Fundamental Gap Comparison

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Abstract We give a survey on the optimal lower and upper bound estimate for the fundamental gap (the difference between the first two eigenvalues of the Laplacian). Many of the results involve comparing to a good model. In particular, we discuss the Neumann gap estimate for manifolds with Ricci curvature bounded from below and the recent Dirichlet fundamental gap estimate for convex domains in Euclidean spaces and spheres using modulus of continuity and two-point maximal principle.

1. Introduction

Given a bounded smooth domain $\Omega \subset M^n$ of Riemannian manifold, the eigenvalue equation of the Laplacian on Ω is

$$\Delta\phi = -\lambda\,\phi. \tag{1.1}$$

Here we focus on either the Dirichlet or Neumann boundary condition/closed manifolds. Then the eigenvalues consist of an infinite sequence going off to infinity.

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Indeed,

$$\text{Dirichlet } (\phi|_{\partial\Omega} = 0): \quad 0 < \lambda_1 < \lambda_2 \le \lambda_3 \cdots \to \infty$$

$$\text{Neumann } (\phi_\nu|_{\partial\Omega} = 0)/\text{closed manifolds}: \quad 0 = \mu_0 < \mu_1 \le \mu_2 \cdots \to \infty.$$

Being important geometric quantities, there are many works in estimating the eigenvalues, especially the first (nonzero) eigenvalue λ_1, μ_1. For earlier work, see e.g, the survey article [28].

The fundamental (or mass) gap refers to the difference between the first two eigenvalues

$$\Gamma(\Omega) = \begin{cases} \lambda_2 - \lambda_1 > 0 & \text{Dirichlet boundary} \\ \mu_1 - 0 > 0 & \text{Neumann boundary/closed manifold} \end{cases} \tag{1.2}$$

of the Laplacian or more generally for Schrödinger operators. It is a very interesting quantity both in mathematics and physics. One of the most important question here is:

Problem 1. *Find optimal (geometric) upper and lower bounds for the gap.*

One of the basic and key tool is the Min-Max principle.

$$\lambda_k = \inf_V \sup_{\phi\neq 0, \phi\in V} \frac{\int_\Omega |\nabla\phi|^2}{\int_\Omega \phi^2},$$

where V ranges all k-dim subspaces of $H_2^1(\Omega)$, here $H_2^1(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\int_\Omega (|\phi|^2 + |\nabla\phi|^2) d\text{vol}.$$

To get a good upper bound one just needs a good test function; however, to get lower bounds one needs control on all functions.

To obtain optimal bounds, good comparison models are very useful. For the upper bound the model is a ball while for the lower bound the model is a strip and requires convex boundary.

1.1 Neumann boundary/Closed manifolds

For the Neumann boundary condition or closed manifolds the gap is just the first nonzero eigenvalue and is well-studied and developed.

For closed manifolds with Ricci curvature lower bound, the first estimate is due to Lichnerowicz [30]. Namely if M^n is a closed manifold with $\mathrm{Ric}_M \geq (n-1)K$ ($K > 0$), then $\mu_1(M) \geq nK = \mu_1(\mathbb{S}^n_K)$. This gives the first eigenvalue comparison. Similar comparison for higher eigenvalues is not true as $\mathbb{C}P^n$ gives a counterexample. Obata [36] proved the rigidity result that equality holds iff M^n is isometric to $\mathbb{S}^n_K$. Escobar [20] established the Lichnerowicz-Obata result for manifolds with convex boundary.

Lichnerowicz estimate does not give any information when $K \leq 0$. For closed manifolds M^n with $\mathrm{Ric}_M \geq 0$, Li-Yau [29] first proved that $\mu_1(M) \geq \frac{1}{2}\left(\frac{\pi}{D}\right)^2$, where D is the diameter of M. Zhong-Yang [59] obtained the optimal estimate that $\mu_1(M) \geq \left(\frac{\pi}{D}\right)^2$. Equality is achieved iff M is isometric to S^1 [22].

For the proof, while the Lichnerowicz estimate can be shown very quickly by integrating the Bochner inequality applied to the first eigenfunction, Zhong-Yang's estimate, building on Li-Yau's gradient estimate for the eigenfunction, requires a very careful choice of auxiliary function and is much more complicated.

