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Perelman's W-functional on manifolds with conical singularities

Xianzhe Dai and Changliang Wang

In this paper, we develop the theory of Perelman's W-functional on manifolds with isolated conical singularities. In particular, we show that the infimum of W-functional over a certain weighted Sobolev space on manifolds with isolated conical singularities is finite, and the minimizer exists, if the scalar curvature satisfies certain condition near the singularities. We also obtain an asymptotic order for the minimizer near the singularities.

1. Introduction

Let (M, g) be a smooth compact Riemannian manifold without boundary. We recall some Riemannian functionals introduced by G. Perelman to study Ricci flows[Per02]. The $\mathcal{F}$-functional is defined by

$$\mathcal{F}(g, f) = \int_M (R_g + |\nabla f|^2)e^{-f} d\mathrm{vol}_g, \tag{1.1}$$

where R_g is the scalar curvature of the metric g, and f is a smooth function on M. Let $u = e^{-\frac{f}{2}}$, then the $\mathcal{F}$-functional becomes

$$\mathcal{F}(g, u) = \int_M (4|\nabla u|^2 + R_g u^2) d\mathrm{vol}_g. \tag{1.2}$$

The Perelman's λ-functional is defined by

$$\lambda(g) = \inf \left\{ \mathcal{F}(g, u) \mid \int_M u^2 d\mathrm{vol}_g = 1 \right\}. \tag{1.3}$$

Clearly, from (1.3) and (1.2), $\lambda(g)$ is the smallest eigenvalue of the Schrödinger operator $-4\Delta_g + R_g$. Starting from this point of view, we have extended Perelman's theory for the λ-functional to a class of singular manifolds, namely manifolds with isolated conical singularities in [DW18].

To take into account of scale, Perelman also introduces W-functional and μ-functional on smooth compact manifolds in [Per02]. They play a crucial role in the study of singularities of Ricci flow. The W-functional is given by

$$(1.4)\qquad \begin{aligned} W(g,f,\tau) &= \int_M [\tau(R_g + |\nabla f|^2) + f - n] \frac{1}{(4\pi\tau)^{\frac{n}{2}}} e^{-f} d\mathrm{vol}_g \\ &= \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \tau \mathcal{F}(g,f) + \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M (f-n) e^{-f} d\mathrm{vol}_g, \end{aligned}$$

where f is a smooth function, and $\tau > 0$ is a scale parameter. As in $\mathcal{F}$-functional, let $u = e^{-\frac{f}{2}}$, then the W-functional becomes

$$(1.5)\quad W(g,u,\tau) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M [\tau(R_g u^2 + 4|\nabla u|^2) - 2u^2 \ln u - nu^2] d\mathrm{vol}_g.$$

The μ-functional is defined by
(1.6)

$$\mu(g,\tau) = \inf \left\{ W(g,u,\tau) \mid u \in C^\infty(M),\ u > 0,\ \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M u^2 d\mathrm{vol}_g = 1 \right\},$$

for each $\tau > 0$. It is well-know that for each fixed $\tau > 0$ the existence of finite infimum follows from the Log Sobolev inequality on smooth compact Riemnnian manifolds, while the regularity of the minimizer follows from the elliptic estimates and Sobolev embedding. The nonlinear log term makes it trickier than the eigenvalue problem (see, e.g. §11.3 in [AH10] for details). For noncompact manifolds, the story is different, and the W-functional on noncompact manifolds was studied in [Zha12]. On the other hand, Ricci flow and Perelman's theory on 2-spheres with conical singularities were studied in [PSSW14].

In this paper we study the W-functional and μ-functional on compact Riemannian manifolds with isolated conical singularities. Recall that, by a compact Riemannian manifold with isolated conical singularities we mean a singular manifold (M, g, S) whose singular set S consists of finite many points and its regular part $(M \setminus S, g)$ is a smooth Riemannian manifold. Moreover, near the singularities, the metric is asymptotic to a (finite) metric cone $C_{(0,1]}(N)$ where N is a compact smooth Riemannian manifold with metric h_0 which will be called a cross section (see §1 for the precise definition). Our main result is the following theorem.

Theorem 1.1. *Let (M^n, g, S) $(n \geq 3)$ be a compact Riemannian manifold with isolated conical singularities. If the scalar curvature of the cross section* *at the conical singularity $R_{h_0} > (n-2)$ on N, then for each fixed $\tau > 0$,*

$$\inf\left\{ W(g,u,\tau) \mid u \in H^1(M), u > 0, \left\| \frac{1}{(4\pi\tau)^{\frac{n}{4}}} u \right\|_{L^2(M)} = 1 \right\} > -\infty. \tag{1.7}$$

Here $H^1(M)$ is the weighted Sobolev space defined in (3.3).

Moreover, there exists $u_0 \in C^\infty(M \setminus S)$ that realizes the infimum in (1.7)*. Furthermore, if (M^n, g, S) satisfies the asymptotic condition AC_1 defined in* (2.1)*, then near each singularity, the minimizer satisfies*

$$u_0 = o(r^{-\alpha}), \quad as \quad r \to 0, \tag{1.8}$$

for any $\alpha > \frac{n}{2} - 1$. Here r is the radial variable on each conical neighborhood of the singularities, and $r = 0$ corresponds to the singular points.

Remark 1.2. The same result holds for $n = 2$ without the geometric condition on the cross section, just as in the case of the $\mathcal{F}$-functional in [DW18], Cf. Remark 1.2 there. The proof requires only slight adaptation of the arguments presented here; namely one uses the L^p $(1 < p < 2)$ Sobolev inequality (Proposition 3.3) to control the logarithmic term.

Remark 1.3. In a recent paper [Ozu19], T. Ozuch studied Perelman's functionals on cones and showed that the infimum in (1.7) is finite.

In [DW18], we have shown that the infimum of the $\mathcal{F}$-functional over the weighted Sobolev space $H^1(M^n)$ $(n \geq 3)$ is finite if $R_{h_0} > (n-2)$ on the cross section. For 2-dimensional manifolds with conical singularities, no assumption is needed and the finiteness of the infimum of the $\mathcal{F}$-functional essentially follows from Cheeger [Che79]. In order to control the term involving $\ln u$ in the W-functional, similar as in the smooth compact case, one uses Log Sobolev inequality on compact manifolds with isolated conical singularities, which follows from a L^2 Sobolev inequality on compact manifolds with isolated conical singularities. Then we conclude that the infimum in (1.7) is finite. The L^2 Sobolev inequality on compact manifolds with isolated conical singularities is established in [DY]. Clearly, it suffices to establish the inequality on a metric cone. For this, a Hardy inequality on model cones, which follows from the classical weighted Hardy inequality, will play an important role.

