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Eta Invariant and Conformal Cobordism

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Abstract. In this note we study the problem of conformally flat structures bounding conformally flat structures and show that the eta invariants give obstructions. These lead us to the definition of an Abelian group, the conformal cobordism group, which classifies the conformally flat structures according to whether they bound conformally flat structures in a conformally invariant way. The eta invariant gives rise to a homomorphism from this group to the circle group, which can be highly nontrivial. It remains an interesting question of how to compute this group.

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1. Introduction

This note is inspired by a question from my colleague Darren Long (see the very interesting paper [4]). Motivated in part by considerations in physics, Long and Reid considered the question of whether a closed orientable hyperbolic 3-manifold can be the totally geodesic boundary of a compact hyperbolic 4-manifold. The eta invariant is shown to be an obstruction [4]. Similarly, they considered the question of whether every flat 3-manifold is a cusp cross-section of a complete finite volume 1-cusped hyperbolic 4-manifolds and showed that again the eta invariant is an obstruction.

In this note we study the question in the conformal category and show that there are more structures here. More precisely, we first define a notion of conformal cobordism among conformally flat structures according to whether one bounds a conformally flat structure in a conformally invariant way. The suitable boundary restriction here turns out to be that of the total umbilicity, which is the conformally invariant analogue of total geodesity (cf. Lemma 2.2). Again, the eta invariant gives us an obstruction for a conformally flat structure to be null cobordant in the conformal category in that it will have to be integers.

Hyperbolic manifolds are of course conformally flat. Note also that a cusp cross-section of a complete finite volume 1-cusped hyperbolic manifolds is totally umbilic. Thus, our result puts the two results of [4] mentioned above into a uniform framework.

These consideration leads to the definition of an Abelian group, the conformal cobordism group of conformally flat structures, which classifies the conformally flat structures according to whether it bounds a conformally flat structure in a conformally invariant way. The eta invariant then gives rise to a homomorphism from this conformal cobordism group to the circle group. The homomorphism can be highly nontrivial. For example, in dimension three it has dense image. It remains a very interesting question of how to compute this group.

2. Conformally Flat Structures

Let M be a smooth manifold of dimension n. A conformal structure on M is specified by an equivalence class of Riemannian metrics:

$$[g] = \{\bar{g} \mid \bar{g} = \mathrm{e}^{2\phi} g,\ \phi \in C^{\infty}(M)\}.$$

A conformal structure $[g]$ is called conformally flat if, for any $x \in M$, there is a coordinate neighborhood U of x, such that

$$g|_U = \lambda \left(\sum \mathrm{d}x_i^2 \right),$$

where λ is a (positive) function on U.

We would be interested in conformally invariant notions about submanifolds. Consider a submanifold $N \subset M$. Its second fundamental forms with respect to the metrics $\bar{g}$, g will be denoted by $\bar{A}$ and A, respectively.

PROPOSITION 2.1. *For any normal vector field v of N, the difference of the bilinear forms $g(\bar{A}, v)$ and $g(A, v)$ is a scalar multiple of the metric. Hence the multiplicities of the eigenvalues of the second fundamental form of $N \subset M$ are conformal invariants. In particular the total umbilicity of a hypersurface is a conformally invariant notion.*

Recall that a submanifold is called totally umbilic if all eigenvalues of its second fundamental form are equal.

Proof. Let $\bar{g} = \mathrm{e}^{2\phi} g$ and $\bar{\nabla}$, ∇ the Levi-Civita connection of $\bar{g}$, g, respectively. Then their difference is a (2, 1)-tensor given by

$$S(X, Y) = \bar{\nabla}_X Y - \nabla_X Y = (X\phi)Y + (Y\phi)X - g(X, Y)\nabla\phi.$$

Thus for a submanifold $N \subset M$, its second fundamental forms with respect to the metrics $\bar{g}$, g are related by

$$\bar{A}(X, Y) = A(X, Y) - g(X, Y)(\nabla\phi)_n, \tag{2.1}$$

where

$$\nabla\phi = (\nabla\phi)_t + (\nabla\phi)_n$$

is the decomposition into the tangential and normal components. It follows then that

$$g(\bar{A}(X, Y), v) = g(A(X, Y), v) - g(X, Y)g((\nabla\phi)_n, v)$$

differs only by a scalar. □

This leads us to the following notion of conformal cobordism. We will restrict ourselves to compact oriented manifolds for the rest of the note.