When $K \leq 0$, Li-Yau [29] also obtained an explicit lower bound for closed manifolds and manifolds with convex boundary. The general optimal lower bound is given by the following gap comparison.

Theorem 2. *Let $\Omega \subset M^n$ be a bounded convex domain or closed manifold, with diameter $\leq D$, and $\mathrm{Ric}_M \geq (n-1)K$. Then*

$$\mu_1(\Omega) \geq \bar{\mu}_1(n, K, D), \tag{1.3}$$

where $\bar{\mu}_1(n, K, D)$ is the first non-zero eigenvalue of the model operator

$$L_{n,K,D}(\varphi) = \varphi''(s) - (n-1)\varphi' \,\mathrm{tn}_K(s)$$

on the interval $[-\frac{D}{2}, \frac{D}{2}]$ with the (Neumann) boundary condition $\varphi'(\pm\frac{D}{2}) = 0$.

Here we denote

$$\mathrm{tn}_K(s) = \begin{cases} \sqrt{K}\tan(\sqrt{K}s), & K > 0 \\ 0, & K = 0 \\ -\sqrt{-K}\tanh(\sqrt{-K}s) & K < 0. \end{cases} \tag{1.4}$$

Remark 3. *Estimate* (1.3) *was first proved by P. Kröger* [25] *for closed manifolds using gradient estimate. Mufa Chen and Fengyu Wang gave a proof using probabilistic 'coupling method'* [17]*. Andrews and Clutterbuck* [4]*, gave a proof using a* *new method, the so called modulus of continuity for heat equation. This gives a simple and unified proof for all cases. Then Yuntao Zhang and Kui Wang* [58] *gave a proof using the modulus of continuity for the eigenvalue equation instead of heat equation.*

Remark 4. *Estimate* (1.3) *includes Lichnerowicz and Zhong-Yang estimates as special cases, when* $K = 0$, $\bar{\mu}_1(n, 0, D) = \left(\frac{\pi}{D}\right)^2$, *and when* $K > 0$, $\bar{\mu}_1(n, K, \frac{\pi}{\sqrt{K}}) = nK$. *In general* $\bar{\mu}_1(n, K, D)$ *can not be computed explicitly. See* [42, 52] *for some nice explicit estimates.*

Remark 5. *There has been many further generalizations of* (1.3). *For weighted Laplacians, there is the work of Bakry-Qian* [13] *using gradient estimates and Andrews-Ni* [6] *using modulus of continuity.*

For the p-Laplacian, Matei extended the estimates for $K > 0$ [33], *Valtorta for* $K = 0$ [48], *and Naber-Valtorta did the general case* [34], *see also* [57]. *For the weighted p-Laplacian it is studied in* [49, 50].

For $RCD(K, N)$ *spaces Y. Jiang - H.C.Zhang* [24] *proved the gap estimate using gradient estimate; Cavalletti-Mondino proved it using optimal transport* [16].

On the other hand, instead of pointwise Ricci curvature lower bound, one can just assume the part below some positive constant (or zero) is small in $L^p (p > \frac{n}{2})$, *and similar estimates hold, see* [12, 21, 38, 41].

For closed Kähler manifolds, the Lichnerowicz estimate can be improved for Laplacian and p-Laplacian, see [15, 46].

Remark 6. *Equality in* (1.3) *is achieved iff when* $n = 1, K = 0$ *or* $K > 0, D = \frac{\pi}{\sqrt{K}}$. *For all* n, K, D, *there are convex domains (even closed manifolds) whose* μ_1 *is arbitrary close to* $\bar{\mu}_1$. *So estimate* (1.3) *is sharp.*

There are also important results concerning sharp upper bounds of the fundamental gap. Aithal-Santhanam [1], generalizing the work of Szëgo [45], Weinberger [51], Ashbaugh-Benguria [10] for bounded domain in the space of constant sectional curvature $\mathbb{M}^n_K$, showed that, for bounded domain Ω in a rank-1 symmetric space (with some size restriction in the compact case)

$$\mu_1(\Omega) \leq \mu_1(B(r_1))$$

where $B(r_1)$ is a ball such that $\mathrm{vol}(B(r_1)) = \mathrm{vol}(\Omega)$ and equality holds iff $\Omega = B(r_1)$.