Then we use the direct method in the calculus of variations to show the existence of a minimizer of the W-functional, following similar strategy as in [AH10] for the smooth compact case. However, there are new difficulties in the singular case that we need to overcome. For example, the scalar curvature, which appears in the W-functional, blows up at the singularities. Thus in order to deal with the limit for the term involving $\ln u$ in the W-functional, instead of using the compactness of classical Sobolev embedding, we need to use the compactness of certain weighted Sobolev embedding obtained in Proposition 3.6 below. Then the regularity of the minimizer follows from the classical elliptic equation theory, since this is a local problem.

Finally, we use certain weighted Sobolev embedding and weighted elliptic estimates to obtain the asymptotic behavior (1.8) for the minimizer. These weighted Sobolev embedding and weighted elliptic estimates follow from classical Sobolev embedding, interior elliptic estimates, and an useful scaling technique. The scaling technique can be applied in this problem because of the obvious homogeneity of a model cone along the radial direction. And the scaling technique has been demonstrated to be very useful in studying weighted norms and weighted spaces on non-compact manifolds. For a brief survey about its applications, we refer to §1 in [Bar86].

In a subsequent paper [DW19] we will generalize some of the results here to the case of non-isolated conical singularity.

2. Manifolds with isolated conical singularities

As mentioned in the introduction, roughly speaking, a compact Riemannian manifold with isolated conical singularities is a singular manifold (M, g) whose singular set S consists of finite many points and its regular part $(M \setminus S, g)$ is a smooth Riemannian manifold. Moreover, near the singularities, the metric is asymptotic to a (finite) metric cone $C_{(0,1]}(N)$ where N is a compact smooth Riemannian manifold with metric h_0. More precisely,

Definition 2.1. *We say $(M^n, d, g, x_1, \ldots, x_k)$ is a compact Riemannian manifold with isolated conical singularities at $x_1, \ldots, x_k$, if*

1) *(M, d) is a compact metric space,*
2) *$(M_0, g|_{M_0})$ is an n-dimensional smooth Riemannian manifold, and the Riemannian metric g induces the given metric d on M_0, where $M_0 = M \setminus \{x_1, \ldots, x_k\}$,*
3) *for each singularity x_i, $1 \leq i \leq k$, their exists a neighborhood $U_{x_i} \subset M$ of x_i such that $U_{x_i} \cap \{x_1, \ldots, x_k\} = \{x_i\}$, $(U_{x_i} \setminus \{x_i\}, g|_{U_{x_i} \setminus \{x_i\}})$ is isometric to $((0, \varepsilon_i) \times N_i, dr^2 + r^2 h_r)$ for some $\varepsilon_i > 0$ and a compact smooth manifold N_i, where r is a coordinate on $(0, \varepsilon_i)$ and h_r is*

a smooth family of Riemannian metrics on N_i *satisfying* $h_r = h_0 + o(r^{\alpha_i})$ *as* $r \to 0$*, where* $\alpha_i > 0$ *and* h_0 *is a smooth Riemannian metric on* N_i*.*

Moreover, we say a singularity p *is a cone-like singularity, if the metric* g *on a neighborhood of* p *is isometric to* $dr^2 + r^2 h_0$ *for some fixed metric* h_0 *on the cross section* N*.*

In our case, as usual, one does analysis away from the singular set. And in the above definition, we only require the zeroth order asymptotic condition $h_r = h_0 + o(r^\alpha)$, as $r \to 0$, for the family of metrics h_r on the cross section N with parameter $r > 0$. However, in some problems we need certain higher order asymptotic conditions for h_r as follows. We say that a compact Riemannian manifold (M^n, g, x) with a single conical singularity at x satisfies the condition AC_k, if

$$r^{i-1} |\nabla^i (h_r - h_0)| \le C_i < +\infty, \tag{2.1}$$

for some constant C_i, and each $1 \le i \le k$, near x.

Remark 2.2. For simplicity, in the rest of this paper, we will only work on manifolds with a single conical point as there is no essential difference between the case of a single singular point and that of multiple isolated singularities. All our work and results for manifolds with a single conical point go through for manifolds with isolated conical singularities.

For the simplicity of notations, we will use (M^n, g, x) to denote a compact Riemannian manifold with a single conical singularity at x, because the metric d is determined by the Riemannian metric g.

3. Sobolev and weighted Sobolev embedding

In this section, we recall certain weighted Sobolev spaces on compact Riemannian manifolds with conical singularities. Then we establish the identification of some of them with the usual (unweighted) Sobolev spaces. Moreover, we also review and establish some Sobolev and weighted Sobolev embedding on compact Riemannian manifolds with isolated conical singularities.

Various weighted Sobolev spaces and their properties have been introduced and intensively studied in different settings, e.g. on complete noncompact manifolds with certain asymptotic behavior at infinity (see, e.g.

[Bar86], [Can81], [CBC81], [CLW12], [CSCB78] [Loc81], [LP87], [LM85], [McO79], [NW73], and [Wan18]), or on various interesting bounded domains in $\mathbb{R}^n$ (see, e.g. [KMR97], [Kuf85], [Tri78], and [Tur00]).

We recall the weighted Sobolev norms and spaces and weighted C^k norms and spaces on compact Riemannian manifolds with isolated conical singularities studied in [Beh13]. They are given as in (3.2), (3.4), and (3.5) below. Similar weighted Sobolev spaces on compact Riemannian manifolds with isolated tame conical singularities have been introduced and studied in [BP03]. A general discussion from the Melrose calculus viewpoint is given in [Ma91], which also includes nonisolated conical singularity.

Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x, and U_x be a conical neighborhood of x such that $(U_x \setminus \{x\}, g|_{U_x\setminus\{x\}})$ is isometric to $((0,\epsilon_0)\times N, dr^2 + r^2 h_r)$. Let $\chi \in C^\infty(M\setminus\{x\})$ be a positive weight function satisfying

$$\chi(y) = \begin{cases} 1 & \text{if } \ y \in M\setminus U_x, \\ \frac{1}{r} & \text{if } \ y = (r,\theta) \in U_x \subset M, \quad \text{and} \quad r < \frac{\epsilon_0}{4}, \end{cases} \tag{3.1}$$

and $0 < (\chi(y))^{-1} \leq 1$ for all $y \in M \setminus \{x\}$.

