

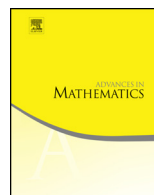


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Local Sobolev constant estimate for integral Ricci curvature bounds[☆]



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ABSTRACT

We obtain a local Sobolev constant estimate for integral Ricci curvature, which enables us to extend several important tools such as the maximal principle, the gradient estimate, the heat kernel estimate and the L^2 Hessian estimate to manifolds with integral Ricci lower bounds, without the non-collapsing conditions.

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1. Introduction

Integral curvature is a very natural notion as it occurs in diverse situations, for example, the Chern–Gauss–Bonnet formula, the isospectral problem, and numerous variational problems. Moreover, integral curvature bounds have recently been discovered in various geometric situations, such as the L^2 bound of the curvature tensor for non-collapsed manifolds with bounded Ricci curvature, and the (almost) L^4 bound of the Ricci curvature for the Kähler–Ricci flow as well as the (real) Ricci flow (under certain conditions) [9,17,27,5,26,28,4]. In [23], the important Laplacian comparison and volume comparison are generalized to integral Ricci lower bound. Combining this with D. Yang’s estimate [30] on the local Sobolev constant, the Cheeger–Colding–Naber theory has now been successfully extended to integral Ricci curvature bound in the noncollapsed case, with important consequences [24,27]. In the collapsed case a local Sobolev constant estimate was missing. Here we provide the missing piece and extend many of the basic estimates for integral curvature in [24,27] to the collapsed case.

For each $x \in M^n$ let $\rho(x)$ denote the smallest eigenvalue for the Ricci tensor $\text{Ric} : T_x M \rightarrow T_x M$, and $\text{Ric}_-^H(x) = ((n - 1)H - \rho(x))_+ = \max\{0, (n - 1)H - \rho(x)\}$, the amount of Ricci curvature lying below $(n - 1)H$. Let

$$\|\text{Ric}_-^H\|_{p,R} = \sup_{x \in M} \left(\int_{B(x,R)} (\text{Ric}_-^H)^p \, d\text{vol} \right)^{\frac{1}{p}}. \tag{1.1}$$

Then $\|\text{Ric}_-^H\|_p$ measures the amount of Ricci curvature lying below a given bound, in this case, $(n - 1)H$, in the L^p sense. Clearly $\|\text{Ric}_-^H\|_{p,R} = 0$ iff $\text{Ric}_M \geq (n - 1)H$. It is often convenient to work with the following **scale invariant** curvature quantity (with $H = 0$):

$$k(x, p, R) = R^2 \left(\int_{B_R(x)} \rho_-^p \right)^{\frac{1}{p}}, \quad k(p, R) = \sup_{x \in M} k(x, p, R). \tag{1.2}$$

The main result of the paper is

Theorem 1.1. *For $p > n/2$, there exists $\varepsilon = \varepsilon(p, n) > 0$ such that if M^n has $k(p, 1) \leq \varepsilon$, then for any $x \in M$, $r \leq 1$ with $\partial B_1(x) \neq \emptyset$, the normalized Dirichlet isoperimetric constant has the estimate*

$$\text{ID}_n^*(B_r(x)) \leq 10^{2n+4}r, \tag{1.3}$$

where

$$\text{ID}_n^*(B_r(x)) = \text{vol}(B_r(x))^{\frac{1}{n}} \cdot \sup_{\Omega} \left\{ \frac{\text{vol}(\Omega)^{1-\frac{1}{n}}}{\text{vol}(\partial\Omega)} \right\}.$$

Here the supremum runs over all subdomains $\Omega \subset B_r(x)$ with smooth boundary and $\partial\Omega \cap \partial B_r(x) = \emptyset$.

See Section 2.2 for a discussion of isoperimetric constants.

Remark 1.2. The smallness of $k(p, 1)$ is necessary. Namely the result is not true if we only assume that $k(p, 1)$ is bounded; see Section 6 for detail. Also the result is not true when $p \leq \frac{n}{2}$ [3].

Remark 1.3. In the presence of the non-collapsing condition $\text{vol } B_r(x) \geq cr^n$, our scale invariant curvature quantity $k(x, p, r) \leq c^{-1/p} r^{2-\frac{n}{p}} \|\text{Ric}_-\|_{p, B_r(x)}$, which is always small when $\|\text{Ric}_-\|_{p, B_1(x)}$ is bounded and r is small. This has been very nicely applied in [27, 28]. Namely when applying to the study of tangent cones, with local volume growth, one only needs to assume that $\|\text{Ric}_-\|_{p, B_1(x)}$ is bounded. Note also that when $k(p, r)$ is small for some r , it gives control on $k(p, r)$ for all r , see Remark 2.2 for detail.

Remark 1.4. In terms of the usual isoperimetric constant, our estimate reads

$$\sup_{\Omega} \left\{ \frac{\text{vol}(\Omega)^{1-\frac{1}{n}}}{\text{vol}(\partial\Omega)} \right\} \leq \frac{10^{2n+4}r}{\text{vol}(B_r(x))^{\frac{1}{n}}}.$$

We emphasize that it is very important that the volume dependence here is $\text{vol}(B_r(x))^{\frac{1}{n}}$. It is of the right scale invariance, and corresponds to the optimal Sobolev constant. We note that a local isoperimetric constant estimate is given independently in a recent paper [25] but with weaker result and under much stronger assumptions.

From (2.9) and (2.10), the theorem above immediately gives

Corollary 1.5. Under the same assumption as in Theorem 1.1, we have the Cheeger’s constant

$$\text{ID}_{\infty}(B_r(x)) \leq 10^{2n+4}r$$

and the Sobolev inequality

$$\left(\int_{B_r(x)} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq 10^{2n+4}r \int_{B_r(x)} |\nabla f|, \tag{1.4}$$

for all $f \in C_0^{\infty}(B_r(x))$ where $r \leq 1$.

See Definition 2.3 for the definition of Cheeger’s constant. By Cheeger’s inequality [8], the first eigenvalue $\lambda_1 \geq \frac{1}{4\text{ID}_\infty^2}$. Thus we also obtain an eigenvalue lower bound. We emphasize that in the above Sobolev inequality we use the averaged integral (volume normalized).

Remark 1.6. Under the pointwise Ricci curvature lower bound, estimates of the type above (namely local or Dirichlet) for Cheeger’s constant and isoperimetric constant are proved in [6,2]. For integral Ricci curvature lower bound, D. Yang [30] obtained a local Sobolev constant estimate under the additional assumption that the manifold is non-collapsed, see Theorem 2.8. Paeng [21] proved a local Cheeger’s constant estimate for integral Ricci curvature under some strong assumption.

Remark 1.7. When M is closed, the global (Neumann) normalized isoperimetric constant (see Section 2.2 for definition) for integral Ricci curvature was already obtained in [13], see Theorem 2.9. The proof for global one does not apply to local one here since it uses a result from geometric measure theory which only works for closed manifolds or domains with convex boundary.

The local Sobolev inequality enables us to obtain many applications. First we can extend the maximal principle and gradient estimate in [24] to the collapsed case. Namely we have the following maximal principle.

Theorem 1.8. *Let M be an n -dimensional Riemannian manifold, and $p > n/2$. There is an $\varepsilon = \varepsilon(n, p) > 0$ and $C = C(n, p, q) > 1$ such that if $k(p, 1) \leq \varepsilon$ and $R \leq 1$ then any function $u : \Omega \subset B(x, R) \rightarrow \mathbb{R}$ with $\Delta u \geq f$ satisfies*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + R^2 \cdot C \cdot \|f_-\|_{q,\Omega}^*,$$

for any $q > \frac{n}{2}$. Here the normalized L^q norm $\|f_-\|_{q,\Omega}^*$ is introduced at the beginning of the next section.

Also we have the gradient estimate.

Theorem 1.9. *Let M be an n -dimensional Riemannian manifold, and $p > n/2$. There is an $\varepsilon(n, p) > 0$ and $C(n, p) > 1$ such that if $k(p, 1) \leq \varepsilon$ and $R \leq 1$ and u is a function on $B_1(x)$ satisfying*

$$\Delta u = f,$$

then

$$\sup_{B_{\frac{R}{2}}(x)} |\nabla u|^2 \leq C(n, p)R^{-2} \left[(\|u\|_{2,B_R(x)}^*)^2 + (\|f\|_{2p,B_R(x)}^*)^2 \right]. \tag{1.5}$$

With the (relative) local Sobolev constant estimate (1.4), one gets heat kernel upper bound, see e.g. [16, (2.17)]. With this and the volume doubling (2.5), Zhang–Zhu obtained Li–Yau’s gradient estimate [31]. Hence one has parabolic Harnack inequality and local Li–Yau heat kernel lower bound, see Theorem 5.5. Consequently we derive the following mean value inequality, extending the one in [27] to the collapsed case.

Theorem 1.10. *For any integer n and $p > \frac{n}{2}$ there exist $\varepsilon = \varepsilon(n, p) > 0$ and $C = C(n, p) > 1$ such that the following holds. Given M a complete n -dimensional Riemannian manifold satisfying $k(p, 1) \leq \varepsilon$, let u be a nonnegative function satisfying*

$$\frac{\partial}{\partial t} u \geq \Delta u - f,$$

where f is a nonnegative space–time function, then, for $q > \frac{n}{2}$,

$$\int_{B_{\frac{1}{2}r}(x)} u(\cdot, 0) d \text{vol} \leq C u(x, r^2) + C(n, p, q) r^2 \sup_{t \in [0, r^2]} \|f(t)\|_{q, B_r(x)}^* \tag{1.6}$$

for all $x \in M$, $r \leq 1$.

With the above tools at our disposal, we can then extend the L^2 Hessian estimate for parabolic approximation of Colding–Naber to integral curvature, see Section 5 for detail. In the noncollapsed case it is established in [27], see also [32].

We expect further applications of our results e.g. to the Cheeger–Colding–Naber theory, which will be discussed in a future paper.

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2. Preliminary

In this section we fix notations and recall the previous work [23], [24] that will play a fundamental role here. We also give a review on the isoperimetric and Sobolev constants and their relations, and introduce the normalized version.

For functions f on M , the L^p norm and normalized L^p norm on a ball $B(x, r) \subset M$ is denoted

$$\|f\|_{p, B(x, r)} = \left(\int_{B(x, r)} |f|^p \right)^{\frac{1}{p}}, \quad \|f\|_{p, B(x, r)}^* = \left(\int_{B(x, r)} |f|^p \right)^{\frac{1}{p}}.$$

(The notation of the volume form of g is often omitted in this paper.) $\|f\|_p, \|f\|_p^*$ denote the norm, normalized norm of f on M .

2.1. Volume comparison for integral curvature

For simplicity, we state the case when $H = 0$. Let M^n be a complete Riemannian manifold of dimension n . Given $x \in M$, let $r(y) = d(y, x)$ be the distance function and $\psi(y) = (\Delta r - \frac{n-1}{r})_+$. The classical Laplacian comparison states that if $\text{Ric}_M \geq 0$, then $\Delta r \leq \frac{n-1}{r}$, i.e., if $\text{Ric}_- \equiv 0$, then $\psi \equiv 0$. In [23] this is generalized to integral Ricci lower bound.