1.2 Dirichlet Boundary Condition

For Dirichlet boundary condition, a sharp upper bound is obtained in the resolution of the Payne-Polya-Weinberger (PPW) Conjecture for $\mathbb{M}^n_K$, the n-dimensional simply connected space of constant curvature K. Indeed, for bounded domain $\Omega \subset \mathbb{M}^n_K$ (note that one does not need a convexity condition here)

$$\Gamma(\Omega) \leq \Gamma(B_{\lambda_1}),$$

where B_{λ_1} is a ball in $\mathbb{M}^n_K$ such that $\lambda_1(B_{\lambda_1}) = \lambda_1(\Omega)$.

The PPW conjecture was proved by Ashbaugh-Benguria [9] for the case when $K = 0$, and by Ashbaugh-Benguria [11] for the case when $K = 1$ (Ω in the upper hemisphere), and finally by Benguria-Linde [14] for the case of $K = -1$.

The question of a sharp lower bound for Dirichlet boundary condition is especially interesting and important. For bounded convex domain $\Omega \subset \mathbb{R}^n$ and Schrödinger operator $-\Delta + V$, where $V \geq 0$, convex, the question carries a special name:

Fundamental Gap Conjecture (van den Berg, Ashbaugh-Benguria, Yau [8, 47, 53]):

$$\Gamma(\Omega, V) \geq \tfrac{3\pi^2}{D^2}, \qquad D = \operatorname{diam} \Omega.$$

The lower bound is approached when $V = 0$, and domain is a thin rectangular box.

Remark 7. *The subject here has a long history, see the excellent survey by Ashbaugh* [7] *for discussion of the fundamental gap and history up to 2006, and the references in* [3]*. We only list some key developments here. In the seminal work* [44] *Singer-Wong-Yau-Yau established the bound* $\Gamma(\Omega, V) \geq \frac{\pi^2}{4D^2}$ *using Li-Yau's gradient estimate method. This is later improved by Yu-Zhong* [56]*, Ling* [31] *to* $\Gamma(\Omega, V) \geq \frac{\pi^2}{D^2}$*. Further improvement to the gap is done by Yau* [54] *which depends on upper bound estimate on the log-concavity of the first eigenfunction. Yau in* [55] *also investigated the case of non-convex potentials. The Fundamental Gap Conjecture is finally resolved by Andrews-Clutterbuck* [3] *by introducing new ideas into the problem, namely the modulus of continuity and concavity. A proof using eigenvalue equation instead of heat equation is given by Ni* [35]*. Cheng-Oden* [18] *obtained gap estimate for domians in $\mathbb{R}^n$ which may not be convex but satisfying interior R-rolling ball condition, Oden-Sung-Wang* [37] *further studied domains in general manifolds with curvature bounds.*

The following question is raised in B. Andrews' excellent survey article [2] among others.

Question 8. *What about convex domains in $\mathbb{S}^n$?*

In a series of joint work the same gap estimate for convex domains of sphere was established.

Theorem 9 ([19, 23, 40])**.** *Let $\Omega \subset \mathbb{S}^n$ be a strictly convex domain with diameter D, $\lambda_i (i = 1, 2)$ be the first two eigenvalues of the Laplacian on Ω with Dirichlet boundary condition. Then*

$$\Gamma(\Omega) = \lambda_2 - \lambda_1 \geq 3\frac{\pi^2}{D^2}. \tag{1.5}$$

The same estimates hold also for Schrödinger operator $-\Delta + V$, where $V \geq 0$ and is convex.

Remark 10. *A proof is given by Seto-Wang-Wei* [40]*, for $n \geq 3, D \leq \pi/2$. Then He-Wei* [23] *showed it for $n \geq 3, D < \pi$. Finally Dai-Seto-Wei* [19] *solved it for all $n \geq 2, D < \pi$.*

Remark 11. *Previous work includes Lee-Wang* [27] *who showed that $\Gamma(\Omega) \geq \frac{\pi^2}{D^2}$ and Ling* [31] *who improved it to $\Gamma(\Omega, V) > \frac{\pi^2}{D^2}$.*

1.3 Further Questions

In [40, Theorem 4.1] the following general gap comparison is proven.