For each $k \in \mathbb{N}$, $p \geq 1$, and $\delta \in \mathbb{R}$, the weighted Sobolev space $W^{k,p}_\delta(M)$ denotes the completion of the space of compactly supported smooth functions on $M\setminus\{x\}$, $C_0^\infty(M\setminus\{x\})$, with respect to the weighted Sobolev norm

$$\|u\|_{W^{k,p}_\delta(M)} = \left(\int_M \left(\sum_{i=0}^k \chi^{p(\delta-i)+n}|\nabla^i u|^p\right) d\mathrm{vol}_g\right)^{\frac{1}{p}}, \tag{3.2}$$

where $\nabla^i u$ denotes the i-times covariant derivative of the function u. For the simplicity of notations, as in [DW18], we set $H^k(M) \equiv W^{k,2}_{k-\frac{n}{2}}(M)$, and

$$\|u\|^2_{H^k(M)} \equiv \int_M \left(\sum_{i=0}^k \chi^{2(k-i)}|\nabla^i u|^2\right) d\mathrm{vol}_g. \tag{3.3}$$

For each $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$, $C^k_{\mathrm{loc}}(M)$ denotes the space of k-th times continuously differentiable functions on $M\setminus\{x\}$. The weighted C^k space $C^k_\delta(M)$ is defined as

$$C^k_\delta(M) = \{u \in C^k_{\mathrm{loc}}(M) \mid \|u\|_{C^k_\delta(M)} < \infty\}, \tag{3.4}$$

where $\|\cdot\|_{C^k_\delta(M)}$ is the weighted C^k norm defined as

$$\|u\|_{C^k_\delta(M)} = \sum_{i=0}^{k} \sup_{y\in M\setminus\{x\}} |\chi^{\delta-i}\nabla^i u(y)|, \tag{3.5}$$

for $u \in C^k_{\mathrm{loc}}$.

As usual, $W^{k,p}(M)$ denotes the completion of $C_0^\infty(M\setminus\{x\})$ with respect to the usual Sobolev norm

$$\|u\|_{W^{k,p}(M)} = \left(\int_M \left(\sum_{i=0}^{k} |\nabla^i u|^p\right) d\mathrm{vol}_g\right)^{\frac{1}{p}}. \tag{3.6}$$

We now recall a weighted Hardy inequality (see, e.g. **330** on p. 245 in [HLP34]), and from which derive a Hardy inequality on metric cones. Later we will see that the Hardy inequality on cone will play an important role for establishing Sobolev embedding on manifolds with isolated conical singularities.

For $p > 1$ and $a \neq 1$, we have

$$\int_0^\infty |f|^p x^{-a} dx \le \left(\frac{p}{|a-1|}\right)^p \int_0^\infty |f'(x)|^p x^{p-a} dx, \tag{3.7}$$

for any $f \in C_0^\infty((0,\infty))$.

This weighted Hardy inequality implies a Hardy inequality on an n-dimensional metric cone $(C(N) = (0,\infty)\times N^{n-1}, g = dr^2 + r^2 h)$ over a smooth compact Riemannian manifold (N^{n-1}, h). Indeed, for $p > 1$ and $k \in \mathbb{N}$ with $pk \neq n$, and any $u \in C_0^\infty(C(N))$,

$$\begin{aligned}
\int_{C(N)} \frac{|u|^p}{r^{pk}} d\mathrm{vol}_g &= \int_N \int_0^\infty \frac{|u|^p(r,\theta)}{r^{pk}} r^{n-1} dr d\mathrm{vol}_h \\
&= \int_N \int_0^\infty |u|^p(r,\theta) r^{n-1-pk} dr d\mathrm{vol}_h \\
&\le \left(\frac{p}{|n-pk|}\right)^p \int_N \int_0^\infty \left|\frac{\partial u}{\partial r}\right|^p (r,\theta) r^{n-1-p(k-1)} dr d\mathrm{vol}_h \\
&\le \left(\frac{p}{|n-pk|}\right)^p \int_{C(N)} \frac{|\nabla u|_g^p}{r^{p(k-1)}} d\mathrm{vol}_g. \qquad (3.8)
\end{aligned}$$

Here, for the first inequality, we used the inequality (3.7) for each $u(r,\theta)$ with fixed θ and $a = pk + 1 - n$. The last inequality follows from $|\nabla u|_g =$

$\left(\left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}|\nabla_N u|_h^2\right)^{\frac{1}{2}}$, where ∇_N is the covariant derivative on N with respect to the metric h.

Then combining with the Kato's inequality, $|\nabla|\nabla^k u|| \leq |\nabla^{k+1}u|$ for any smooth function u and non-negative integer k, this Hardy inequality on metric cones directly implies the following equivalence between the weighted Sobolev norms and the usual Sobolev norms.

Lemma 3.1. *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x. For each $p > 1$ and $k \in \mathbb{N}$ with $p\,i \neq n$ for all $i = 1, 2, \ldots, k$, if (M^n, g, x) satisfies the condition AC_{k-1} near x defined in* (2.1)*, then we have for any $u \in C_0^\infty(M \setminus \{x\})$,*

$$\|u\|_{W^{k,p}(M)} \leq \|u\|_{W^{k,p}_{k-\frac{n}{p}}(M)} \leq C(g, n, p, k)\|u\|_{W^{k,p}(M)}, \tag{3.9}$$

for a constant $C(g, n, p, k)$ depending on g, n, p, and k.

Consequently, we have $W^{k,p}_{k-\frac{n}{p}}(M^n) = W^{k,p}(M^n)$ for each $p > 1$ and $k \in \mathbb{N}$ with $p\,i \neq n$ for all $i = 1, 2, \ldots, k$.

Even though we have obtained that some weighted Sobolev norms are equivalent to the usual Sobolev norms, sometimes it is still more convenient to use weighted Sobolev norms. For example, a certain homogeneity of weighted Sobolev norms on metric cones has been demonstrated to be very useful in §8 in [DW18] and the proof of Proposition 3.4 below. Moreover, we only have equivalence between the usual Sobolev norms and weighted Sobolev norms for special weight indices $\delta = k - \frac{n}{p}$ with k, p, and n satisfying certain conditions. But, in some problems, we have to use weighted Sobolev norms with more general weight indices, e.g. in §4 and §5.