Theorem 2.1 (Laplacian and Volume Comparison [23,24]). *Let M^n be a complete Riemannian manifold of dimension n . If $p > \frac{n}{2}$, then*

$$\|\psi\|_{2p, B(x,r)} \leq \left(\frac{(n-1)(2p-1)}{2p-n} \|\text{Ric}_-\|_{p, B(x,r)} \right)^{\frac{1}{2}}. \tag{2.1}$$

Equivalently

$$\|\psi\|_{2p, B(x,r)}^* \leq \left(\frac{(n-1)(2p-1)}{2p-n} \|\text{Ric}_-\|_{p, B(x,r)}^* \right)^{\frac{1}{2}}. \tag{2.2}$$

Consequently we have the following volume comparison estimate: for any $r_2 \geq r_1 > 0$,

$$\left(\frac{\text{vol}(B_{r_2}(x))}{r_2^n} \right)^{\frac{1}{2p}} - \left(\frac{\text{vol}(B_{r_1}(x))}{r_1^n} \right)^{\frac{1}{2p}} \leq C(n, p) r_2^{1-\frac{n}{2p}} (\|\text{Ric}_-\|_{p, B(x,r_2)})^{\frac{1}{2}}. \tag{2.3}$$

In other words,

$$\left(\frac{\text{vol}(B_{r_1}(x))}{\text{vol}(B_{r_2}(x))} \right)^{\frac{1}{2p}} \geq \left(\frac{r_1}{r_2} \right)^{\frac{n}{2p}} \left[1 - C(n, p) (k(x, p, r_2))^{\frac{1}{2}} \right], \tag{2.4}$$

where $C(n, p)$ is a constant depending on n, p . Hence there exists $\varepsilon_0 = \varepsilon_0(p, n) > 0$ such that, if $k(x, p, r_0) \leq \varepsilon_0$, then

$$\frac{\text{vol}(B_r(x))}{\text{vol}(B_{r_0}(x))} \geq \frac{1}{2} \left(\frac{r}{r_0} \right)^n, \forall r < r_0. \tag{2.5}$$

Remark 2.2. As pointed out in [24, Section 2.3], if $k(x, p, r_2) \leq \varepsilon_0$ for the ε_0 above, then (2.5) implies,

$$k(x, p, r_1) \leq 2^{1/p} \left(\frac{r_1}{r_2} \right)^{2-\frac{n}{p}} \cdot k(x, p, r_2) \leq 2^{1/p} k(x, p, r_2), \quad \forall r_1 \leq r_2. \tag{2.6}$$

Hence $k(x, p, r_1) \rightarrow 0$ as $r_1 \rightarrow 0$ and $k(x, p, r_1) \leq \varepsilon_0(p, n)$ when $r_1 \leq 2^{\frac{1}{n-2p}} r_2$. On the other hand, when $k(p, r_1) \leq \varepsilon_0(p, n)$, then

$$k(p, r_2) \leq 2^{\frac{n+1}{p}} \left(\frac{r_2}{r_1}\right)^2 k(p, r_1) \quad \text{for all } r_2 \geq r_1. \tag{2.7}$$

Hence when $k(p, r)$ is small for some r , it gives control on $k(p, r)$ for all r .

Note also that if one has a lower bound for the Ricci curvature $\text{Ric} \geq (n - 1)H$ then the quantity $k(p, R)$ will be small for sufficiently small R . Namely general lower bound can be reduced to zero lower bound in the local analysis.

2.2. Dirichlet and Neumann isoperimetric and Sobolev constants

In this subsection we review the definitions of the isoperimetric and Sobolev constants and their relations, and introduce the normalized form. For details, see [18,7], though we use a different convention here.

Definition 2.3. For a complete noncompact Riemannian manifold M^n or a compact Riemannian manifold M^n with $\partial M \neq \emptyset$, for $n \leq \alpha \leq \infty$, the Dirichlet (also referred as local) α -isoperimetric constant of M is defined by

$$\text{ID}_\alpha(M) = \sup_{\Omega} \frac{\text{vol}(\Omega)^{1-\frac{1}{\alpha}}}{\text{vol}(\partial\Omega)},$$

where Ω is an open submanifold of M with compact closure and smooth boundary such that $\partial\Omega \cap \partial M = \emptyset$.

When $\alpha = n$, $\text{ID}_n(M)$ is scale invariant. When $\alpha = \infty$, this is Cheeger’s constant [8], which scales like $\text{vol}^{1/n}$. The Dirichlet α -isoperimetric constant controls the local volume growth: for given a geodesic ball $B(x, r) \subset M$, $\text{vol} B(x, r) \geq \left(\frac{r}{\alpha \text{ID}_\alpha(M)}\right)^\alpha$ for $n \leq \alpha < \infty$.

Definition 2.4. The Dirichlet α -Sobolev constant of M is defined by

$$\text{SD}_\alpha(M) = \sup_f \frac{\|f\|_{\frac{\alpha}{\alpha-1}}}{\|\nabla f\|_1},$$

where f ranges over $C_c^\infty(M)$.

Definition 2.5. When M is compact with or without boundary, the Neumann α -isoperimetric constant of M is defined by

$$\text{IN}_\alpha(M) = \sup_{\Gamma} \frac{\min\{\text{vol}(D_1), \text{vol}(D_2)\}^{1-\frac{1}{\alpha}}}{\text{vol}(\Gamma)},$$

where Γ varies over compact $(n - 1)$ -dim submanifold of M which divide M into two disjoint open submanifolds D_1, D_2 of M .

From the definition, if $\Omega \subset M$, $\partial\Omega \cap \partial M = \emptyset$, and $\text{vol}(\Omega) \leq \frac{1}{2} \text{vol}(M)$, then

$$\text{ID}_\alpha(\Omega) \leq \text{IN}_\alpha(M). \tag{2.8}$$

Definition 2.6. The Neumann α -Sobolev constant of M is defined by

$$\text{SN}_\alpha(M) = \sup_f \frac{\inf_{a \in \mathbb{R}} \|f - a\|_{\frac{\alpha}{\alpha-1}}}{\|\nabla f\|_1},$$

where f ranges over $C^\infty(M)$.

Theorem 2.7 ([12, 8], see also [18, 7]). For all $n \leq \alpha \leq \infty$,

$$\text{ID}_\alpha(M) = \text{SD}_\alpha(M), \quad \text{IN}_\alpha(M) \geq \text{SN}_\alpha(M) \geq \frac{1}{2} \text{IN}_\alpha(M).$$

For convenience we consider the normalized Dirichlet and Neumann α -isoperimetric and α -Sobolev constants:

$$\begin{aligned} \text{ID}_\alpha^*(M) &= \text{ID}_\alpha(M) \text{vol}(M)^{1/\alpha}, & \text{SD}_\alpha^*(M) &= \text{SD}_\alpha(M) \text{vol}(M)^{1/\alpha}, \\ \text{IN}_\alpha^*(M) &= \text{IN}_\alpha(M) \text{vol}(M)^{1/\alpha}, & \text{SN}_\alpha^*(M) &= \text{SN}_\alpha(M) \text{vol}(M)^{1/\alpha}. \end{aligned}$$

Observe that

$$\text{ID}_\alpha^*(M) \geq \text{ID}_\infty(M), \quad \text{IN}_\alpha^*(M) \geq \text{IN}_\infty(M), \tag{2.9}$$

and

$$\begin{aligned} \text{SD}_\alpha^*(M) &= \sup_f \frac{\|f\|_{\frac{\alpha}{\alpha-1}}^*}{\|\nabla f\|_1^*}, \text{ where } f \text{ ranges over } C_c^\infty(M), \\ \text{SN}_\alpha^*(M) &= \sup_f \frac{\inf_{a \in \mathbb{R}} \|f - a\|_{\frac{\alpha}{\alpha-1}}^*}{\|\nabla f\|_1^*}, \text{ where } f \text{ ranges over } C^\infty(M). \end{aligned}$$

By [Theorem 2.7](#), we have

$$\text{ID}_\alpha^*(M) = \text{SD}_\alpha^*(M). \tag{2.10}$$

These normalized quantities are very useful in studying the collapsed case, see below. They are used in [\[29\]](#) in proving a Neumann type maximal principle without volume lower bound.

In [\[30, Theorem 7.4\]](#) D. Yang obtained a Dirichlet isoperimetric constant estimate in the non-collapsing case when $\|Ric_-\|_p^*$ is small. Namely

Theorem 2.8. *Given $p > n/2$ and $v > 0$, there is an $\varepsilon(n, p, v) > 0$ such that if $B_1(x) \subset M^n$ has $\text{vol } B_1(x) \geq v$ and $k(p, 1) \leq \varepsilon$, then $\text{ID}_n(B_{\frac{1}{2}}(x)) \leq C(n, p, v)$.*

For closed Riemannian manifold M^n , Gallot [13, Theorem 3] showed that the normalized Neumann α -isoperimetric constant is bounded from above when $\text{diam}(M)^2 \|Ric_-\|_p^*$ is small ($\leq \varepsilon(n, p)$) for $p > n/2$, and $\alpha > n$. Petersen–Sprouse [22] obtained the bound for $\alpha = n$. Namely

Theorem 2.9. *Given $p > n/2$ and $D > 0$, there is an $\varepsilon(n, p, D) > 0$ such that if $\text{diam } M^n \leq D$ and $\|Ric_-\|_p^* \leq \varepsilon$, then $\text{IN}_n^*(M) \leq C(n, p, D)$.*

3. Local isoperimetric constant estimate for closed manifolds

For the local analysis, we need local (Dirichlet) Sobolev constant bound. From (2.8), we automatically get a local estimate when the volume of the domain is small relative to the whole manifold. We show that the measure can only have small concentration whenever $\|Ric_-\|_p^*$ is small.