Theorem 12 (Seto-Wang-Wei)**.** *Let Ω be a bounded convex domain with diameter D in a Riemannian manifold M^n with $\mathrm{Ric}_M \geq (n-1)K$. Assume ϕ_1 satisfies the log-concavity estimates, $\forall x, y \in \Omega$, with $x \neq y$,*

$$\langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2}) \rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2}) \rangle \leq 2 \left(\log \bar{\phi}_1 \right)' \left(\frac{d(x,y)}{2} \right), \tag{1.6}$$

where γ is the unit speed minimizing geodesic with $\gamma(-\frac{d}{2}) = x$, $\gamma(\frac{d}{2}) = y$, $d = d(x, y)$. Then we have the gap comparison

$$\lambda_2 - \lambda_1 \geq \bar{\lambda}_2(n, D, K) - \bar{\lambda}_1(n, D, K), \tag{1.7}$$

where $\bar{\lambda}_i(n,D,K)$ and $\bar{\phi}_i$ are the eigenvalue and eigenfunction of

$$L_{n,K,D}(\phi) = \phi'' - (n-1)\,\mathrm{tn}_K(s)\phi' \tag{1.8}$$

on $[-\frac{D}{2}, \frac{D}{2}]$ with Dirichlet boundary condition.

Remark 13. *The log-concavity condition* (1.6) *implies that ϕ_1 is more log-concave than $\bar{\phi}_1$, but it contains more information than just Hess $\log \phi_1 \le$ Hess $\log \bar{\phi}_1$.*

The condition (1.6) *is automatically satisfied in the Neumann boundary/closed manifolds case as the first eigenfunctions are just constants. In this sense, Theorem 12 is a generalization of Theorem 2.*

The log-concavity condition (1.6) is only proven for convex domains in $\mathbb{M}^n_K$ for $K \ge 0$. A natural question is

Question 14. *Is* (1.6) *true for convex domains in $\mathbb{H}^n$?*

It is known that in this case the first eigenfunction may not be log-concave [43]. Another related question is for convex domains in $\mathbb{C}P^n$: is the first eigenfunction log-concave? This is conjectured to be true by Prof. Zhiqin Lu.

For traingles in $\mathbb{R}^2$ with diameter D, Lu-Rowlett [32] showed the Dirichelt gap is $\ge \frac{64\pi^2}{9D^2}$ and equality holds iff it is an equilateral triangle.

Question 15. *What about triangles in the upper hemisphere of $\mathbb{S}^2$?*

Question 16. *What is a good explicit estimate for $\bar{\lambda}_2(n,D,K) - \bar{\lambda}_1(n,D,K)$ in terms of K? In the author's work* [19] *an estimate was given when KD^2 is small.*

Here are some other related questions in Dirichlet gap low bound estimates for convex domain in $\mathbb{R}^n$. In [5] the first eigenfunction of the Laplacian with Robin boundary condition for convex domain in $\mathbb{R}^n$ is studied. Surprisingly, it is shown that the Robin ground state is not log-concave for small Robin parameter on a large class of convex domain. It is asked in [5] if similar gap comparison estimate still holds true for Robin boundary condition. For convex domain in $\mathbb{R}^n$, it is known that the first eigenfunction of the p-Laplacian is log-concave [39], but similar gap estimate for the p-Laplacian is still unknown.

In the next two sections we discuss the strategy of proving Theorem 1.9. Even though the basic strategy is the same as in [3], the sphere case is much more complicated and subtle. In particular the natural model (1.8) does not give an optimal estimate when $n = 2$ and a different model is considered.

2. Basic Approach and Modulus of Continuity

The basic outline of approach to obtaining the lower bound estimate of the fundamental gap for convex domain in sphere is as follows:

- Step 1: Reduction to a Neumann gap problem. This follows the original approach of Singer-Wong-Yau-Yau [44].
- Step 2: Establishing the super log-concavity of the first eigenfunction. This is the new ingredient introduced first by Andrews-Clutterbuck for the Euclidean case and is the most technical part of the proof.
- Step 3: The gap estimate $\lambda_2 - \lambda_1 \geq 3\frac{\pi^2}{D^2}$ by two-point maximum principle.