Another application of the Hardy inequality obtained in (3.8) is the following Sobolev inequality on the metric cone $(C(N) = (0, \infty) \times N^{n-1}, g = dr^2 + r^2 h)$.

Lemma 3.2. *For $1 < p < n$, and any $u \in C_0^\infty(C(N))$, we have*

$$\|u\|_{L^q(C(N))} \leq C\|\nabla u\|_{L^p(C(N))}, \tag{3.10}$$

for a constant C only depending on the cross section (N^{n-1}, h) and p, where $q = \frac{np}{n-p}$.

Sketch of the proof of Lemma 3.2: The Sobolev inequality in Lemma 3.2 has been established in [DY] for the case $p = 2$. And no essential difference between $p = 2$ case and general case in Lemma 3.2. For proof we refer to [DY]. The basic idea is to choose a finite sufficiently small open cover for the cross section (N^{n-1}, h) so that each piece can be embedded into Euclidean unit sphere $\mathbb{S}^{n-1}$ and the metric h restricted onto each small piece is equivalent to standard metric on the Euclidean unit sphere. Then on the cone over each small piece the metric h is equivalent to standard Euclidean metric on $\mathbb{R}^n$. Then we choose a partition of unity $\{\rho_i\}_{i=1}^N$ subject to the open cover chose for (N^{n-1}, h). If we let $\pi : C(N) \to N$ be the natural projection. Then an important observation pointed in [DY] is the pointwise estimate:

$$|\nabla(\pi^* \rho_i)|(r, \theta) \leq C_i r^{-1}, \tag{3.11}$$

where C_i is a constant, and ∇ is the covariant derivative with respect to $g = dr^2 + r^2 h$ on the cone. Then combining (3.8) and (3.11), one can easily obtain Sobolev inequality in Lemma 3.2.

By applying the Kato's inequality again, Lemma 3.2 implies the following Sobolev inequalities and Sobolev embedding on compact Riemannian manifolds with isolated conical singularities.

Proposition 3.3. *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x. For each $1 < p < n$, we have*

1) *for any $u \in C_0^\infty(M \setminus \{x\})$*

$$\|u\|_{W^{l,q}(M)} \leq C(M, g, p, k)\|u\|_{W^{k,p}(M)}, \tag{3.12}$$

for any $1 \leq q \leq q_l$, where $C(M, g, p, k)$ is a constant, and $l < k$ and q_l satisfy $\frac{1}{q_l} = \frac{1}{p} - \frac{k-l}{n} > 0$,

2) *hence continuous embedding $W^{k,p}(M) \subset W^{l,q}(M)$, for any $1 \leq q \leq q_l$,*

Thus, the Sobolev embedding on compact manifolds with isolated conical singularities relies on the weighted L^p-Hardy inequality (3.7) for $p > 1$, which is known not to be true in the case of $p = 1$. So in general, we do not have Sobolev embeddings on manifolds with isolated conical singularities in the case of $p = 1$. However, in [Beh13], Behrndt has established *weighted* Sobolev embeddings for all $p \geq 1$ on compact manifolds with isolated conical singularities as follows by using a homogeneity of weighted Sobolev norms on metric cones and a scaling technique, which is used in [Bar86] in the case of asymptotically Euclidean manifolds.

Proposition 3.4 ([Beh13], Theorem 2.5). *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x satisfying the condition AC_{k-1} defined in* (2.1).

1) *For each $1 \leq p < n$, $\delta \in \mathbb{R}$, we have for any $u \in C_0^\infty(M \setminus \{x\})$*

$$\|u\|_{W^{l,q}_\delta(M)} \leq C\|u\|_{W^{k,p}_\delta(M)}, \tag{3.13}$$

for any $1 \leq q \leq q_l$, a constant $C = C(g, n, p, k, l)$ is a constant, $l < k$, and q_l satisfy $\frac{1}{q_l} = \frac{1}{p} - \frac{k-l}{n} > 0$. Therefore, we have continuous embedding $W^{k,p}_\delta(M) \subset W^{l,q}_\delta(M)$, for $l < k$ and $q \leq q_l$.

2) *For any $u \in W^{k,p}_\delta(M)$ with $k > \frac{n}{p} + l$, we have*

$$\|u\|_{C^l_\delta(M)} \leq C\|u\|_{W^{k,p}_\delta(M)}, \tag{3.14}$$

for a constant $C = C(g, n, k)$. Therefore, we have continuous embedding $W^{k,p}_\delta(M) \subset C^l_\delta(M)$. Moreover,

$$|\nabla^l u(r, x)| = o(r^{-l+\delta}) \quad as \quad r \to 0. \tag{3.15}$$

Remark 3.5. The local version of weighted Sobolev inequality in (3.13) has been established in Theorems 2.1 and 2.2 in [CLW12] by a different method.

The weighted Sobolev embeddings in Proposition 3.4 are special cases of embeddings obtain in Theorem 2.5 in [Beh13] with the same weight index δ. Here we also obtain the asymptotic behavior in (3.15) for functions in certain weighted Sobolev spaces similarly as in Theorem 1.2 in [Bar86]. This asymptotic behavior will be used in obtaining an asymptotic behavior for the minimizing function of the W-functional near the singularities on manifolds with isolated conical singularities.

In Theorem 3.3 in [BP03], on a compact Riemannian manifold with isolated tame conical singularities M^n of dimension n, the continuous embedding $W^{k,2}_{k-\frac{n}{2}}(M^n) \subset L^q(M^n)$ for $k \geq 1$ and $2 \leq q \leq \frac{2n}{n-2}$ with $n \geq 5$ has been shown, and these can be considered as special cases of Proposition 3.4, since $\|u\|_{L^q(M)} \leq \|u\|_{W^{0,q}_{k-\frac{n}{2}}(M)}$ for all $k \geq 1$ and $2 \leq q \leq \frac{2n}{n-2}$ with $n \geq 5$.

Finally, we show the following compactness property for a weighted Sobolev embedding obtained in Proposition 3.4. This compactness property will be used in showing the existence of the minimizer of the W-functional.