DEFINITION. We say that two conformally flat structures $(M_1, [g_1])$, $(M_2, [g_2])$ are conformally cobordant if there is a conformally flat manifold $(W, [h])$ such that

(1) $\partial W = M_1 \cup (-M_2)$ and $[h|_{M_i}] = [g_i]$.
(2) M_i's are totally umbilic submanifolds of W.

Here $-M_2$ denotes M_2 with the opposite orientation. For simplicity we will usually suppress the metric and denote conformal cobordism by $M_1 \overset{c.c.}{\sim} M_2$. Let $\mathcal{C}_n$ be the set of all conformally flat structures in dimension n where the underlying manifolds are closed. The conformal cobordism defines a relation on $\mathcal{C}_n$ which is symmetric and transitive. However, unlike the usual cobordism, it is not clear that the relation is reflexive, for the silly reason that the product of a conformally flat manifold with an interval is in general not conformally flat. This can be readily remedied by extending the relation so that two conformally flat manifolds are related if either they are equal or they are conformally cobordant. This defines an equivalence relation on $\mathcal{C}_n$.

The equivalence classes of conformally cobordant conformally flat n-manifolds forms an Abelian semi-group under the disjoint union. The *conformal cobordism group* $\Omega_n^{\mathcal{C}}$ is defined to be the Grothendieck group of this Abelian semi-group. Sitting inside this Abelian group is the subgroup generated by all the hyperbolic structures of dimension n, $\Omega_n^{\mathcal{C}}(H) \subset \Omega_n^{\mathcal{C}}$.

We also consider the connected sum of conformally flat structures. If M_1, M_2 are conformally flat manifolds of dimension n and $D_j \subset M_j$ are round disks, let $M_{0j} = M_j - (\text{int}\, D_j)$. Since one can invert in a round sphere, it is clear that one can glue M_{01}, M_{02} along their boundaries so that $M_1 \# M_2$ also admits a conformally flat structure.

We end this section with the following lemma which will be used later.

LEMMA 2.2. *A two-sided hypersurface is totally umbilic if and only if it is totally geodesic with respect to a conformally equivalent metric.*

Proof. If a hypersurface is totally geodesic with respect to a conformally equivalent metric, then the eigenvalues of its second fundamental form with respect to this conformally equivalent metric are all zero. Hence it is totally umbilic by the conformal invariance of the multiplicity.

Conversely, if $N \subset (M, g)$ is two-sided and totally umbilic, we construct a conformal deformation of g such that N becomes totally geodesic. Since N is two-sided we can construct a smooth function r which coincides with the signed distance to N within a collar neighborhood of N. We use y to denote (local) coordinates of N. Together with r, (r, y) forms a local coordinates of M in the collar neighborhood of N. By the total umbilicity of N, the eigenvalues of its second fundamental form A are all equal to $\lambda(y)$ for some smooth function $\lambda(y)$ on N. Now set $\phi = \lambda(y)r$ in the collar neighborhood of N and extend to all of M smoothly in any manner. Then N will be totally geodesic with respect to $\bar{g} = e^{2\phi}g$. Indeed, according to (2.1), we just need to compute the normal component of $\nabla\phi$:

$$(\nabla\phi)_n = \lambda(y)\frac{\partial}{\partial r}.$$

Our claim follows. □

Remark. Clearly the deformation can be localized in a neighborhood of the hypersurface.