Proposition 3.1. *Suppose $\text{diam}(M) = D$. There exists $\varepsilon = \varepsilon(n, p) > 0$ such that if*

$$D^2 \|Ric_-\|_p^* \leq \varepsilon, \tag{3.1}$$

then for any $a \leq a_0$ where $a_0 = a_0(n)$ solves

$$\frac{\frac{1}{2} - a_0}{\frac{1}{2} + a_0} = \left(\frac{3}{4}\right)^{\frac{1}{n}} \tag{3.2}$$

we have

$$\text{vol}(B_{aD}(x)) \leq \frac{1}{2} \text{vol}(M), \forall x \in M. \tag{3.3}$$

Proof. For any $x \in M$ we choose a dual point $x' \in M$ with $\text{dist}(x, x') = \frac{D}{2}$. Then, for any radius $r < \frac{D}{2}$,

$$\frac{\text{vol}(B_r(x))}{\text{vol}(M)} \leq \frac{\text{vol}(B_r(x))}{\text{vol}(B_{\frac{D}{2}+r}(x'))} \leq 1 - \frac{\text{vol}(B_{\frac{D}{2}-r}(x'))}{\text{vol}(B_{\frac{D}{2}+r}(x'))}. \tag{3.4}$$

Therefore it suffices to show that for $r = aD$ with $a \leq a_0$, the last term above is greater than or equal to $\frac{1}{2}$. By (2.4) the last term can be estimated as follows

$$\left(\frac{\text{vol}(B_{\frac{D}{2}-r}(x'))}{\text{vol}(B_{\frac{D}{2}+r}(x'))}\right)^{\frac{1}{2p}} \geq \left(\frac{\frac{D}{2} - r}{\frac{D}{2} + r}\right)^{\frac{n}{2p}} \left[1 - C(n, p) \left(k(x', p, \frac{D}{2} + r)\right)^{\frac{1}{2}}\right]. \tag{3.5}$$

If $k(x', p, D) = D^2 \|\text{Ric}_-\|_p^* \leq \varepsilon_0$, by (2.6),

$$k(x', p, \frac{D}{2} + r) \leq 2^{1/p} k(x', p, D) = 2^{1/p} D^2 \|\text{Ric}_-\|_p^*.$$

Hence if we assume that

$$D^2 \|\text{Ric}_-\|_p^* \leq \varepsilon \leq \varepsilon_0, \tag{3.6}$$

then

$$\frac{\text{vol}(B_{\frac{D}{2}-r}(x'))}{\text{vol}(B_{\frac{D}{2}+r}(x'))} \geq \left(\frac{\frac{D}{2}-r}{\frac{D}{2}+r}\right)^n \left[1 - C(n, p) 2^{1/2p} \varepsilon^{\frac{1}{2}}\right]^{2p}. \tag{3.7}$$

Plug in $r = aD$ with $a \leq a_0$, the choice of a_0 implies that

$$\left(\frac{\frac{D}{2}-aD}{\frac{D}{2}+aD}\right)^n \geq \frac{3}{4}.$$

Now set

$$\frac{3}{4} \left[1 - C(n, p) 2^{1/2p} \varepsilon^{\frac{1}{2}}\right]^{2p} \geq \frac{1}{2}.$$

Clearly there exists $\varepsilon(n, p)$ such that this holds for all $\varepsilon \leq \varepsilon(n, p)$. \square

Combining this with Theorem 2.9 and (2.8), we have

Theorem 3.2. *Given $p > n/2$ and $D > 0$, there is an $\varepsilon(n, p, D) > 0, r_0 = r_0(n)$ such that if $\text{diam } M^n \leq D$ and $\|\text{Ric}_-\|_p^* \leq \varepsilon$, then $\text{ID}_n^*(B_r(x)) \leq C(n, p, D)$ for all $x \in M$ and $r \leq r_0$.*

Similarly we have a local version of Proposition 3.1 which will be needed in the next section.

Theorem 3.3. *There exists $\varepsilon = \varepsilon(n, p) > 0$ and $r_0 = r_0(n) > 0$ such that the following holds. Let (M, g) be a complete noncompact Riemannian manifold satisfying $k(p, 1) \leq \varepsilon$, then we have*

$$\frac{\text{vol}(B_{r_0}(x))}{\text{vol}(B_1(x))} \leq \frac{1}{2}, \forall x \in M. \tag{3.8}$$

Proof. For any $x \in M, r < \frac{1}{3}$, choose a point x' with $d = \text{dist}(x, x') = \frac{1-r}{2} \geq \frac{1}{3}$. Then we have

$$B_r(x) \subset B_1(x) \setminus B_{d-r}(x') \subset B_{d+r}(x') \subset B_1(x).$$

As above we calculate

$$\frac{\text{vol}(B_r(x))}{\text{vol}(B_1(x))} \leq \frac{\text{vol}(B_r(x))}{\text{vol}(B_{d+r}(x'))} \leq 1 - \frac{\text{vol}(B_{d-r}(x'))}{\text{vol}(B_{d+r}(x'))}.$$

To estimate the last term recall that

$$\left(\frac{\text{vol}(B_{d-r}(x'))}{\text{vol}(B_{d+r}(x'))}\right)^{\frac{1}{2p}} \geq \left(\frac{d-r}{d+r}\right)^{\frac{n}{2p}} \left[1 - C(n, p)(k(x', p, d+r))^{\frac{1}{2}}\right]. \tag{3.9}$$

Since $d+r \leq 1$, by (2.6), if $k(x', p, 1) \leq \varepsilon_0$, we have $k(x', p, d+r) \leq 2^{1/p}k(x', p, 1)$. Hence when $\varepsilon \leq \varepsilon_0$, we get

$$\left(\frac{\text{vol}(B_{d-r}(x'))}{\text{vol}(B_{d+r}(x'))}\right)^{\frac{1}{2p}} \geq \left(\frac{d-r}{d+r}\right)^{\frac{n}{2p}} \left(1 - C(n, p)2^{1/2p}\varepsilon^{\frac{1}{2}}\right).$$

Now we choose a_0 such that

$$\frac{1 - a_0}{1 + a_0} = \left(\frac{3}{4}\right)^{\frac{1}{n}},$$

then for any $r \leq \frac{1}{3}a_0$, we have, since $d \geq \frac{1}{3}$,

$$\left(\frac{d-r}{d+r}\right)^n \geq \frac{3}{4}.$$

Choose $\varepsilon \leq \varepsilon_0$ such that

$$\left(1 - C(n, p)2^{1/2p}\varepsilon^{\frac{1}{2}}\right)^{2p} \geq \frac{2}{3}. \tag{3.10}$$

Then

$$\frac{\text{vol}(B_r(x))}{\text{vol}(B_1(x))} \leq 1 - \left(\frac{d-r}{d+r}\right)^n \left(1 - C(n, p)2^{1/2p}\varepsilon^{\frac{1}{2}}\right)^{2p} \leq 1 - \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}.$$

The proof is complete by choosing $r_0 = \frac{1}{3}a_0$ and any $0 < \varepsilon \leq \varepsilon_0$ satisfying (3.10). \square

4. Local isoperimetric constant estimate for complete manifolds

In this section we first obtain an estimate on the weak Cheeger’s constant with an error using Laplacian comparison for integral curvature and an idea of Gromov [14, Page 9–10]. This will then enable us to prove Theorem 1.1 by using a covering argument of Anderson [2].

Recall the following lemma of Gromov which is stated for closed manifold in [14], but also works for complete manifolds.

Lemma 4.1 ([14]). *Let M^n be a complete Riemannian manifold and H be any hypersurface dividing M into two parts M_1, M_2 . For any Borel subsets $W_i \subset M_i$, there exists x_1 in one of W_i , say W_1 , and a subset W in the other one, W_2 , such that*

$$\text{vol}(W) \geq \frac{1}{2} \text{vol}(W_2) \tag{4.1}$$

and any $x_2 \in W$ has a unique minimal geodesic connecting to x_1 which intersects H at some z such that

$$\text{dist}(x_1, z) \geq \text{dist}(x_2, z). \tag{4.2}$$

Using Laplacian comparison estimate we have

Lemma 4.2. *Let H, W and x_1 be as in above lemma. Then*

$$\text{vol}(W) \leq 2^{n-1} D \left[\text{vol}(H') + \text{vol}(B_D(x_1)) \| \text{Ric}_- \|_{p, B_D(x_1)}^{*\frac{1}{2}} \right] \tag{4.3}$$

where $D = \sup_{x \in W} \text{dist}(x_1, x)$ and H' is the set of intersection points with H of geodesics $\gamma_{x_1, x}$ for all $x \in W$.

Proof. Let $\Gamma \subset S_{x_1}$ be the set of unit vectors such that $\gamma_v = \gamma_{x_1, x_2}$ for some $x_2 \in W$. We compute the volume in the polar coordinate at x_1 . Write $dv = \mathcal{A}(\theta, t) d\theta \wedge dt$ in the polar coordinate $(\theta, t) \in S_{x_1} \times \mathbb{R}^+$. Recall that [23]

$$\frac{\partial}{\partial t} \frac{\mathcal{A}}{t^{n-1}} \leq \psi \frac{\mathcal{A}}{t^{n-1}}$$

where $\psi = \max(0, \Delta r(\theta, t) - \frac{n-1}{t})$ denotes the error term of Laplacian comparison. We thus have

$$\mathcal{A}(\theta, r) \leq 2^{n-1} \mathcal{A}(\theta, t) + 2^{n-1} \int_t^r \psi(\theta, s) \mathcal{A}(\theta, s) ds, \quad \forall \frac{r}{2} \leq t \leq r. \tag{4.4}$$

For any $\theta \in \Gamma$, let $r(\theta)$ be the radius such that $\exp_{x_1}(r\theta) \in H$. Then, by assumption,

$$W \subset \{ \exp_{x_1}(r\theta) | \theta \in \Gamma, r(\theta) \leq r \leq 2r(\theta) \}.$$

Thus,

$$\text{vol}(W) \leq \int_{\Gamma} \int_{r(\theta)}^{2r(\theta)} \mathcal{A}(\theta, t) dt d\theta$$

$$\begin{aligned} &\leq 2^{n-1} \int_{\Gamma} \int_{r(\theta)}^{2r(\theta)} \left(\mathcal{A}(\theta, r(\theta)) + \int_{r(\theta)}^t \psi(\theta, s) \mathcal{A}(\theta, s) ds \right) d\theta dt \\ &\leq 2^{n-1} D \int_{\Gamma} \mathcal{A}(\theta, r(\theta)) d\theta + 2^{n-1} D \int_{\Gamma} \int_0^D \psi(\theta, t) \mathcal{A}(\theta, t) d\theta dt \end{aligned}$$

On the other hand,

$$\text{vol}(H') = \int_{\Gamma} \frac{\mathcal{A}(\theta, r(\theta))}{\cos \alpha(\theta)} d\theta \geq \int_{\Gamma} \mathcal{A}(\theta, r(\theta)) d\theta$$

where $\alpha(\theta)$ is the angle between H and radial geodesic $\exp_{x_1}(t\theta)$. Thus,

$$\text{vol}(W) \leq 2^{n-1} D \text{vol}(H') + 2^{n-1} D \left(\int_{\Gamma} \int_0^D \psi^{2p} \mathcal{A} d\theta dt \right)^{\frac{1}{2p}} \left(\int_{\Gamma} \int_0^D \mathcal{A} d\theta dt \right)^{1 - \frac{1}{2p}}.$$

Through the Laplacian estimate (2.1) we get

$$\text{vol}(W) \leq 2^{n-1} D \text{vol}(H') + 2^{n-1} D \text{vol}(B_D(x_1))^{1 - \frac{1}{2p}} \left(\int_{B_D(x_1)} |Ric_-|^p dv \right)^{\frac{1}{2p}} \tag{4.5}$$

the required estimate. \square

Now we can obtain an estimate on the weak Cheeger’s constant with an error.