Proposition 3.6. *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x. The embedding $W^{1,1}_{1-n}(M) \subset L^q(M)$ is compact for any $1 \le q < \frac{n}{n-1}$.*

Proof. The embedding follows from $W^{1,1}_{1-n}(M) \subset W^{0,q}_{1-n} \subset L^q(M)$ for $1 \le q < \frac{n}{n-1}$. The first inclusion is given in Proposition 3.4. The second inclusion follows from $q(1-n)+n > 0$ and the definition of weighted Sobolev norms (3.2). Thus, we only need to show the compactness of the embedding. For that we will use the idea of the proof of Lemma 3.2 described right after the lemma.

Choose $0 < \epsilon < \frac{\epsilon_0}{10}$ sufficiently small so that $C_{3\epsilon}(N) = (0, 3\epsilon) \times N \subset M$ is a conical neighborhood of x, and on $C_{2\epsilon}(N)$,

$$\frac{1}{2}(g_0 = dr^2 + r^2 h_0) \le (g = dr^2 + r^2 h_r) \le 2(g_0 = dr^2 + r^2 h_0).$$

Then choose a smooth function ϕ_1 on $M \setminus \{x\}$ with $\phi_1 \equiv 1$ on $C_\epsilon(N) \subset M \setminus \{x\}$, $\mathrm{supp}(\phi_1) \subset C_{2\epsilon}(N)$, $0 \le \phi_1 \le 1$, and $\phi_1|_{C_{2\epsilon}(N)} = \phi_1(r)$ is a radial function. And set $\phi_2 = 1 - \phi_1$ on $M \setminus \{x\}$.

Let $\{u_m\}_{m=1}^{\infty} \subset C_0^\infty(M \setminus \{x\}) \subset W^{1,1}_{1-n}(M)$ be a sequence with bounded $W^{1,1}_{1-n}(M)$ norm, i.e.

$$\|u_m\|_{W^{1,1}_{1-n}(M)} = \int_M (|\nabla u_m| + \chi |u_m|) d\mathrm{vol}_g \le A, \tag{3.16}$$

for some uniform constant A, where χ is the weight function given in (3.1).

We choose a finite sufficiently small open cover $\{U_i\}_{i=1}^{i_0}$ of N^{n-1}, such that U_i can be embedded into the Euclidean unit sphere $\mathbb{S}^{n-1}$, and

$$\frac{1}{2} g_{\mathbb{S}^{n-1}} \le h_0|_{U_i} \le 2 g_{\mathbb{S}^{n-1}}, \tag{3.17}$$

for all $1 \le i \le i_0$. Consequently, $C_{2\epsilon}(U_i) = (0, 2\epsilon) \times N$ can be embedded into $\mathbb{R}^n$ as $\Phi_i : C_{2\epsilon}(U_i) \to B_1(0) \subset \mathbb{R}^n$, and

$$\frac{1}{4}\Phi_i^*(g_{\mathbb{R}^n}) \le (g = dr^2 + r^2 h_r)|_{C_{2\epsilon}(U_i)} \le 4\Phi_i^*(g_{\mathbb{R}^n}), \tag{3.18}$$

for all $1 \le i \le i_0$, where $B_1(0)$ is the unit ball centered at the origin in $\mathbb{R}^n$.

We also choose a partition of unity $\{\rho_i\}_{i=1}^{i_0}$ subject to the open cover $\{U_i\}_{i=1}^{i_0}$ of N^{n-1}. Then for each $1 \le i \le i_0$, and $m \in \mathbb{N}$, $(\pi^*(\rho_i) \cdot \phi_1 \cdot u_m) \circ$

$\Phi_i^{-1} \in C_0^\infty(\overline{B_1(0)})$, and

$$\begin{aligned}
&\|(\pi^*(\rho_i)\cdot\phi_1\cdot u_m)\circ\Phi_i^{-1}\|_{W^{1,1}(\overline{B_1(0)})} \\
&= \int_{\overline{B_1(0)}} \left(|\nabla((\pi^*(\rho_i)\cdot\phi_1\cdot u_m)\circ\Phi_i^{-1})|_{g_{\mathbb{R}^n}} + |(\pi^*(\rho_i)\cdot\phi_1\cdot u_m)\circ\Phi_i^{-1}|\right) d\mathrm{vol}_{g_{\mathbb{R}^n}} \\
&\leq \int_{\overline{B_1(0)}} [(4|\nabla(\pi^*(\rho_i))|_g|\phi_1\cdot u_m|)\circ\Phi_i^{-1} + (\pi^*(\rho_i)4|\nabla(\phi_1\cdot u_m)|_g)\circ\Phi_i^{-1}] d\mathrm{vol}_{g_{\mathbb{R}^n}} \\
&\quad + \int_{\overline{B_1(0)}} |(\pi^*(\rho_i)\cdot\phi_1\cdot u_m)\circ\Phi_i^{-1}| d\mathrm{vol}_{g_{\mathbb{R}^n}} \\
&\leq 4^{n+1}C\int_{C_{2\epsilon}(N)} \left(\frac{1}{r}|u_m| + |\nabla u_m|_g + |u_m|\right) d\mathrm{vol}_g \\
&\leq 4^{n+1}C\int_M (|\nabla u_m|_g + \chi|u_m|) d\mathrm{vol}_g \\
&= 4^{n+1}C\|u_m\|_{W^{1,1}_{1-n}(M)} \leq 4^{n+1}C\cdot A,
\end{aligned}$$

where C and A are constants independent of m and i.

Then we choose a finite open cover $\{V_j\}_{j=1}^{j_0}$ for the compact manifold $M\setminus C_\epsilon(N)$ with smooth boundary $(N,\epsilon^2 h_\epsilon)$ such that the metric g on M restricted on each V_j is quasi-isometric to the standard n-dimensional unit ball or a subset of the unit ball, say $\Psi_j : V_j \to B_1(0)\subset\mathbb{R}^n$. We also choose a partition of unity $\{\psi_j\}_{j=1}^{j_0}$ subject to the open cover. Then for each $1\leq j\leq j_0$, and $m\in\mathbb{N}$, $(\psi_j\cdot\phi_2\cdot u_m)\circ\Psi_j^{-1}\in C_0^\infty(\overline{B_1(0)})$, and

$$\begin{aligned}
\|(\psi_j\cdot\phi_2\cdot u_m)\circ\Psi_j^{-1}\|_{W^{1,1}(\overline{B_1(0)})} &\leq C'\|u_m\|_{W^{1,1}(M)} \\
&\leq C'\|u_m\|_{W^{1,1}_{1-n}(M)} \leq C'\cdot A,
\end{aligned} \tag{3.19}$$

for constants C' and A independent of m and i.