Totally umbilic hypersurface appears naturally. For example, the cusp sections of a complete hyperbolic manifold are such hypersurfaces. Another example would be the conformal infinity of conformally compact Einstein spaces that appear in the AdS/CFT correspondence.

3. The Eta Invariant

The eta invariant was introduced by Atiyah–Patodi–Singer in their seminal work on index formula for manifolds with boundary [1]. For an odd dimensional oriented Riemannian manifold (M^n, g), the full signature operator is defined to be

$$A = \tau(d + \delta) = d\tau + \tau d, \tag{3.2}$$

where $\tau = (\sqrt{-1})^{\frac{n+1}{2}+p(p-1)}*$ on p-forms and is essentially the Hodge star operator. The full signature operator depends on the metric through the Hodge star operator. One has $\tau^2 = 1$.

The full signature operator A could be further restricted to act on the space of even (resp. odd) forms, with the two restrictions related in terms of conjugation by τ. Thus our A essentially decomposes into two copies of $A|_{\Lambda^{\text{even}}}$. Following [1, I, p. 63] we refer to $A|_{\Lambda^{\text{even}}}$ as the signature operator for odd-dimensional manifolds, still denoted by A. The signature operator A is a self-adjoint elliptic operator.

Assuming now M is compact without boundary. For $s \in \mathbb{C}$, $\mathrm{Re}(s) \gg 0$, put

$$\eta_A(s) = \sum_{\lambda \neq 0} \mathrm{sgn}(\lambda)|\lambda|^{-s},$$

where the summation runs over all nonzero eigenvalues of A, and $\mathrm{sgn}(\lambda)$ denotes the sign of λ. Then $\eta_A(s)$ extends to a meromorphic function on $\mathbb{C}$ which is holomorphic near $s = 0$. The eta invariant of A is defined by $\eta(A) = \eta_A(0)$.

Let W be an even if dimensional compact oriented manifold with boundary $\partial W = M$. If h is a Riemannian metric on W, which is of product type near the boundary, then the celebrated Atiyah–Patodi–Singer index formula gives

$$\mathrm{sign}(W) = \int_W \mathrm{L}\left(\frac{R}{2\pi}\right) - \eta(A), \tag{3.3}$$

where L denotes the L-polynomial and R the curvature tensor of h, and A is the signature operator on M associated with $g = h|_M$.

Although one usually requires the product metric structure near the boundary, the restriction can be relaxed to that of the boundary being totally geodesic. This is because, in the general case where the metric may not be the product near the boundary, the extra term coming in is the integral of a transgression of the L-polynomial which depends on the second fundamental form and can be shown to vanish in the totally geodesic case. More precisely, let h be an arbitrary metric on W and $g = h|_M$. If we denote by x the geodesic distance to M in a collar neighborhood, then by the Gauss lemma, we have $h = \mathrm{d}x^2 + g(x)$ in a collar neighborhood, say $[0, 1] \times M$, and if we denote y to be the local coordinates on M, $g(x) = a_{ij}(x, y)\,\mathrm{d}y_i\,\mathrm{d}y_j$. One easily verifies that

$$a_{ij}(x, y) = a_{ij}(y) - 2xS_{ij} + \mathrm{O}(x^2) \tag{3.4}$$

where $S_{ij} = \langle A(\partial/\partial y_i,\ \partial/\partial y_j), \partial/\partial x\rangle$. Now we extend h to a metric $\bar{h}$ on $\bar{W} = [-1, 0] \times M \cup W$ so that, on $[-1, -\frac{1}{2}] \times M$, $\bar{h} = \mathrm{d}x^2 + g$ (say, by extending the $a_{ij}(x, y)$). Applying the index formula of Atiyah–Patodi–Singer to $\bar{W}$ yields

$$\begin{aligned}\mathrm{sign}(W) = \mathrm{sign}(\bar{W}) &= \int_{\bar{W}} \mathrm{L}\left(\frac{R}{2\pi}\right) - \eta(A)\\ &= \int_W \mathrm{L}\left(\frac{R}{2\pi}\right) + \int_{[-1,0]\times M} \mathrm{L}\left(\frac{R}{2\pi}\right) - \eta(A)\end{aligned}$$