Corollary 4.3. *Let H be any hypersurface dividing M into two parts. For any ball $B = B_r(x)$ we have*

$$\begin{aligned} &\min(\text{vol}(B \cap M_1), \text{vol}(B \cap M_2)) \\ &\leq 2^{n+1} r \left[\text{vol}(H \cap B_{2r}(x)) + \text{vol}(B_{2r}(x)) \|Ric_-\|_{p, B_{2r}(x)}^{*\frac{1}{2}} \right]. \end{aligned} \tag{4.6}$$

Proof. Put $W_i = B \cap M_i$ in above lemma and notice that $D \leq 2r$ and $H' \subset H \cap B_{2r}(x)$. \square

Corollary 4.4. *Given a hypersurface H dividing M^n into two parts, there exists $\varepsilon = \varepsilon(p, n)$ such that if $k(x, p, 1) \leq \varepsilon$, then for a metric ball $B = B_r(x)$, $r \leq \frac{1}{2}$, which is divided equally by H , we have*

$$\text{vol}(B_r(x)) \leq 2^{n+3} r \text{vol}(H \cap B_{2r}(x)). \tag{4.7}$$

Proof. The previous corollary gives

$$\text{vol}(B) \leq 2^{n+2}r \left[\text{vol}(H \cap B_{2r}(x)) + \text{vol}(B_{2r}(x)) \left(\int_{B_{2r}(x)} |Ric_-|^p dv \right)^{\frac{1}{2p}} \right].$$

If $k(x, p, 1) \leq 2^{-1/p}\varepsilon_0$, by (2.6), we have $k(x, p, r) \leq \varepsilon_0$ for all $r \leq 1$. Hence by (2.5),

$$\text{vol}(B_{2r}(x)) \leq 2^{n+1} \text{vol}(B).$$

Again if $k(x, p, 1) \leq 2^{-1/p}2^{-2(2n+3)}$, then $k(x, p, r) \leq 2^{-2(2n+3)}$ for all $r \leq 1$. Hence

$$\text{vol}(B) \leq 2^{n+2}r \left(\text{vol}(H \cap B_{2r}(x)) + 2^n 2^{-(2n+3)} \text{vol}(B)r^{-1} \right),$$

which gives

$$\text{vol}(B) \leq 2^{n+3}r \text{vol}(H \cap B_{2r}(x)).$$

Therefore choosing $\varepsilon = \min\{2^{-1/p}\varepsilon_0, 2^{-1/p}2^{-2(2n+3)}\}$ suffices. \square

This estimate and volume doubling gives an estimate on the local isoperimetric constant via Vitali Covering Lemma.

Proof of Theorem 1.1. First of all we show that the isoperimetric constant estimate (1.3) holds for some small radius $r_0 = r_0(n)$, under the assumption $k(p, 1) \leq \varepsilon_1$ for some small constant $\varepsilon_1 = \varepsilon_1(p, n)$. By Theorem 3.3, we may assume that ε_1 is chosen such that there exists $r_0 = r_0(n)$ with $\frac{\text{vol}(B_{2r_0}(x))}{\text{vol}(B_{\frac{1}{10}}(x))} \leq \frac{1}{2}, \forall x \in M$. Now given any $y \in M$, let Ω be a smooth subdomain of $B_{r_0}(y)$. We may assume that Ω is connected and its boundary $H = \partial\Omega$ divides M into two parts Ω and Ω^c . For any $x \in \Omega$, let r_x be the smallest radius such that

$$\text{vol}(B_{r_x}(x) \cap \Omega) = \text{vol}(B_{r_x}(x) \cap \Omega^c) = \frac{1}{2} \text{vol}(B_{r_x}(x)).$$

Since $\Omega \subset B_{2r_0}(x)$ and $\text{vol}(B_{2r_0}(x)) \leq \frac{1}{2} \text{vol}(B_{\frac{1}{10}}(x))$, we have $r_x \leq \frac{1}{10}$. Take ε_1 as in Corollary 4.4, then by (4.7)

$$\text{vol}(B_{r_x}(x)) \leq 2^{n+3}r_x \text{vol}(H \cap B_{2r_x}(x)). \tag{4.8}$$

The domain Ω has a covering

$$\Omega \subset \bigcup_{x \in \Omega} B_{2r_x}(x).$$

By Vitali Covering Lemma, cf. [20, Section 1.3], we can choose a countable family of disjoint balls $B_i = B_{2r_{x_i}}(x_i)$ such that $\cup_i B_{10r_{x_i}}(x_i) \supset \Omega$. Moreover, we assume ε_1 is chosen such that $k(p, r) \leq \varepsilon_0$ for all $r \leq 1$, then by the volume doubling property (2.5),

$$\frac{\text{vol}(B_{2r_x}(x))}{\text{vol}(B_{10r_x}(x))} \geq \frac{1}{2 \cdot 5^n}.$$

Hence

$$\sum_i \text{vol}(B_i) \geq \frac{1}{2 \cdot 5^n} \sum_i \text{vol}(B_{10r_{x_i}}(x_i)) \geq \frac{1}{2 \cdot 5^n} \text{vol}(\Omega).$$

Applying the volume doubling property (2.5) again gives

$$\sum_i \text{vol}(B_{r_{x_i}}(x_i)) \geq \frac{1}{4 \cdot 10^n} \text{vol}(\Omega). \tag{4.9}$$

Moreover, since the balls B_i are disjoint, combining with (4.8) gives,

$$\text{vol}(\partial\Omega) \geq \sum_i \text{vol}(B_i \cap H) \geq 2^{-n-3} \sum_i r_{x_i}^{-1} \text{vol}(B_{r_{x_i}}(x_i)). \tag{4.10}$$

These two estimates lead to

$$\begin{aligned} \frac{\text{vol}(\Omega)^{\frac{n-1}{n}}}{\text{vol}(\partial\Omega)} &\leq 10^{n-1} 2^{n+5} \frac{\left(\sum_i \text{vol}(B_{r_{x_i}}(x_i))\right)^{\frac{n-1}{n}}}{\sum_i r_{x_i}^{-1} \text{vol}(B_{r_{x_i}}(x_i))} \\ &\leq 10^{2n+4} \frac{\sum_i \text{vol}(B_{r_{x_i}}(x_i))^{\frac{n-1}{n}}}{\sum_i r_{x_i}^{-1} \text{vol}(B_{r_{x_i}}(x_i))} \\ &\leq 10^{2n+4} \sup_i \frac{\text{vol}(B_{r_{x_i}}(x_i))^{\frac{n-1}{n}}}{r_{x_i}^{-1} \text{vol}(B_{r_{x_i}}(x_i))} \\ &= 10^{2n+4} \sup_i \left(\frac{r_{x_i}^n}{\text{vol}(B_{r_{x_i}}(x_i))} \right)^{\frac{1}{n}}. \end{aligned}$$

On the other hand, since $\text{dist}(y, x_i) \leq r_0$, we have

$$B_{r_0}(y) \subset B_{2r_0}(x_i).$$

Now $r_{x_i} \leq \frac{1}{10}$, applying the volume doubling property (2.5) again,

$$\text{vol}(B_{r_{x_i}}(x_i)) \geq \frac{10^n r_{x_i}^n}{2} \text{vol}(B_{\frac{1}{10}}(x_i)) \geq \frac{10^n r_{x_i}^n}{2} \text{vol}(B_{r_0}(y)).$$

Substituting into above calculation we get

$$\frac{\text{vol}(\Omega)^{\frac{n-1}{n}}}{\text{vol}(\partial\Omega)} \leq \frac{10^{2n+4}}{\text{vol}(B_{r_0}(y))^{\frac{1}{n}}},$$

the desired estimate.

We next make a scaling argument to show that the estimate (1.3) remains hold for any radius $r \leq 1$, under the assumption $k(p, 1) \leq \varepsilon_2$ for a smaller constant $\varepsilon_2 = \varepsilon_2(p, n) > 0$. Put $r_1 = \frac{r}{r_0} \leq \frac{1}{r_0}$. After a scaling, it is sufficient to check that

$$k(p, r_1) \leq \varepsilon_1.$$

Choose ε_2 such that $\varepsilon_2 \leq \varepsilon_0$, so (2.5) holds for all $r \leq 1$. Now if $r_1 \leq 1$, by (2.6)

$$k(p, r_1) \leq 2^{1/p}k(p, 1) \leq 2^{1/p}\varepsilon_2.$$

On the other hand, if $1 \leq r_1 \leq \frac{1}{r_0}$, then by (2.7),

$$k(p, r_1) \leq 2^{\frac{n+1}{p}} r_1^2 k(p, 1) \leq 2^{\frac{n+1}{p}} r_0^{-2} \varepsilon_2.$$

Combining the two cases we can choose $\varepsilon_2 = \varepsilon_2(p, n)$ as

$$\varepsilon_2 = \min\{2^{-\frac{1}{p}}\varepsilon_1, 2^{-\frac{n+1}{p}} r_0^2 \varepsilon_1, \varepsilon_0\}.$$

The theorem is now proved by setting $\varepsilon = \varepsilon_2$. \square

Combining Theorem 1.1 with (2.10), we have

Corollary 4.5. *If $k(p, 1) \leq \varepsilon$ for the ε in Theorem 1.1, then,*

$$\|f\|_{\frac{n}{n-1}, B_1(x)}^* \leq 10^{2n+4} \|\nabla f\|_{1, B_1(x)}^*, \quad \forall f \in C_0^\infty(B_1(x)), \tag{4.11}$$

Applying (4.11) to $f^{\frac{2(n-1)}{n-2}}$ and using the Hölder inequality gives

$$\|f\|_{\frac{2n}{n-2}, B_1(x)}^* \leq \frac{2(n-1)}{n-2} 10^{2n+4} \|\nabla f\|_{2, B_1(x)}^*, \quad \forall f \in C_0^\infty(B_1(x)). \tag{4.12}$$

This is essential in the applications.

By a scaling argument, we have

Corollary 4.6. *If $k(p, 1) \leq \varepsilon$ for the ε in Theorem 1.1, then, for any $r \leq 1$,*

$$\|f\|_{\frac{n}{n-1}, B_r(x)}^* \leq C(n)r \|\nabla f\|_{1, B_r(x)}^*, \quad \forall f \in C_0^\infty(B_r(x)), \tag{4.13}$$

and

$$\|f\|_{\frac{2n}{n-2}, B_r(x)}^* \leq C(n)r \|\nabla f\|_{2, B_r(x)}^*, \quad \forall f \in C_0^\infty(B_r(x)). \tag{4.14}$$

Corollary 4.7. *If $k(p, 1) \leq \varepsilon$ for the ε in [Theorem 1.1](#), then, for any $r \leq 1$, the first eigenvalue of Dirichlet Laplace has lower bound*

$$\lambda_1(B_r(x)) \geq C(n)^{-1}r^{-2}. \tag{4.15}$$

Proof. Suppose $\Delta f = -\lambda f$ for some $\lambda > 0$ and f with $\int f^2 dv = 1$ and $f = 0$ on $\partial B_r(x)$. Then

$$1 = \int_{B_r(x)} f^2 dv \leq \left(\int_{B_r(x)} f^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C(n)r^2 \int_{B_r(x)} |\nabla f|^2 = C(n)r^2\lambda.$$

Thus $\lambda \geq C(n)^{-1}r^{-2}$ for any eigenvalue $\lambda > 0$. \square

5. Applications

With this new local Sobolev constant estimate many of the results for integral curvature in [\[24,27\]](#) can be easily extended to the collapsed case. In particular, we have maximum principle, gradient estimate for harmonic function and heat kernel, excess estimate, L^2 estimate for the Hessian of the harmonic and parabolic approximation of the distance function.