Then for each fixed $1\leq q<\frac{n}{n-1}$, by the compactness of usual Sobolev embedding on the closed unit ball in $\mathbb{R}^n$, we can choose a subsequence of $\{u_m\}_{i=1}^\infty$, which is still denoted by $\{u_m\}$, such that $\{\pi^*(\rho_1)\cdot\phi_1\cdot u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^q(M)$. And do this for $i=2,\dots,i_0$, and then $j=1,\dots,j_0$, and the subsequences from each step. Finally, we can obtain a subsequence of the original sequence $\{u_m\}$, which is still denoted by $\{u_m\}$, such that all $\{\pi^*(\rho_i)\cdot\phi_1\cdot u_m\}$ for $1\leq i\leq i_0$ and all $\{\psi_j\cdot\phi_2\cdot u_m\}$ for $1\leq j\leq j_0$ are Cauchy sequences in $L^q(M)$. Therefore, $\{u_m\}$ is a Cauchy sequence in $L^q(M)$, since

$$\begin{aligned} \text{(3.20)} \qquad & \|u_m - u_{m'}\|_{L^q(M)} \\ & \leq \sum_{i=1}^{i_0} \|\pi^*(\rho_i)\cdot\phi_1\cdot u_m - \pi^*(\rho_i)\cdot\phi_1\cdot u_{m'}\|_{L^q(M)} \\ & \quad + \sum_{j=1}^{j_0} \|\psi_j\cdot\phi_2\cdot u_m - \psi_j\cdot\phi_2\cdot u_{m'}\|_{L^q(M)}. \end{aligned}$$

This completes the proof, since $C_0^\infty(M \setminus \{x\})$ is dense in $W^{1,1}_{1-n}(M)$. □

4. Finite lower bound of W-functional

In this section, we show that on a manifold with a single conical singularity (M^n, g, x) the W-functional has a finite lower bound over all functions in $H^1(M)$. By the work in [DW18] about the λ-functional on these manifolds, the key here is to obtain a bound for the term $\int_M u^2 \log u d\mathrm{vol}_g$ in the definition of the W-functional.

By using the L^2 Sobolev inequality on compact manifolds with isolated conical singularities obtained in Proposition 3.3 for the particular case of $k = 1, p = 2$, it is well-known that we can derive the following Logarithmic Sobolev inequality (see, e.g. Lemma 5.8 in [CLN06]).

Lemma 4.1. *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x. For any $a > 0$, there exists a constant $C(a, g)$ such that if $u \in W^{1,2}(M)$ with $u > 0$ and $\|u\|_{L^2(M)} = 1$, then*

$$\text{(4.1)} \qquad \int_M u^2 \ln u d\mathrm{vol}_g \leq a \int_M |\nabla u|^2 d\mathrm{vol}_g + C(a, g).$$

Then for any $a > 0$, and $u \in H^1(M) \equiv W^{1,2}_{1-\frac{n}{2}}(M) \subset W^{1,2}(M)$ with $u > 0$ and $\left\| \frac{1}{(4\pi\tau)^{\frac{n}{4}}} u \right\|_{L^2(M)} = 1$, we have

$$
\begin{aligned}
W(g,u,\tau) &= \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_M [\tau(R_g u^2 + 4|\nabla u|^2) - 2u^2\ln u - nu^2] d\mathrm{vol}_g \\
&\geq \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_M \tau(R_g u^2 + 4|\nabla u|^2) d\mathrm{vol}_g - a\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\int_M |\nabla u|^2 d\mathrm{vol}_g \\
&\quad - \frac{n}{2}\ln(4\pi\tau) - C(a,g) - n \\
&= \frac{\tau}{(4\pi\tau)^{\frac{n}{2}}}\int_M \left(R_g u^2 + \left(4 - \frac{a}{\tau}\right)|\nabla u|^2\right) d\mathrm{vol}_g \\
&\quad - \frac{n}{2}\ln(4\pi\tau) - C(a,g) - n. \qquad (4.2)
\end{aligned}
$$

Moreover, for each fixed $\tau > 0$, by Remark 1.3 in [DW18], we can choose a sufficiently small $a > 0$ such that

$$
\inf\left\{\int_M \left(R_g u^2 + \left(4 - \frac{a}{\tau}\right)|\nabla u|^2\right) d\mathrm{vol}_g \mid u \in H^1(M),\ u > 0,\ \frac{\|u\|_{L^2(M)}}{(4\pi\tau)^{\frac{n}{4}}} = 1\right\} > -\infty,
$$

if $R_{h_0} > (n-2)$ on the cross section of at the conical singularity.

Thus, we have

Theorem 4.2. *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x. If the scalar curvature of the cross section at the conical singularity $R_{h_0} > (n-2)$ on N, then for each fixed $\tau > 0$,*

$$
\inf\left\{W(g,u,\tau) \mid u \in H^1(M), u > 0, \left\|\frac{1}{(4\pi\tau)^{\frac{n}{4}}}u\right\|_{L^2(M)} = 1\right\} > -\infty. \tag{4.3}
$$

Moreover, there exists $u_0 \in C^\infty(M \setminus \{x\})$ that realizes the infimum.

Proof. We have seen that the infimum is finite with the condition $R_{h_0} > (n-2)$. Now we show the existence of the minimizer u_0 by using direct methods in the calculus of variations by the following two steps.

Step 1. Let

$$
m = \inf\left\{W(g,u,\tau) \mid u \in H^1(M), u > 0, \left\|\frac{1}{(4\pi\tau)^{\frac{n}{4}}}u\right\|_{L^2(M)} = 1\right\} > -\infty, \tag{4.4}
$$

and $\{u_i\}_{i=1}^{\infty}$ be a minimizing sequence, i.e.

$$u_i > 0, \quad \left\| \frac{1}{(4\pi\tau)^{\frac{n}{4}}} u_i \right\|_{L^2(M)} = 1, \quad \text{for all} \quad i, \tag{4.5}$$

and

$$\lim_{i\to\infty} W(g, u_i, \tau) = m. \tag{4.6}$$

By the work in [DW18], there exist constants $A = A(g)$, $C_1 = C_1(g, A)$, and $C_2 = C_2(g, A)$, such that for any $u \in H^1(M)$

$$C_1\|u\|_{H^1(M)} \leq \int_M ((R_g + A)u^2 + 4|\nabla u|^2) d\mathrm{vol}_g \leq C_2\|u\|_{H^1(M)}. \tag{4.7}$$

Here, the left inequality follows from Theorem 5.1 in [DW18] with the condition $R_{h_0} > (n-2)$, and the right inequality follows from the definition of the weighted Sobolev norm $\|\cdot\|_{H^1(M)}$ and the fact that M and the cross section N are compact.