For Step 1, following Singer-Wong-Yau-Yau, let $w(x) = \dfrac{\phi_2(x)}{\phi_1(x)}$, where $\phi_i,\ i = 1, 2$ are the normalized eigenfunctions corresponding to the first and second eigenvalue, respectively:

$$\Delta\phi_i = -\lambda_i\phi_i, \quad \phi_i|_{\partial\Omega} = 0.$$

Moreover $\phi_1 > 0$ in the interior.

Then

$$\Delta w = -(\lambda_2 - \lambda_1)w - 2\langle \nabla \log \phi_1, \nabla w\rangle, \tag{1.9}$$

where w satisfies the Neumann boundary condition, c.f. [44].

Note that equation (1.9) can be interpreted as an eigenvalue equation for the weighted Laplacian with respect to the measure $\phi_1^2 d\text{vol}$. Thus we are reduced to estimating the Neumann gap of a (weighted) Laplacian.

It is clear from (1.9) that treating the second term there (which involves the gradient of the log of the first eigenfunction) is important. Indeed it turns out that it is the critical part in obtaining the sharp lower bound for the fundamental gap. The result is the so called super log-concavity of the first eigenfunction

The super log-concavity of the first eigenfunction is first introduced by Andrews-Clutterbuck [3] in the Eucildean case. For convex domains Ω of a Riemannian manifold, and any $x, y \in \Omega$, let γ be a normal geodesic connecting x with y. Indeed we will parametrize γ by the interval $[-\frac{d}{2}, \frac{d}{2}]$ where $d = d(x, y)$. Thus $\gamma(-\frac{d}{2}) = x,\ \gamma(\frac{d}{2}) = y$. Then we have

Theorem 17 (Dai-Seto-Wei, [19]). *Given $\Omega \subset \mathbb{M}^n_K$ a bounded strict convex domain with diameter D, let $\phi_1 > 0$ be a first eigenfunction of the Laplacian on Ω. Assume $K \geq 0$. Then for $\forall x, y \in \Omega$, with $x \neq y$,*

$$\begin{aligned}&\langle \nabla \log \phi_1(y), \gamma'(\tfrac{d}{2})\rangle - \langle \nabla \log \phi_1(x), \gamma'(-\tfrac{d}{2})\rangle \\ &\qquad \leq -2\tfrac{\pi}{D}\tan\left(\tfrac{\pi d}{2D}\right) + (n-1)\,\mathrm{tn}_K(\tfrac{d}{2}).\end{aligned}$$

In particular,

$$Hess\,(\log \phi_1) \leq -\left(\frac{\pi^2}{D^2} - \frac{n-1}{2}K\right)\, id.$$

Remark 18. *Seto-Wang-Wei* [40] *previously established another super log-concavity result for the first eigenfunction (the sphere model). As a consequence Seto-Wang-Wei then proved the spherical version of the fundamental gap estimate when the dimension $n \geq 3$. The log-concavity given above is worse than the sphere model when $n \geq 3$ but better than sphere model when $n = 2$.*

We will come back to the super log-concavity in the next section, but assuming it, we now discuss the proof of the fundamental gap estimate Theorem 9. Namely, for all n, when $K \geq 0$,

$$\lambda_2 - \lambda_1 \geq 3\tfrac{\pi^2}{D^2}.$$

As before $w(x) = \dfrac{\phi_2(x)}{\phi_1(x)}$ and also $\bar{w}(s) = \dfrac{\bar{\phi}_2(s)}{\bar{\phi}_1(s)}$, where

$$\Delta \phi_i = -\lambda_i \phi_i, \quad \phi_i|_{\partial\Omega} = 0$$

and $\bar{\phi}_1, \bar{\phi}_2$ are the corresponding eigenfunctions for the 1-dimensional model, namely

$$\begin{cases} \bar{\phi}_i'' + \bar{\lambda}_i \bar{\phi}_i = 0 \\ \bar{\phi}_i(\pm D/2) = 0 \end{cases} \tag{1.10}$$

Clearly $\bar{\lambda}_1 = \frac{\pi^2}{D^2}, \bar{\lambda}_2 = \frac{4\pi^2}{D^2}$ and hence $\bar{\lambda}_2 - \bar{\lambda}_1 = \frac{3\pi^2}{D^2}$ the desired lower bound. (Of course the eigenfunctions are also explicitly given by the trig functions but we found it more illuminating to work with the eigenvalue equations.)