Now, on $[-1, 0] \times M$, $L = dQ$ where Q is the transgression of L. More specifically, if we denote by $h_0 = \mathrm{d}x^2 + g$ the product metric on $[-1, 0] \times M$, ω, ω_0 the connection 1-forms of h, h_0, respectively (reverting back to the Cartan formalism) and Ω, Ω_0 the corresponding curvature 2-forms, then

$$Q = r \int_0^1 \mathrm{L}(\omega - \omega_0, \Omega_t, \ldots, \Omega_t)\,\mathrm{d}t,$$

where r is an operator on the invariant polynomials whose action on the degree k component is given by multiplication by k, and $\Omega_t = \mathrm{d}\omega_t + \omega_t \wedge \omega_t$ is the curvature form of $\omega_t = t\omega + (1 - t)\omega_0$. Thus

$$\int_{[-1,0]\times M} \mathrm{L} = -\int_M Q|_{x=0}.$$

On the other hand, at $x = 0$, one verifies that

$$\theta = \omega - \omega_0 = \begin{pmatrix} 0 & \alpha_1 & \cdots & \alpha_n \\ -\alpha_1 & \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \vdots & & \vdots \\ -\alpha_n & \beta_{n1} & \cdots & \beta_{nn} \end{pmatrix}$$

where $\alpha_i = -S(\partial/\partial y_i, \partial/\partial y_j)\, \mathrm{d}y_j$ and $\beta_{ij} = b_{ij}\, \mathrm{d}x$. Thus, $Q|_M$ depends only on the second fundamental form. Since L is homogeneous, when M is totally geodesic, we have $\int_M Q|_{x=0} = 0$ as claimed.

Alternatively, one may use (3.5) to see that the metric is approximately a product up to the second order when M is totally geodesic and argue that the original argument of Atiyah–Patodi–Singer goes through in this case, see the remark after Theorem 4.2 in [1, I].

We recall the following fact from [1, II, pp. 420–421].

LEMMA 3.1. *The eta invariant $\eta(A)$ is a conformal invariant. That is, if $\bar{g} = \mathrm{e}^{2\phi} g$ is a metric conformal to g, and if we denote by A_g and $A_{\bar{g}}$ the corresponding signature operators, we have*

$$\eta(A_g) = \eta(A_{\bar{g}}). \tag{3.5}$$

Proof. For completeness and also for comparing with Theorem 3.2, we include the following proof from [1, II]. Choose a function $a(t)$ such that $a(t) = 0$ for small t and $a(t) = 1$ near $t = 1$. Consider the metric $h = \mathrm{e}^{2a(t)\phi}(\mathrm{d}t^2 + g)$ on $W = M \times [0,\ 1]$, which is of product type near the boundaries. Therefore, by the Atiyah–Patodi–Singer index formula, we have

$$0 = \mathrm{sign}(W) = \int_W \mathrm{L}\left(\frac{R}{2\pi}\right) + (\eta(A_{\bar{g}} - \eta(A_g)).$$

Now, since the L-polynomial is expressible in terms of the Pontryagin forms which depends only on the Weyl curvature tensor, the differential form $\mathrm{L}(R/2\pi)$ is conformally invariant. But h is conformal to the product metric, yielding $\int_W \mathrm{L}(R/2\pi) = 0$. □

Thus, since the eta invariant is independent of the choice of the metric in the conformal class, we will denote it by $\eta(M, [g])$. Therefore it defines a function on the space $\mathcal{C}_n$ of conformally flat structures which is smooth except for simple (integer) jump discontinuity on the variation of the metric g.