Denote $C_s(\Omega)$ the normalized local Sobolev constant of $\Omega \subset M^n$,

$$\|f\|_{\frac{2n}{n-2}, \Omega}^* \leq C_s(\Omega)\|\nabla f\|_{2, \Omega}^*, \quad \forall f \in C_0^\infty(\Omega). \tag{5.1}$$

Note that $C_s(\Omega)$ scales like diameter.

Recall the following maximal principle [\[24, Corollary 3.2\]](#).

Theorem 5.1. *Let M be an n -dimensional Riemannian manifold, and $p > n/2$. For any function $u : \Omega \subset M \rightarrow \mathbb{R}$ with $\Delta u \geq -f$, where f is non-negative on Ω , we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + C(n, p) \cdot C_s^2(\Omega) \cdot \|f\|_{p, \Omega}^*.$$

Combining this with [\(4.14\)](#) gives [Theorem 1.8](#).

Now we derive the following gradient estimate.

Theorem 5.2. *Let M be an n -dimensional Riemannian manifold, and $p > n/2$. If u is a function on $B_R(x)$ satisfying*

$$\Delta u = f,$$

then

$$\begin{aligned} \sup_{B_{\frac{R}{2}}(x)} |\nabla u|^2 &\leq C(n, p) R^{-2} \cdot \frac{\text{vol } B_R(x)}{\text{vol } B_{\frac{3}{4}R}(x)} \cdot \left[(\|u\|_{2, B_R(x)}^*)^2 + (\|f\|_{2p, B_R(x)}^*)^2 \right] \\ &\cdot \left[(R^{-2} C_s^2(B_R(x)))^{\frac{2p}{2p-n}} \left(1 + k(p, R)^{\frac{2p}{2p-n}} \right) + R^{-2} C_s^2 (1 + R^{-2} C_s^2 k(p, R)) \right]^{n/2} \end{aligned}$$

The estimate follows from the standard Nash–Moser iteration, by using the L^p integrability of Ric and f^2 for $p > \frac{n}{2}$. On the other hand, as we do not assume the harmonicity of u (i.e. $f = 0$), and Ricci curvature pointwise lower bound, the proof requires several extra estimates and the Laplacian comparison estimate (2.2). This full general version is often needed in applications. Since a proof is not in the literature, we give a detailed proof here.

Proof. By scaling we may assume $R = 1$. Recall the Bochner formula,

$$\frac{1}{2} \Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla f \rangle + \text{Ric}(\nabla u, \nabla u) \geq \langle \nabla u, \nabla f \rangle - |\text{Ric}_-| |\nabla u|^2. \tag{5.2}$$

Put

$$v = |\nabla u|^2 + \|f^2\|_p^*.$$

Note that when f is constant, one can iterate with $v = |\nabla u|^2$ and the proof simplifies.

For any function $\eta \in C_0^\infty(B_1(x))$ and constant $q > 1$, compute

$$\begin{aligned} \int |\nabla(\eta v^{q/2})|^2 &= - \int \eta v^q \Delta \eta - 2 \int \eta v^{q/2} \langle \nabla \eta, \nabla v^{q/2} \rangle - \int \eta^2 v^{q/2} \Delta v^{q/2} \\ &= \int (2|\nabla \eta|^2 - \eta \Delta \eta) v^q - 2 \int v^{q/2} \langle \nabla \eta, \nabla(\eta v^{q/2}) \rangle \\ &\quad - (1 - \frac{2}{q}) \int |\nabla(\eta v^{q/2}) - v^{q/2} \nabla \eta|^2 - \frac{q}{2} \int \eta^2 v^{q-1} \Delta v. \end{aligned}$$

By regrouping,

$$\begin{aligned} \int |\nabla(\eta v^{q/2})|^2 &= \frac{q}{2(q-1)} \int \left((1 + \frac{2}{q}) |\nabla \eta|^2 - \eta \Delta \eta \right) v^q \\ &\quad - \frac{1}{q-1} \int v^{q/2} \langle \nabla \eta, \nabla(\eta v^{q/2}) \rangle - \frac{q^2}{4(q-1)} \int \eta^2 v^{q-1} \Delta v \\ &\leq \frac{1}{2} \int |\nabla(\eta v^{q/2})|^2 + \frac{1}{2} \frac{q^2 + q - 1}{(q-1)^2} \int |\nabla \eta|^2 v^q \\ &\quad - \frac{q}{2(q-1)} \int \eta v^q \Delta \eta - \frac{q^2}{4(q-1)} \int \eta^2 v^{q-1} \Delta v. \end{aligned}$$

Hence,

$$\int |\nabla(\eta v^{q/2})|^2 \leq \frac{q^2 + q - 1}{(q - 1)^2} \int |\nabla\eta|^2 v^q - \frac{q}{q - 1} \int \eta v^q \Delta\eta - \frac{q^2}{2(q - 1)} \int \eta^2 v^{q-1} \Delta v. \tag{5.3}$$

Now plugging (5.2) into (5.3), we have

$$\begin{aligned} \int |\nabla(\eta v^{q/2})|^2 &\leq \frac{q^2 + q - 1}{(q - 1)^2} \int |\nabla\eta|^2 v^q - \frac{q}{q - 1} \int \eta v^q \Delta\eta \\ &\quad + \frac{q^2}{q - 1} \int \eta^2 v^q |\text{Ric}_-| - \frac{q^2}{q - 1} \int \eta^2 v^{q-1} \langle \nabla u, \nabla f \rangle. \end{aligned}$$

For the last term, we have

$$\begin{aligned} &\int \eta^2 v^{q-1} \langle \nabla u, \nabla f \rangle \\ &= - \int \eta^2 v^{q-1} f^2 - 2 \int \eta f v^{q-1} \langle \nabla u, \nabla \eta \rangle - (q - 1) \int \eta^2 f v^{q-2} \langle \nabla u, \nabla v \rangle \\ &\geq - \int 6\eta^2 v^{q-1} f^2 - \int |\nabla\eta|^2 v^q - \frac{2(q - 1)}{q^2} \cdot \frac{1}{8} \int \eta^2 |\nabla v^{q/2}|^2 \\ &\geq - \frac{2(q - 1)}{q^2} \cdot \frac{1}{4} \int |\nabla(\eta v^{q/2})|^2 - \int 6\eta^2 v^{q-1} f^2 - \left(1 + \frac{q - 1}{2q^2}\right) \int |\nabla\eta|^2 v^q. \end{aligned}$$

To control $\Delta\eta$, we choose a more specific cut-off function. For $0 < r < 1$, let $\varphi \in C_0^\infty(R)$ be a cut-off function such that $0 \leq \varphi \leq 1$, $\varphi(t) \equiv 1$ for $t \in [0, r]$, $\varphi(t) \equiv 0$ for $t \geq 1$, and $\varphi' \leq 0$. Then define

$$\eta(y) = \varphi(d(x, y)), \tag{5.4}$$

where $d(x, y)$ is the distance function from x . Thus $|\nabla\eta| = |\varphi'|$, and

$$\begin{aligned} \Delta\eta &= \varphi'' + \varphi' \Delta d = \varphi'' + \varphi' \left(\Delta d - \frac{n - 1}{d} + \frac{n - 1}{d}\right) \\ &\geq \varphi'' + \varphi' \left(\psi + \frac{n - 1}{d}\right) \geq -|\varphi''| - \frac{|\varphi'|}{r} - |\varphi'| \psi, \end{aligned}$$

where $\psi = (\Delta d - \frac{n-1}{d})_+$.

Therefore we have, for $q \geq \frac{n}{n-2}$,

$$\begin{aligned} &\int |\nabla(\eta v^{q/2})|^2 \\ &\leq C(n)q \int \left[\left(|\varphi''| + \frac{|\varphi'|}{r} + |\varphi'| \psi \right) \eta v^q + |\varphi'|^2 v^q + \eta^2 f^2 v^{q-1} + \eta^2 |\text{Ric}_-| v^q \right]. \end{aligned}$$

Notice that this formula remains valid for $q = 1$. In fact

$$|\nabla(\eta v^{1/2})|^2 = \left| v^{1/2} \nabla \eta + \eta \frac{|\nabla u|}{v^{1/2}} \nabla |\nabla u| \right|^2 \leq 2v |\nabla \eta|^2 + 2\eta^2 |\text{Hess } u|^2,$$

and

$$\begin{aligned} \int \eta^2 |\text{Hess } u|^2 &= - \int \nabla_i u (2\eta \nabla_j \eta \nabla_i \nabla_j u + \eta^2 \nabla_i \Delta u + \eta^2 R_{ij} \nabla_j u) \\ &\leq \frac{1}{2} \int \eta^2 |\text{Hess } u|^2 + 3 \int |\nabla \eta|^2 v + \int \eta^2 f^2 + \int \eta^2 |\text{Ric}_-| v. \end{aligned}$$

Denote $\mu = \frac{n}{n-2}$. Applying the Sobolev inequality (5.1), we obtain for $q \geq \frac{n}{n-2}$ and $q = 1$,

$$\begin{aligned} &\left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{1/\mu} \\ &\leq C_s^2(B_1(x)) C(n) q \int_{B_1(x)} \left[\left(|\varphi''| + \frac{|\varphi'|}{r} + |\varphi'| \psi \right) \eta v^q + |\varphi'|^2 v^q \right. \\ &\quad \left. + f^2 \eta^2 v^{q-1} + |\text{Ric}_-| \eta^2 v^q \right]. \end{aligned} \tag{5.5}$$

The integration involving Ricci curvature can be estimated as follows. For $p > \frac{n}{2}$,

$$\begin{aligned} \int_{B_1(x)} |\text{Ric}_-| \eta^2 v^q &\leq \|\text{Ric}_-\|_p^* \cdot \left(\int_{B_1(x)} (\eta^2 v^q)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\leq \|\text{Ric}_-\|_p^* \left(\int_{B_1(x)} \eta^2 v^q \right)^{\frac{p-1}{p} a} \cdot \left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{(1-a) \frac{p-1}{p}} \\ &\leq \|\text{Ric}_-\|_p^* \left[\epsilon \left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{\frac{1}{\mu}} + \epsilon^{-\frac{(1-a)\mu}{a}} \cdot \left(\int_{B_1(x)} \eta^2 v^q \right) \right], \end{aligned}$$

where $a = a(n, p) = \frac{2p-n}{2(p-1)} > 0$ is determined via

$$a + (1-a)\mu = \frac{p}{p-1}.$$

Here we used Young’s inequality

$$xy \leq \epsilon x^\gamma + \epsilon^{-\frac{\gamma}{\gamma-1}} y^{\gamma^*}, \quad \forall x, y \geq 0, \gamma > 1, (\gamma^*)^{-1} + \gamma^{-1} = 1,$$

where

$$\gamma = \frac{p}{(1-a)(p-1)\mu}, \quad \gamma^* = \frac{p}{(p-1)a}.$$

By setting $\epsilon = (4C(n)qC_s^2\|\text{Ric}_-\|_p^*)^{-1}$, we conclude

$$\begin{aligned} & C(n)qC_s^2 \int_{B_1(x)} \eta^2 |\text{Ric}_-| v^q \\ & \leq \frac{1}{4} \left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{\frac{1}{\mu}} + C(n,p) (qC_s^2 \|\text{Ric}_-\|_p^*)^{\frac{2p}{2p-n}} \cdot \left(\int_{B_1(x)} \eta^2 v^q \right). \end{aligned} \tag{5.6}$$

