = -2 \int_{S_{r\to\infty}} \left(\frac{1}{2} R_i - R_i\right) dS^i = \int_{S_{r\to\infty}} R_i dS^i.$$

Note that, by using the local gradient estimate of Shi (see Theorem 13.1 in [8]), we have that $R_i = O(|x|^{-\sigma})$ for $\sigma = 2 + \tau$. Hence, we have

$$m'(t) = 0$$

when $n = 3$. The same is true if $\tau > n - 3$ for any dimension $n \geq 3$.

Similarly, if the initial metric satisfies the additional decay condition 1.8, then one has a better estimate $R_i = O(|x|^{-\sigma})$ for $\sigma = 3 + \tau$. Therefore, $m'(t) = 0$ provided $\tau + 3 > n - 1$. This will be the case if $n \leq 6$ and $\tau > \frac{n-2}{2}$, or $\tau > \min\{n - 4, \frac{n-2}{2}\}$.

On the other hand, if $R = O(r^{-q})$, $q > n$, then Theorem 10 applies and we once again have $m'(t) = 0$.

Combining the results in Sect. 3, and 4, we have proved Theorem 1.

Let us now make an observation. Using the divergence theorem we have

$$m'(t) = \frac{1}{4\omega_n} \int_{B_{r\to\infty}} \Delta R dv_{g(t)}.$$

Comparing this with the formula

$$\Delta R = R_t - 2|Rc|^2,$$

and

$$\frac{d}{dt} \int_{B_r} R dv_{g(t)} = \int_{B_r} R_t dv_{g(t)} - \int_{B_r} R^2 dv_{g(t)},$$

we obtain that

$$m'(t) = \frac{1}{4\omega_n} \int_{B_{r\to\infty}} (R_t - 2|Rc|^2) dv_{g(t)} = \lim_{r\to\infty} \frac{1}{4\omega_n} \left[\frac{d}{dt} \int_{B_r} R dv_{g(t)} + \int_{B_r} R^2 dv_{g(t)} - 2 \int_{B_r} |Rc|^2 dv_{g(t)}\right]. \quad (4.27)$$

This yields the following result.

Proposition 12. *Under the Ricci flow for ALE metrics with (4.25) we have*

$$\frac{d}{dt}\int_M R dv_{g(t)} = \int_M \left(2|Rc|^2 - R^2\right) dv_{g(t)},$$

provided that $n = 3$ *or* $\tau > n - 3$.

5. Uniform Convergence

We now turn our attention to Theorem 2. First, we introduce the notion of *uniform convergence* in our context. To this end, we now discuss the weighted Sobolev spaces $W_\tau^{k,q}$ and a certain related space $\mathcal{M}_\tau$ on an ALE space M, see [2, 11].

For $q \geq 1$ and $\tau \in \mathbb{R}$, the weighted Lebesque space $L_\tau^q(M)$ consists of locally integrable functions u on M for which the norm

$$\|u\|_{q,\tau} = \left(\int_M |r^{-\tau}u|^q r^{-n} d\text{vol}\right)^{1/q}$$

is finite. For nonnegative integer k, the weighted Sobolev space $W_\tau^{k,q}(M)$ is the set of u for which $|\nabla^i u| \in L_\tau^q(M)$ for $0 \leq i \leq k$, with the norm

$$\|u\|_{q,k,\tau} = \sum_{i=0}^{k} \|\nabla^i u\|_{q,\tau}.$$

For $\tau > \frac{n-2}{2}$, we define $\mathcal{M}_\tau$ to be the set of all C^∞ metrics g on M such that, in some asymptotic coordinates,

$$g_{ij} - \delta_{ij} \in W_\tau^{1,q}(M), \qquad R(g) \in L^1(M). \tag{5.28}$$

We equip $\mathcal{M}_\tau$ with the norm

$$\|g\|_{\mathcal{M}_\tau} = \|g_{ij} - \delta_{ij}\|_{W_\tau^{1,q}} + \|R(g)\|_{L^1}. \tag{5.29}$$

Now, consider the Ricci flow $(M, g(t))$ which we assume to exist for all time $0 \leq t < \infty$. Furthermore, suppose that each $(M, g(t))$ is ALE of asymptotic order $\tau > \frac{n-2}{2}$ and that the scalar curvature $R(t)$ of $g(t)$ is integrable on M (so that the ADM mass is well defined), i.e. $g(t) \in \mathcal{M}_\tau$.

Definition 13. We say that $g(t)$ converges uniformly to $g_\infty \in \mathcal{M}_\tau$ as $t \to \infty$ if $g(t)$ converges to g_∞ in $\mathcal{M}_\tau$. i.e.,

$$\lim_{t\to\infty} \|g(t) - g_\infty\|_{\mathcal{M}_\tau} = 0.$$

We now prove Theorem 2.

Proof. This follows from the argument of [11, Lemma 9.4]. The key here is the following identity, first observed in [17, 2]. In terms of the asymptotic coordinate,

$$R(g) = \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + O(r^{-2\tau-2}), \tag{5.30}$$

where the $O(r^{-2\tau-2})$ is controlled by the $W_\tau^{1,q}$-norm of g.

Let η be a cut off function which is identically 1 for large r and 0 for $r \leq 1$ and inside. Then, by the Divergence Theorem

$$m(g) = \lim_{R\to\infty} \frac{1}{4\omega_n} \int_{S_R} \eta(\partial_i g_{ij} - \partial_j g_{ii}) * dx_j$$
$$= \int_M \left(-\eta\nabla^*\beta + \langle\beta, \nabla\eta\rangle\right) d\text{vol},$$

where $\beta = (\partial_i g_{ij} - \partial_j g_{ii})\partial_j$ is the mass density vector. Theorem 2 now follows from the formula above and (5.30). □

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