THEOREM 3.2. *If $(M, [g])$ is conformally cobordant to zero, then the eta invariant of its signature operator is an integer: $\eta(M, [g]) \equiv 0 \pmod{\mathbb{Z}}$. More generally, if $M_1 \overset{c.c.}{\sim} M_2$, then their eta invariants differ by an integer:*

$$\eta(M_1) - \eta(M_2) \equiv 0 \pmod{\mathbb{Z}}.$$

Proof. By definition, if $(M, [g])$ is conformally cobordant to zero, then it bounds a conformally flat manifold $(W, [h])$ such that $[h|_M] = [g]$ and M is a totally umbilic hypersurface of W. According to Lemma 2.2., there is a metric in the conformal class $[h]$, say h, such that M is totally geodesic with respect to h. Therefore we can apply the Atiyah–Patodi–Singer index formula and obtain

$$\mathrm{sign}(W) = \int_W \mathrm{L}\left(\frac{R}{2\pi}\right) - \eta(M, [g]) = -\eta(M, [g])$$

again by the conformal invariance of the Pontryagin forms. The second statement is proved similarly. □

Thus, as in [4], the eta invariant is an obstruction for conformally flat structures bounding in the conformal category. We also note that the eta invariant is well behaved in the connect sum.

THEOREM 3.3. *The function defined on* $\mathcal{C}_n$ *by* $\Phi(M, [g]) = \mathrm{e}^{2\pi i\eta(M,[g])}$ *is smooth in the variation of metric and is multiplicative under connected sum operation:*

$$\Phi(M_1 \# M_2) = \Phi(M_1)\Phi(M_2).$$

Further it gives rise to a group homomorphism from the conformal cobordism group to the circle group $\Phi : \Omega_n^{\mathcal{C}} \longrightarrow S^1$.

Proof. First of all, the smoothness follows since the exponentiation kills the integer jump. Now for the connected sum, by the conformal flatness of M_j, through a conformal deformation of metrics, one can make the metrics on M_j to be the Euclidean metric near any particular point. Let $D_j(3r) \subset M_j$ be round disks of radius $3r$ and $M_{0j} = M_j - (\mathrm{int}\, D_j(2r))$. Since the neck region $N_j(3r, r) = D_j(3r) - (\mathrm{int}\, D_j(r))$ is conformally equivalent to a cylinder, by a conformal deformation of metrics, one can make the metrics on M_j to be product on $N_j(3r, r)$. Then $M_1 \# M_2 = M_{01} \cup M_{02}$. Now we use the gluing law of eta invariant [2] to deduce

$$\mathrm{e}^{2\pi i\eta(M_1\#M_2)} = \mathrm{e}^{2\pi i\eta(M_{01})}\mathrm{e}^{2\pi i\eta(M_{02})},$$
$$\mathrm{e}^{2\pi i\eta(M_j)} = \mathrm{e}^{2\pi i\eta(M_{0j})}\mathrm{e}^{2\pi i\eta(D_j(2r))},$$
$$\mathrm{e}^{2\pi i\eta(D_1(2r))}\mathrm{e}^{2\pi i\eta(D_2(2r))} = \mathrm{e}^{2\pi i\eta(\mathrm{S}^n)},$$

where $\eta(\mathrm{S}^n)$ is the eta invariant of the standard sphere which is zero. Combining these equations we obtain the desired multiplicativity.

Finally Φ descends to the conformal cobordism group by Theorem 3.2. That it is a group homomorphism follows trivially since the group operation is given by the disjoint union. □

As pointed in [5] (cf. also [4]) the eta invariant of hyperbolic 3-manifolds takes values in a dense set of the real line. Thus, the homomorphism Φ can be highly nontrivial.

Remark. For conformally flat spin manifolds one can consider the eta invariant of the Dirac operator, which also give an obstruction to bound in the conformal category.

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