For the term $\int_{B_1(x)} \eta^2 f^2 v^{q-1}$, since $v \geq \|f^2\|_p^*$, we have

$$\int_{B_1(x)} \eta^2 f^2 v^{q-1} \leq \frac{1}{\|f^2\|_p^*} \int_{B_1(x)} \eta^2 f^2 v^q \leq \left(\int_{B_1(x)} (\eta^2 v^q)^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}}.$$

Now the same argument as above with $\epsilon = (4C(n)C_s^2q)^{-1}$ gives

$$\begin{aligned} & C(n)C_s^2q \int_{B_1(x)} \eta^2 f^2 v^{q-1} \\ & \leq \frac{1}{4} \left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{\frac{1}{\mu}} + C(n,p) (C_s^2q)^{\frac{2p}{2p-n}} \cdot \left(\int_{B_1(x)} \eta^2 v^q \right). \end{aligned} \tag{5.7}$$

For the term with ψ , using the Hölder inequality and the Laplacian comparison estimate (2.2),

$$\begin{aligned} & \int_{B_1(x)} \psi \eta |\varphi'| v^q \leq \|\psi\|_{2p}^* \cdot \|\eta \varphi' v^q\|_{\frac{2p}{2p-1}}^* \\ & \leq C(n,p) (\|\text{Ric}_-\|_p^*)^{1/2} \cdot \|\eta \varphi' v^q\|_{\frac{2p}{2p-1}}^*. \end{aligned}$$

Note that for $b = \frac{p(n-2)}{n(2p-1)} < 1$,

$$\begin{aligned} \|\eta \varphi' v^q\|_{\frac{2p}{2p-1}}^* &= \left[\int_{B_1(x)} (\eta^2 v^q)^{b\mu} (|\varphi'|^2 v^q)^{\frac{p}{2p-1}} \right]^{\frac{2p-1}{2p}} \\ &\leq \left[\left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^b \left(\int_{B_1(x)} (|\varphi'|^2 v^q)^{\frac{np}{np+2p-n}} \right)^{\frac{np+2p-n}{n(2p-1)}} \right]^{\frac{2p-1}{2p}} \end{aligned}$$

$$\begin{aligned} &\leq \left[\left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^b \left(\int_{B_1(x)} |\varphi'|^2 v^q \right)^{\frac{p}{2p-1}} \right]^{\frac{2p-1}{2p}} \\ &\leq \epsilon \left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{1/\mu} + \frac{1}{4\epsilon} \int_{B_1(x)} |\varphi'|^2 v^q. \end{aligned}$$

Here we used the fact that $\frac{np}{np+2p-n} < 1$ since $p > n/2$.

Choose $\epsilon = (4C(n)C_s^2 q C(n, p) (\|\text{Ric}_-\|_p^*)^{1/2})^{-1}$, we have

$$\begin{aligned} &C(n)C_s^2 q \int_{B_1(x)} \psi \eta |\varphi'| v^q \tag{5.8} \\ &\leq \frac{1}{4} \left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{\frac{1}{\mu}} + (qC(n)C_s^2)^2 C^2(n, p) \|\text{Ric}_-\|_p^* \int_{B_1(x)} |\varphi'|^2 v^q. \end{aligned}$$

Plugging the three estimates (5.6), (5.7), (5.8) into the inequality (5.5) gives

$$\begin{aligned} &\left(\int_{B_1(x)} (\eta^2 v^q)^\mu \right)^{1/\mu} \\ &\leq 4C_s^2 C(n)q \left[\int_{B_1(x)} \left(|\varphi''| + \frac{|\varphi'|}{r} \right) \eta v^q + (1 + qC_s^2 C^2(n, p) \|\text{Ric}_-\|_p^*) \int_{B_1(x)} |\varphi'|^2 v^q \right] \\ &\quad + C(n, p) (C_s^2 q)^{\frac{2p}{2p-n}} \left(1 + (\|\text{Ric}_-\|_p^*)^{\frac{2p}{2p-n}} \right) \left(\int_{B_1(x)} \eta^2 v^q \right). \end{aligned}$$

Define $q_k = \mu^k$, $k \geq 0$, and $r_k = (\frac{3}{4} - \sum_{i=0}^k 2^{-i-3})$. Choose cut-off functions $\eta_k = \varphi_k \circ d \in C_0^\infty(B_{r_k}(x))$ such that

$$\eta_k \equiv 1, \quad \text{on } B_{r_{k+1}}(x); \quad |\varphi'_k| \leq 2^{k+5}, \quad |\varphi''_k| \leq 2^{2k+10}.$$

Then substituting η_k into the estimate and running the iteration for any $k \geq 0$ we get

$$\|v\|_{\infty, B_{\frac{1}{2}}(x)}^* \leq C(n, p) A^{n/2} \|v\|_{1, B_{\frac{3}{4}}(x)}^*,$$

where

$$A = C_s^2 (1 + C_s^2 \|\text{Ric}_-\|_p^*) + C_s^{\frac{4p}{2p-n}} \left(1 + (\|\text{Ric}_-\|_p^*)^{\frac{2p}{2p-n}} \right).$$

Finally observe that, by integrating by parts, for $\eta \in C_0^\infty(B_1(x))$ with $\eta \equiv 1$ in $B_{\frac{3}{4}}(x)$, and $|\nabla\eta| \leq 5$, we have

$$\begin{aligned} \int_{B_{\frac{3}{4}}(x)} |v| &\leq \frac{\text{vol } B_1(x)}{\text{vol } B_{\frac{3}{4}}(x)} \int_{B_1(x)} \eta^2 (|\nabla u|^2 + \|f^2\|_p^*) \\ &\leq \frac{\text{vol } B_1(x)}{\text{vol } B_{\frac{3}{4}}(x)} \left[\|f^2\|_p^* + \int_{B_1(x)} \eta^2 (u^2 + f^2) + 8 \int_{B_1(x)} |\nabla\eta|^2 u^2 \right] \\ &\leq 201 \cdot \frac{\text{vol } B_1(x)}{\text{vol } B_{\frac{3}{4}}(x)} \cdot (\|u\|_2^*)^2 + 2(\|f\|_{2p}^*)^2. \end{aligned}$$

This gives the gradient estimate. \square

Combining (2.5) and (4.14) with Theorem 5.2 gives Theorem 1.9.

Later on we will need Harnack inequality for harmonic function. Hence we also give a gradient estimate for $\ln u$ as in Cheng–Yau’s gradient estimate [10]. In the proof we need Li–Schoen’s trick of bounding high power by lower power [19].

Theorem 5.3. *Assume as in above theorem. Let u be a positive harmonic function in $B_R(x)$, then*

$$\sup_{B_{\frac{R}{2}}(x)} |\nabla \ln u|^2 \leq C(n, p, R^{-2}C_s^2, k(p, R)) R^{-2} \frac{\text{vol } B_R(x)}{\text{vol } B_{\frac{1}{2}R}(x)}.$$

Proof. By scaling we may assume $R = 1$. Let $h = \ln u, v = |\nabla h|^2$. Then $\Delta h = -v$. From the Bochner formula,

$$\begin{aligned} \frac{1}{2} \Delta |\nabla h|^2 &= |\text{Hess } h|^2 + \langle \nabla h, \nabla \Delta h \rangle + \text{Ric}(\nabla h, \nabla h) \\ &\geq \frac{v^2}{n} - \langle \nabla h, \nabla v \rangle - |\text{Ric}_-|v. \end{aligned}$$

For any $\eta \in C_0^\infty(B_1(x))$, $l \geq 0$, multiply above by $v^l \eta^2$ and integrate on $B_1(x)$ gives,

$$\frac{1}{2} \int v^l \eta^2 \Delta v \geq \int \frac{\eta^2 v^{l+2}}{n} - \int v^l \eta^2 \langle \nabla h, \nabla v \rangle - \int v^{l+1} \eta^2 |\text{Ric}_-|. \tag{5.9}$$

We compute

$$\int v^l \eta^2 \langle \nabla h, \nabla v \rangle = - \int v^{l+1} \eta^2 \Delta h - l \int v^l \eta^2 \langle \nabla h, \nabla v \rangle - 2 \int v^{l+1} \eta \langle \nabla h, \nabla \eta \rangle.$$

Hence

$$\begin{aligned} \int v^l \eta^2 \langle \nabla h, \nabla v \rangle &= \frac{1}{l+1} \int v^{l+2} \eta^2 - \frac{2}{l+1} \int v^{l+1} \eta \langle \nabla h, \nabla \eta \rangle \\ &\leq \frac{2}{l+1} \int v^{l+2} \eta^2 + \frac{1}{l+1} \int v^{l+1} |\nabla \eta|^2. \end{aligned} \tag{5.10}$$

By (5.3),

$$\begin{aligned} &\frac{(l+1)^2}{2l} \int v^l \eta^2 \Delta v \\ &\leq - \int \left| \nabla \left(\eta v^{\frac{l+1}{2}} \right) \right|^2 + \frac{(l+1)^2 + l}{l^2} \int v^{l+1} |\nabla \eta|^2 - \frac{l+1}{l} \int \eta v^{l+1} \Delta \eta \end{aligned} \tag{5.11}$$

Plugging (5.11) and (5.10) into (5.9), we have

$$\begin{aligned} - \int \left| \nabla \left(\eta v^{\frac{l+1}{2}} \right) \right|^2 &\geq \frac{(l+1)^2}{l} \left(\frac{1}{n} - \frac{2}{l+1} \right) \int v^{l+2} \eta^2 + \frac{l+1}{l} \int \eta v^{l+1} \Delta \eta \\ &\quad - \frac{2l^2 + 4l + 1}{l^2} \int v^{l+1} |\nabla \eta|^2 - \frac{(l+1)^2}{l} \int v^{l+1} \eta^2 |\text{Ric}_-|. \end{aligned}$$

When $l \geq 2n - 1$, choose η as in (5.4), we have

$$\int \left| \nabla \left(\eta v^{\frac{l+1}{2}} \right) \right|^2 \leq C(n)l \int \left[\left(|\varphi''| + \frac{|\varphi'|}{r} + |\varphi'|\psi \right) \eta v^{l+1} + v^{l+1} |\varphi'|^2 + v^{l+1} \eta^2 |\text{Ric}_-| \right]$$

Use Sobolev inequality (5.1) and estimate as in (5.6), (5.8) and iterate from $l = 2n - 1$ as in Theorem 5.2, we have

$$\|v\|_{\infty, B_{\frac{1}{2}}(x)}^* \leq C(n, p) A^{n/2} \|v\|_{2n-1, B_{\frac{3}{4}}(x)}^*, \tag{5.12}$$

where

$$A = C_s^2 (1 + C_s^2 \|\text{Ric}_-\|_p^*) + C_s^{\frac{4p}{2p-n}} \left(1 + (\|\text{Ric}_-\|_p^*)^{\frac{2p}{2p-n}} \right).$$

Since we have volume doubling, by the proof of Theorem 2.1 in [19], we can lower the power $2n - 1$ in (5.12) by adjusting the size of the balls. Namely we have

$$\|v\|_{\infty, B_{\frac{1}{4}}(x)}^* \leq C(n, p, C_s^2, \|\text{Ric}_-\|_p^*) \|v\|_{1, B_{\frac{5}{4}}(x)}^*.$$

For the L^1 bound, since $v = -\Delta h$,

$$\int_{B_1(x)} \eta^2 v = - \int_{B_1(x)} \eta^2 \Delta h = 2 \int_{B_1(x)} \eta \langle \nabla \eta, \nabla h \rangle \leq \frac{1}{2} \int_{B_1(x)} \eta^2 v + 2 \int_{B_1(x)} |\nabla \eta|^2,$$

where $\eta \in C_0^\infty(B_1(x))$ is a cut-off function with $\eta = 1$ on $B_{\frac{4}{5}}(x)$ and $|\nabla \eta| \leq 6$.