Then by (4.2) and (4.7), there exists a constant B such that

$$\|u_i\|_{H^1(M)} \leq B, \tag{4.8}$$

for all i. Thus, by Theorem 3.1 in [DW18], there exists a subsequence of the minimizing sequence $\{u_i\}$, which is still denoted by $\{u_i\}$, weakly converges to u_0 in $H^1(M)$, and strongly converges to u_0 in $L^2(M)$ for some $u_0 \in H^1(M)$. Consequently, $u_0 \geq 0$ a.e., and $\left\| \frac{1}{(4\pi\tau)^{\frac{n}{4}}} u_0 \right\|_{L^2(M)} = 1$.

Step 2. Now we will show that $W(g, u_0, \tau) \leq \lim_{i\to\infty} W(g, u_i, \tau) = m$, and then u_0 is a minimizer.

For any $u, v \in H^1(M)$, let

$$(u, v)_A \equiv \int_M ((R_g + A)u \cdot v + 4\langle \nabla u, \nabla v \rangle) d\mathrm{vol}_g. \tag{4.9}$$

Then by (4.7), $(u, v)_A$ is an inner product on $H^1(M)$, and it induces a norm $\|\cdot\|_A$ that is equivalent to $H^1(M)$ norm. Then we have

$$\begin{aligned} \|u_i\|_A^2 &= \|u_0\|_A^2 + 2(u_0, u_i - u_0)_A + \|u_i - u_0\|_A^2 \\ &\geq \|u_0\|_A^2 + 2(u_0, u_i - u_0)_A. \end{aligned} \tag{4.10}$$

Becasue $u_0 \in H^1(M)$ and u_i weakly converges to u_0 in $H^1(M)$, one has

$$\lim_{i\to\infty} (u_0, u_i - u_0)_A = 0. \tag{4.11}$$

Thus,

$$\lim_{i\to\infty} \|u_i\|_A^2 \geq \|u_0\|_A^2. \tag{4.12}$$

Then combining with $\lim\limits_{i\to\infty} \|u_i\|_{L^2(M)} = \|u_0\|_{L^2(M)}$, we obtain

$$\begin{aligned}\lim_{i\to\infty} \int_M &[\tau(R_g u_i^2 + 4|\nabla u_i|^2) - n u_i^2] d\mathrm{vol}_g \\ &\geq \int_M [\tau(R_g u_0^2 + 4|\nabla u_0|^2) - n u_0^2] d\mathrm{vol}_g.\end{aligned} \tag{4.13}$$

So it suffices to show that $\int_M u_i^2 \ln u_i d\mathrm{vol}_g \to \int_M u_0^2 \ln u_0 d\mathrm{vol}_g$, as $i \to \infty$, for a subequence of the minimizing sequence $\{u_i\}$. As in the proof of Proposition 11.10 in [AH10], $\nabla(u^2 \ln u) = (2u \ln u + u)\nabla u$, and for any $\gamma > 0$ there exists constants $a, b > 0$ such that $|u \ln u| \leq a + b u^{1+\gamma}$. Then for sufficiently small $\gamma > 0$, we have

$$\begin{aligned}\int_M |\nabla(u_i^2 \ln u_i)| d\mathrm{vol}_g &\leq \int_M |u_i + 2u_i \ln u_i| \cdot |\nabla u_i| d\mathrm{vol}_g \\ &\leq \left(\int_M |2a + u_i|^2 d\mathrm{vol}_g\right)^{\frac{1}{2}} \left(\int_M |\nabla u_i|^2\right)^{\frac{1}{2}} \\ &\quad + 2b \left(\int_M |u_i|^{2+2\gamma}\right)^{\frac{1}{2}} \left(\int_M |\nabla u_i|^2 d\mathrm{vol}_g\right)^{\frac{1}{2}} \\ &\leq C_3,\end{aligned} \tag{4.14}$$

for a constant C_3 independent of i. Here, we use the Sobolev embedding $H^1(M) \subset W^{1,2}(M) \subset L^q(M)$ for $1 \leq q \leq \frac{2n}{n-2}$. And also

$$\begin{aligned}\int_M \chi |u_i^2 \ln u_i| d\mathrm{vol}_g &\leq \int_M \chi |u_i| \cdot |a + b u_i^{1+\gamma}| d\mathrm{vol}_g \\ &\leq \left(\int_M \chi^2 |u_i|^2 d\mathrm{vol}_g\right)^{\frac{1}{2}} \left(\int_M |a + b u_i^{1+\gamma}|^2 d\mathrm{vol}_g\right)^{\frac{1}{2}} \\ &\leq \|u_i\|_{H^1(M)} \left(a(\mathrm{Vol}_g(M))^{\frac{1}{2}} + b \left(\int_M |u_i|^{2+2\gamma} d\mathrm{vol}_g\right)^{\frac{1}{2}} \right) \\ &\leq C_4,\end{aligned} \tag{4.15}$$

for a constant C_4 independent of i.

Thus,

$$\|u_i^2 \ln u_i\|_{W^{1,1}_{1-n}(M)} \leq C_3 + C_4, \tag{4.16}$$

for all i (including $i = 0$). Then by Proposition 3.6, the sequence $v_i := u_i^2 \ln u_i$ has a subsequence that converges to some v_0 in $L^1(M)$. Moreover, $u_i^2 \ln u_i$ has a subsequence that converges to $u_0^2 \ln u_0$ almost everywhere on M, since u_i converges to u_0 in $L^2(M)$. Thus $v_0 = u_0^2 \ln u_0$ in $L^1(M)$, and so by passing to a subsequence we have $\lim_{i\to\infty} \int_M u_i^2 \ln u_i d\mathrm{vol}_g = \int_M u_0^2 \ln u_0 d\mathrm{vol}_g$.