An easy computation gives

$$\begin{aligned}\Delta w &= -(\lambda_2 - \lambda_1)w - 2\langle \nabla \log \phi_1, \nabla w\rangle, \\ \bar{w}'' &= -(\bar{\lambda}_2 - \bar{\lambda}_1)\bar{w} - 2(\log \bar{\phi}_1)'\bar{w}'\end{aligned}$$

where the Neumann boundary conditions are satisfied.

Consider the quotient of the oscillations

$$Q(x,y) = \frac{w(x) - w(y)}{\bar{w}\left(\frac{d(x,y)}{2}\right)}$$

on $\overline{\Omega} \times \overline{\Omega} \setminus \Delta$, where $\Delta = \{(x,x) | x \in \overline{\Omega}\}$ is the diagonal. One applies the maximal principle to $Q(x,y)$.

Assume the maximum of Q is attained at (x_0, y_0) and $x_0 \neq y_0$. Denote $d_0 = \frac{d(x_0,y_0)}{2}$, $m = Q(x_0, y_0)$. Then, at the maximum point (x_0, y_0), we have $\nabla_E Q = 0$ for any $E \in T_{x_0}\Omega \oplus T_{y_0}\Omega$. It follows then that

$$\nabla w(y_0) = \nabla w(x_0) = -\frac{m}{2}\bar{w}'(d_0/2)e_n.$$

Here $e_n = \gamma'$. Furthermore,

$$\nabla^2_{E,E} Q = \frac{1}{\bar{w}(d_0)}\left(\nabla^2_{E,E}(w(y_0) - w(x_0)) - m\nabla^2_{E,E}\bar{w}(d_0)\right) \leq 0$$

Let $e_1, \cdots, e_{n-1}$ be an orthonormal set of parallel vector fields along γ which are perpendicular to γ. Adding up $\nabla^2_{E_i,E_i} Q \leq 0,\ i = 1, \cdots, n$, where $E_i = e_i \oplus e_i$ for $i = 1, \cdots, n-1$ and $E_n = e_n \oplus (-e_n)$, we derive

$$0 \geq \frac{\Delta w(x_0) - \Delta w(y_0)}{\bar{w}} - \frac{m}{\bar{w}}\sum_{i=1}^{n} \nabla^2_{E_i,E_i}\bar{w}. \tag{1.11}$$

To extract useful information from the inequality above, we now state a new form of the Laplacian comparison result due to [4] and reformulated in this form by [40]. The usual Laplace comparison is for one point distance function, i.e., in $d(x,y)$, we fix x (or y).

If we let both vary, one has

Theorem 19. *M^n with $\mathrm{Ric}_M \geq (n-1)K$, then (in weak sense)*

$$\sum_{i=1}^{n-1} \nabla^2_{E_i,E_i} d(x,y) \quad \leq \quad -2(n-1)\,\mathrm{tn}_K(\frac{d(x,y)}{2})$$

where $e_i \perp \nabla d$ are orthonormal and parallel and $E_i = e_i \oplus e_i$.