Hence

$$\|v\|_{1, B_{\frac{4}{5}}(x)}^* \leq 144 \frac{\text{vol } B_1(x)}{\text{vol } B_{\frac{4}{5}}(x)}. \quad \square$$

With [Theorem 1.9](#) one can prove as in [\[24, Theorem 6.4\]](#) the following.

Lemma 5.4. *For any integer n and $p > \frac{n}{2}$ there exist ε and C such that the following holds. Let M be a complete n -dimensional Riemannian manifold satisfying $k(p, 1) \leq \varepsilon$. For any metric ball $B_r(x)$ with $\partial B_r(x) \neq \emptyset$, $r \leq 1$, there exists $\phi \in C_0^\infty(B_r(x))$ satisfying*

$$0 \leq \phi \leq 1, \quad |\nabla \phi|^2 + |\Delta \phi| \leq Cr^{-2}.$$

With the (relative) local Sobolev constant estimate [\(1.4\)](#), one gets heat kernel upper bound, see e.g. [\[16, \(2.17\)\]](#). With this and the volume doubling [\(2.5\)](#), Zhang–Zhu obtained Li–Yau’s gradient estimate [\[31\]](#). Hence one has parabolic Harnack inequality. With this we have the local heat kernel lower bounds as in [\[11, Lemma 2.3\]](#). Namely, we have

Theorem 5.5. *Let M be an n -dimensional Riemannian manifold, and $p > n/2$. There is an $\varepsilon(n, p) > 0$ and $C(n, p) > 1$ such that if $k(p, 1) \leq \varepsilon$, then for any real number s , $0 < r < 1$, $x \in M$ and nonnegative solution u of the heat equation in $Q = (s - r^2, s) \times B_r(x)$,*

$$\sup_{Q_-} u \leq C \inf_{Q_+} u,$$

where $Q_- = (s - \frac{3}{4}r^2, s - \frac{1}{2}r^2) \times B_{\frac{1}{2}r}(x)$, $Q_+ = (s - \frac{1}{4}r^2, s) \times B_{\frac{1}{2}r}(x)$.

The heat kernel $H(x, y, t)$ satisfies the two-sided Gaussian bound

$$\frac{c_1}{\text{vol } B_{\sqrt{t}}(x)} e^{-\frac{d^2(x,y)}{c_2 t}} \leq H(x, y, t) \leq \frac{C_1}{\text{vol } B_{\sqrt{t}}(x)} e^{-\frac{d^2(x,y)}{C_2 t}}$$

for all $t \in (0, 1)$ and $x, y \in M$.

For our purpose, we need two-sided bound on the Dirichlet heat kernels of the balls. Let $H^B(x, y, t)$ be the Dirichlet heat kernel of the ball $B_\rho(x)$ with $t \in (0, 1)$ and $\rho \geq \sqrt{t}$. Note that $H^B(x, y, t) \leq H(x, y, t)$. With the local volume doubling and Poincare inequality, by [\[16, \(3.4\)\]](#), there exist constants (depending only on the constants from the volume doubling and Poincare inequality) a , τ (small), A (large) and c such that

$$H^B(x, y, t) \leq \frac{c}{\text{vol } B_{\sqrt{t}}(x)}$$

for $y \in B_{a\sqrt{t}}(x)$, $t \in (0, \tau)$, $\rho \geq A\sqrt{t}$. By making ε smaller (and a rescaling argument as at the end of the proof of [Theorem 1.1](#)) we obtain

Theorem 5.6 (Dirichlet Heat kernel upper and lower bounds). For any integer n and $p > \frac{n}{2}$ there exist ε and C such that the following holds. Let M be a complete n -dimensional Riemannian manifold satisfying $k(p, 1) \leq \varepsilon$. Let $H^{B_r(x)}(x, y, t)$ be the Dirichlet heat kernel of the ball $B_r(x)$. Then

$$H^{B_r(x)}(x, y, t) \leq \frac{C}{\text{vol } B_{\sqrt{t}}(x)} e^{-\frac{d^2(x,y)}{5t}}, \quad \forall x, y \in M \text{ with } 0 < t \leq 1 \tag{5.13}$$

and

$$H^{B_r(x)}(x, y, t) \geq \frac{C^{-1}}{\text{vol } B_{\sqrt{t}}(x)}, \quad 0 < t \leq \frac{1}{2}r^2, \quad y \in B_{10\sqrt{t}}(x). \tag{5.14}$$

This Dirichlet heat kernel upper and lower bounds give the quantitative mean value inequality.

Proposition 5.7. Under the assumption above, let u be a nonnegative function satisfying

$$\frac{\partial}{\partial t} u \geq \Delta u - f,$$

where f is a nonnegative space–time function. Then, for $q > \frac{n}{2}$,

$$\int_{B_{\frac{1}{2}r}(x)} u(\cdot, 0) d \text{vol} \leq C u(x, r^2) + C(n, p, q) r^2 \sup_{t \in [0, r^2]} \|f(t)\|_{q, B_r(x)}^* \tag{5.15}$$

for all $x \in M, r \leq 1$.

Remark 5.8. For our application it’s crucial that the norm of f is a normalized local norm instead of the global norm in [27]. It recovers Lemma 2.1 in [11], where it is proven when f is constant. The key here is to use Dirichlet heat kernel of balls.

Proof. Compute

$$\begin{aligned} & \frac{d}{dt} \int_{B_r(x)} u(y, t) H^{B_r(x)}(x, y, r^2 - t) d \text{vol}(y) \tag{5.16} \\ &= \int_{B_r(x)} \left[H^{B_r(x)}(x, y, r^2 - t) \left(\frac{\partial}{\partial t} - \Delta \right) u(y, t) \right] d \text{vol}(y) \\ &\geq - \int_{B_r(x)} \left[H^{B_r(x)}(x, y, r^2 - t) f(y, t) \right] d \text{vol}(y) \end{aligned}$$

By the upper bound of $H^{B_r(x)}(x, y, r^2 - t)$, we have, for $q > 1$,

$$\begin{aligned}
 & \int_{B_r(x)} \left[H^{B_r(x)}(x, y, r^2 - t) f(y, t) \right] d \operatorname{vol}(y) \\
 & \leq \|f(t)\|_{q, B_r(x)} \|H^{B_r(x)}(x, y, r^2 - t)\|_{\frac{q}{q-1}, B_r(x)} \\
 & \leq \frac{C}{\operatorname{vol} B_{\sqrt{r^2-t}}(x)} \|f(t)\|_{q, B_r(x)} \left(\int_{B_r(x)} e^{-\frac{q}{q-1} \frac{d^2(x,y)}{5(r^2-t)}} d \operatorname{vol}(y) \right)^{1-\frac{1}{q}} \\
 & \leq C (\operatorname{vol} B_{\sqrt{r^2-t}}(x))^{-\frac{1}{q}} \|f(t)\|_{q, B_r(x)} \\
 & = C \left(\frac{\operatorname{vol} B_r(x)}{\operatorname{vol} B_{\sqrt{r^2-t}}(x)} \right)^{\frac{1}{q}} \|f(t)\|_{q, B_r(x)}^* \\
 & \leq Cr^{\frac{n}{q}} (r^2 - t)^{-\frac{n}{2q}} \|f(t)\|_{q, B_r(x)}^*.
 \end{aligned}$$

In the last step we used the volume doubling property (2.5).

Now integrate (5.16) from 0 to r^2 gives

$$\begin{aligned}
 u(x, r^2) & \geq \int_{B_r(x)} u(y, 0) H^{B_r(x)}(x, y, r^2) d \operatorname{vol}(y) - Cr^{\frac{n}{q}} \int_0^{r^2} (r^2 - t)^{-\frac{n}{2q}} \|f(t)\|_{q, B_r(x)}^* dt \\
 & \geq C^{-1} \int_{B_{\frac{1}{2}r}(x)} u(y, 0) d \operatorname{vol}(y) - C(n, p, q) r^2 \sup_{t \in [0, r^2]} \|f(t)\|_{q, B_r(x)}^*.
 \end{aligned}$$

Here we used the lower bound for $H^{B_r(x)}(x, y, r^2)$ on $B_{\frac{1}{2}r}(x)$ and $q > \frac{n}{2}$. \square

Corollary 5.9. *Assume as above. Let u be a nonnegative function satisfying*

$$\Delta u \leq f.$$

Then, for $q > \frac{n}{2}$,

$$\int_{B_{\frac{1}{2}r}(x)} u d \operatorname{vol} \leq C \left(u(x) + r^2 \|f_+\|_{q, B_r(x)}^* \right) \tag{5.17}$$

for all $x \in M, r \leq 1$.

This is the L^1 Harnack inequality. For the Euclidean case, see e.g. [15, Theorem 4.15]. We would like to thank Ruobin Zhang for this reference.

With the above tools, we can extend Colding–Naber’s L^2 Hessian estimate for the parabolic approximation of the distance function to the integral curvature setting without essential difficulties. In the noncollapsed case it is done in [27], see also [32].

In what follows, we always assume $p > \frac{n}{2}$ and M is a complete n -dimensional Riemannian manifold satisfying $k(p, 1) \leq \varepsilon(n, p)$ for the ε so the results above all hold.