Now we have obtained a minimizer $u_0 \in H^1(M)$, and u_0 is a weak solution of the elliptic equation

$$-4\Delta u + R_g u - \frac{2}{\tau} u \ln u - \frac{n}{\tau} u - \frac{m}{\tau} u = 0, \tag{4.17}$$

where m is the infimum of the W-functional. The regularity of u_0 and $u_0 > 0$ can be shown locally. Thus the proof is the same as the compact smooth case, for details, see, e.g. p. 179 in [AH10]. □

5. Asymptotic behavior of the minimizer

In this section, we obtain an asymptotic order for the minimizer near the singularity by using a weighted elliptic bootstrapping. For this, we need the following weighted L^p elliptic estimate. In the following, we set

$$L := -\Delta + \frac{1}{4} R. \tag{5.1}$$

Proposition 5.1 (cf. Proposition 2.7 (ii) in [Beh13]). *Let (M^n, g, x) be a compact Riemannian manifold with a single conical singularity at x satisfying the condition AC_1 defined in (2.1). If $u \in W^{0,p}_\delta(M)$, and $Lu \in W^{0,p}_{\delta-2}(M)$, then*

$$\|u\|_{W^{2,p}_\delta(M)} \leq C \left(\|Lu\|_{W^{0,p}_{\delta-2}(M)} + \|u\|_{W^{0,p}_\delta(M)} \right), \tag{5.2}$$

for a constant $C = C(g, n, k, \delta)$.

This weighted elliptic estimate follows from the usual interior elliptic estimates and the homogeneity of the operator L the same as the Laplace operator for an exact cone. Combining this weighted elliptic estimate and the weighted Sobolev inequalities in Proposition 3.4 implies the following asymptotic order estimate for the minimizer of the W-functional near the conical singularities.

Theorem 5.2. *Let u be the minimizer of W-functional obtained in Theorem 4.2. If the manifold satisfying the condition AC_1 defined in (2.1), then we have*

$$u = o(r^{-\alpha}), \quad as \quad r \to 0, \tag{5.3}$$

for any $\alpha > \frac{n}{2} - 1$.

Proof. Since u satisfies the second order elliptic equation

$$Lu = \frac{2}{\tau} u \ln u + \frac{n+m}{\tau} u, \tag{5.4}$$

and $u \in W^{1,2}_{1-\frac{n}{2}}(M)$, where m is the infimum of the W-functional, by the weighted Sobolev embedding in Proposition 3.4, we have $u \in W^{0,p}_{1-\frac{n}{2}}(M)$, for any $1 \le p \le \frac{2n}{n-2}$.

Because for each $\gamma > 0$ there exists a constant $a(\gamma)$ such that $|u \ln u| \le a(\gamma) + |u|^{1+\gamma}$, we have $u \ln u \in W^{0,p}_{(1-\frac{n}{2})(1+\gamma)}(M) \subset W^{0,p}_{(1-\frac{n}{2})(1+\gamma)-2}(M)$ for any $1 \le p \le \frac{2n}{(n-2)} \frac{1}{(1+\gamma)}$ and any $\gamma > 0$. So we have $Lu \in W^{0,p}_{(1-\frac{n}{2})(1+\gamma)-2}(M)$, since $u \in W^{0,p}_{1-\frac{n}{2}}(M) \subset W^{0,p}_{(1-\frac{n}{2})(1+\gamma)-2}(M)$.

Thus, by Proposition 5.1, $u \in W^{2,p}_{(1-\frac{n}{2})(1+\gamma)}(M)$ for any $1 \le p \le \frac{2n}{(n-2)} \frac{1}{(1+\gamma)}$ and any $\gamma > 0$. If $2 < n < 6$, then by (2) in Proposition 3.4 we have obtained that $u = o(r^{-\alpha})$ as $r \to 0$ for any $\alpha > \frac{n}{2} - 1$, since $\gamma > 0$ could be arbitrarily small.

If $n \ge 6$, then using Proposition 3.4 again, we have $u \in W^{0,p}_{(1-\frac{n}{2})(1+\gamma)}(M)$ for any $1 \le p \le \frac{2n}{(n-2)(1+\gamma)-4}$ and any $\gamma > 0$, and $u \ln u \in W^{0,p}_{(1-\frac{n}{2})(1+\gamma)^2}(M)$ for any $1 \le p \le \frac{2n}{[(n-2)(1+\gamma)-4]} \frac{1}{(1+\gamma)}$ and any $\gamma > 0$. Then as before we have $Lu \in W^{0,p}_{(1-\frac{n}{2})(1+\gamma)^2-2}(M)$, and by Proposition 5.1, we have $u \in W^{2,p}_{(1-\frac{n}{2})(1+\gamma)^2}(M)$, for any $1 \le p \le \frac{2n}{[(n-2)(1+\gamma)-4]} \frac{1}{(1+\gamma)}$ and any $\gamma > 0$.

For $n = 6$, we can choose $p \ge 1$, such that $2 > \frac{n}{p} = \frac{6}{p} > 2\gamma(1+\gamma)$. And then by (2) in Proposition 3.4 we have obtained that $u = o(r^{-\alpha})$ as $r \to 0$ for any $\alpha > \frac{n}{2} - 1$, since $\gamma > 0$ could be arbitrarily small.

For $6 < n < 10$, because $\frac{2n}{[(n-2)(1+\gamma)-4]} \frac{1}{(1+\gamma)} < \frac{2n}{(n-6)(1+\gamma)^2}$, we can choose $p \ge 1$ such that $2 > \frac{n}{p} > \frac{(n-6)(1+\gamma)^2}{2}$ for sufficiently small $\gamma > 0$. Thus $u = o(r^{-\alpha})$ as $r \to 0$ for any $\alpha > \frac{n}{2} - 1$, since $\gamma > 0$ could be arbitrarily small.

Then for each fixed $n \geq 10$, by repeating this process finitely many times, we can always obtain that $u = o(r^{-\alpha})$ as $r \to 0$ for any $\alpha > \frac{n}{2} - 1$. $\square$

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Department of Mathematics
University of California, Santa Barbara, CA 93106, USA
E-mail address: `dai@math.ucsb.edu`

Max Planck Institute for Mathematics
Vivatsgasse 7, Bonn 53111, Germany
E-mail address: `cwangmath@outlook.com`

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