Fix two points y_-, y_+ in M^n , the excess is

$$e(x) = d(y_-, x) + d(y_+, x) - d(y_-, y_+).$$

Define

$$b_+(x) = d(y_+, x) - d(y_-, y_+), \quad b_-(x) = d(y_-, x).$$

Hence $e(x) = b_+(x) + b_-(x)$. Note that

$$\Delta b_{\pm}(x) \leq \frac{n-1}{d(x, y_{\pm})} + \psi_{\pm},$$

where $\psi_{\pm} = \left(\Delta d(y_{\pm}, x) - \frac{n-1}{d(x, y_{\pm})} \right)_+$ is the error term of the Laplacian comparison.

Denote $d_0 = d(y_-, y_+)$. Without loss of generality, assume $d_0 \leq 1$. Denote by $A_{r_1, r_2} = A_{r_1 d_0, r_2 d_0}(\{y_-, y_+\})$ the annulus for the set $\{y_-, y_+\}$, with $0 < r_1 < r_2$. Then [Corollary 5.9](#) and the Laplacian comparison estimate [\(2.2\)](#) gives

Theorem 5.10. *Fix some small positive constant $\delta > 0$. There exist $\bar{\varepsilon} = \bar{\varepsilon}(n, p, \delta)$ and $C = C(n, p, \delta)$ such that for all $0 < \varepsilon < \bar{\varepsilon}$, $x \in A_{\frac{\delta}{4}, 16}$,*

$$\int_{B_{\varepsilon d_0}(x)} e(y) dy \leq C [e(x) + (\varepsilon d_0)^2 (\|\psi_-\|_{2p, d_0}^* + \|\psi_+\|_{2p, d_0}^*)] \leq C [e(x) + \varepsilon^2 d_0].$$

In particular, if $e(x) \leq \varepsilon^2 d_0$, then

$$e(y) \leq C \varepsilon^{1 + \frac{1}{n+1}} d_0, \quad \forall y \in B_{\frac{1}{2} \varepsilon d_0}(x).$$

Remark 5.11. We obtain the optimal integral bound for the excess as in the pointwise Ricci lower bound case [\[11, Theorem 2.6\]](#), compare [\[27, Corollary 2.19\]](#), [\[32, Lemma 4.9\]](#). For the pointwise estimate, note that Abresch–Gromoll’s original estimate gives $\varepsilon^{1 + \frac{1}{n-1}}$ [\[1\]](#).

As in [\[11\]](#), one can extend [Lemma 5.4](#) to annulus so we have the cut-off function ϕ such that

$$\phi = 1 \text{ in } A_{\frac{\delta}{4}, 8}; \quad \phi = 0 \text{ outside } A_{\frac{\delta}{16}, 16}$$

and

$$|\nabla \phi|^2 + |\Delta \phi| \leq C(n, p, \delta).$$

Define the parabolic approximation functions $\mathbf{b}_{\pm,t}$ and \mathbf{e}_t by

$$\mathbf{b}_{\pm,t}(x) = \int H(x, y, t)\phi(y)b_{\pm}(y)dvol(y)$$

and

$$\mathbf{e}_t(x) = \int H(x, y, t)\phi(y)e(y)dvol(y).$$

Then

$$\mathbf{e}_t = \mathbf{b}_{+,t} + \mathbf{b}_{-,t}.$$

Following [27], we have following estimates for the approximates, which play important role in the Cheeger–Colding–Naber local theory for Gromov–Hausdorff limits.

Theorem 5.12. *There exists $C = C(n, p, \delta)$ such that for all $0 < \epsilon \leq \bar{\epsilon}(n, p, \delta)$, any $x \in A_{\frac{\delta}{2}, 4}$ with $e(x) \leq \epsilon^2 d_0$ and any ϵ -geodesic σ connecting y_-, y_+ , there exists $r \in [\frac{1}{2}, 2]$ with*

1. $|\mathbf{b}_{\pm, r\epsilon^2 d_0^2} - b_{\pm}| \leq C d_0 (\epsilon^2 + \epsilon^{2 - \frac{n}{2p}}).$
2. $\int_{B_{\epsilon d_0}(x)} \left| |\nabla \mathbf{b}_{\pm, r\epsilon^2 d_0^2}|^2 - 1 \right| \leq C (\epsilon + \epsilon^{1 - \frac{n}{2p}}).$
3. $\int_{\delta d_0}^{(1-\delta)d_0} \int_{B_{\epsilon d_0}(\sigma(s))} \left| |\nabla \mathbf{b}_{\pm, r\epsilon^2 d_0^2}|^2 - 1 \right| \leq C (\epsilon^2 + \epsilon^{2 - \frac{n}{p}}).$
4. $\int_{\delta d_0}^{(1-\delta)d_0} \int_{B_{\epsilon d_0}(\sigma(s))} |\text{Hess } \mathbf{b}_{\pm, r\epsilon^2 d_0^2}|^2 \leq \frac{C(1 + \epsilon^{-\frac{n}{p}})}{d_0^2}.$

We will only show the first lemma here to indicate the difference.

Lemma 5.13. *There exists a constant $C = C(n, p, \delta)$ such that*

$$\Delta \mathbf{b}_{\pm,t}, \Delta \mathbf{e}_t \leq C \left(\frac{1}{d_0} + t^{-\frac{n}{4p}} \right)$$

for $t < 1$.

Proof. Since, for $x \in A_{\frac{\delta}{16}, 16}$,

$$\Delta(\phi b_+) = b_+ \Delta \phi + 2 \langle \nabla \phi, \nabla b_+ \rangle + \phi \Delta b_+ \leq C d_0^{-1} + \psi_+,$$

we have

$$\begin{aligned} \Delta \mathbf{b}_{+,t}(x) &= \int_{A_{\frac{\delta}{16}, 16}} \Delta_x H(x, y, t)\phi(y)b_+(y)dvol(y) \\ &= \int_{A_{\frac{\delta}{16}, 16}} \Delta_y H(x, y, t)\phi(y)b_+(y)dvol(y) \end{aligned}$$

$$\begin{aligned}
 &= \int_{A_{\frac{\delta}{16}, 16}} H(x, y, t) \Delta_y(\phi(y)b_+(y)) dvol(y) \\
 &\leq \frac{C}{d_0} + \int_{A_{\frac{\delta}{16}, 16}} H(x, y, t) \psi_+ dvol(y).
 \end{aligned}$$

Using the upper bound for $H(x, y, t)$ and argue as in [Proposition 5.7](#), we have

$$\begin{aligned}
 \int_{A_{\frac{\delta}{16}, 16}} H(x, y, t) \psi_+ dvol(y) &\leq \|\psi_+\|_{2p} \|H(x, y, t)\|_{\frac{2p}{2p-1}} \\
 &\leq C(n, p) \|\psi_+\|_{2p}^* t^{-\frac{n}{4p}}.
 \end{aligned}$$

These give the estimate; the other terms are exactly the same. \square

6. Necessity of smallness of integral Ricci

By exploring Yang’s counter-example [\[30\]](#), we point out that the smallness of integral Ricci curvature, [\(1.2\)](#), is a critical condition in order to get the L^p version of Cheeger–Colding theory.

For any $k > 1$, let $M = (-1, 1) \times T^{n-1}$ be a portion of a complete manifold with a family of warped product metric

$$g_\epsilon = dr^2 + (\epsilon^2 + r^2)^k g_F \tag{6.1}$$

where T is a compact torus with flat metric g_F and $\epsilon > 0$ is the parameter. A direct calculation gives the sectional curvature

$$\begin{aligned}
 K\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) &= -k^2 r^2 (\epsilon^2 + r^2)^{-2}, \\
 K\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial r}\right) &= -k(\epsilon^2 + r^2)^{-1} - k(k-2)r^2(\epsilon^2 + r^2)^{-2},
 \end{aligned}$$

and the Ricci curvature

$$\begin{aligned}
 Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) &= -[(n-2)k^2 r^2 + k(k-2)r^2 + k(\epsilon^2 + r^2)](\epsilon^2 + r^2)^{-2}, \\
 Ric\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) &= -(n-1)[k(k-2)r^2 + k(\epsilon^2 + r^2)](\epsilon^2 + r^2)^{-2}, \\
 Ric\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial r}\right) &= 0,
 \end{aligned}$$

where (x^1, \dots, x^{n-1}) is the local normal coordinate of T^{n-1} .

Fix one $p \in (\frac{n}{2}, \frac{k(n-1)+1}{2})$. In the following calculation, \approx means equivalence up to a multiplication by a constant depending only on n and p . The first observation is

$$|Rm_{g_\epsilon}| \approx |Ric_{g_\epsilon}| \approx k^2 r^2 (\epsilon^2 + r^2)^{-2} + k(\epsilon^2 + r^2)^{-1}. \tag{6.2}$$

Put

$$B_r^* = \{(t, x) \in M \mid -r < t < r\}. \tag{6.3}$$

Then, for any function f ,

$$\int_{B_r^*} f dv_{g_\epsilon} = \int_{-r}^r \int_T f(x, t) (\epsilon^2 + t^2)^{\frac{(n-1)k}{2}} dv_{g_F} dt.$$

Applying to the curvature function we have, whenever $\epsilon \ll r$,

$$\begin{aligned} \int_{B_r^*} |Rm_{g_\epsilon}|^p &\approx k^{2p} \int_0^r \int_T (\epsilon^2 + t^2)^{\frac{(n-1)k}{2} - p} dv_{g_F} dt \\ &\approx \frac{k^{2p}}{k(n-1) + 1 - 2p} \text{vol}(T) r^{k(n-1)+1-2p} \\ &\approx k^{2p-1} \text{vol}(T) r^{k(n-1)+1-2p}, \end{aligned} \tag{6.4}$$

whenever $k \gg 1$ is sufficiently large. Then notice that

$$\text{vol}(B_r^*) \approx \frac{1}{k(n-1) + 1} \text{vol}(T) r^{k(n-1)+1} \approx k^{-1} \text{vol}(T) r^{k(n-1)+1}.$$

Thus we have, whenever $\epsilon \ll r$,

$$r^2 \left(\int_{B_r^*} |Rm_{g_\epsilon}|^p \right)^{\frac{1}{2p}} \approx k^2. \tag{6.5}$$

In particular, the formula remains hold on the limit space,

$$r^2 \left(\int_{B_r(o)} |Rm_{g_0}|^p \right)^{\frac{1}{2p}} \approx k^2, \forall r > 0. \tag{6.6}$$

The family of manifolds (M, g_ϵ) has polynomial volume growth. Hence a generalized local Sobolev constant and Poincaré constant are still bounded.¹ However, since the limit

¹ It should be understood as the $[k(n-1) + 1]$ -Sobolev inequality, but not the n -Sobolev inequality we have proved. Actually, the calculation shows the n -Sobolev constant is not bounded in Yang's example.

collapse at the origin, the volume comparison of geodesic spheres fails. Furthermore, to guarantee the splitting property on the tangent cones of the limit space we eventually need that

$$r^2 \left(\int_{B_r(o)} |\text{Ric}_{g_0}|^p \right)^{\frac{1}{2p}} \rightarrow 0$$

as $r \rightarrow 0$, which never hold. Therefore, the boundedness of L^p norm of curvature tensor is not sufficient to extend Cheeger–Colding theory.

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