As a consequence, one has the *partial Laplace comparison for two-point radial function*. Let $v(x,y) = \varphi(d(x,y))$. If $\mathrm{Ric}_{M^n} \geq (n-1)K$, then

$$\sum_{i=1}^{n-1} \nabla^2_{E_i,E_i} v(x,y) \leq -(n-1)\varphi'\,\mathrm{tn}_K(\frac{r(x,y)}{2}) \text{ if } \varphi' \geq 0$$

$$\sum_{i=1}^{n-1} \nabla^2_{E_i,E_i} v(x,y) \geq -(n-1)\varphi' \operatorname{tn}_K(\frac{r(x,y)}{2}) \text{ if } \varphi' \leq 0$$

In particular, applying it to the second term of the inequality (1.11) and using $\bar{w}' \geq 0$, one can deduce

$$\sum_{i=1}^{n-1} \nabla^2_{E_i,E_i} \bar{w} \leq -(n-1) \operatorname{tn}_K \bar{w}'.$$

On the other hand, $\nabla^2_{E_n,E_n} \bar{w} = \bar{w}''$. Hence

$$\begin{aligned}
0 \quad &\geq \quad -(\lambda_2 - \lambda_1)m + \frac{m\bar{w}'}{\bar{w}}(n-1)\operatorname{tn}_K - \frac{m}{\bar{w}}\bar{w}'' \\
&\quad\quad -\frac{m\bar{w}'}{\bar{w}}(\langle \nabla \log \phi_1(y_0), e_n\rangle - \langle \nabla \log \phi_1(x_0), e_n\rangle) \\
&= \quad -(\lambda_2 - \lambda_1)m + (\bar{\lambda}_2 - \bar{\lambda}_1)m + \frac{m\bar{w}'}{\bar{w}}(-2\tfrac{\pi}{D}\tan\left(\tfrac{\pi d}{2D}\right) + (n-1)\operatorname{tn}_K(\tfrac{d}{2})) \\
&\quad\quad -\frac{m\bar{w}'}{\bar{w}}(\langle \nabla \log \phi_1(y_0), e_n\rangle - \langle \nabla \log \phi_1(x_0), e_n\rangle) \\
&\geq \quad -(\lambda_2 - \lambda_1)m + (\bar{\lambda}_2 - \bar{\lambda}_1)m
\end{aligned}$$

by the super log-concavity, as $m, \bar{w}$ are positive.

When the maximum of Q is attained at (x_0, y_0) and $x_0 = y_0$, it can be proved similarly by a limiting process.

Note that as $y \to x$,

$$Q(x, \gamma'(0)) = \frac{2\langle \nabla w(x), \gamma'(0)\rangle}{\bar{w}'(0)}.$$

We refer to [19, 40] for more detail.

3. Back to super log-concavity

The notion of the modulus of concavity is the crucial concept introduced by [3] in their solution of the original fundamental gap conjecture.

Definition 20. *Given a semi-convex function u on a domain Ω, a function ψ : $[0, +\infty) \to \mathbb{R}$ is called a modulus of concavity for u if for every $x \neq y$ in Ω*

$$\langle \nabla u(y), \gamma'(\tfrac{d}{2})\rangle - \langle \nabla u(x), \gamma'(-\tfrac{d}{2})\rangle \leq 2\psi\left(\frac{d(x,y)}{2}\right).$$

As in Andrews-Clutterbuck, He-Wei, [3, 23], the proof of the super concavity consists of the following steps.

- Step 1: Preservation of modulus of concavity. Namely one shows that there is a way to deform a modulus function by essentially parabolic equations so that the modulus of concavity property is preserved.
- Step 2: Existence of solution. That is, one shows that the solutions to the parabolic equations exist.
- Step 3: Convergence to model solution as $t \to \infty$. One shows that the inital modulus of concavity can be improved via the previous steps so that it approaches the desired modulus of concavity, the model solution.

The preservation of super log-concavity result in our case is stated in the following theorem.

Theorem 21. *Let $\Omega \subset \mathbb{M}^n_K$ be a strictly convex domain with diameter D. Assume $K \geq 0$. Let $u(x,t) = e^{-\lambda_1 t}\phi_1(x)$, where $\Delta\phi_1 = -\lambda_1\phi_1$. Suppose $\psi_0 : [0, \frac{D}{2}] \to \mathbb{R}$ is a Lipschitz continuous function such that $\psi_0 + \frac{n-1}{2}\operatorname{tn}_K$ is a modulus of concavity for $\log\phi_1$. If ψ is a solution of*

$$\begin{aligned} \frac{\partial\psi(s,t)}{\partial t} &\geq \psi''(s,t) + 2\psi(s,t)\psi'(s,t) - \operatorname{tn}_K(s)\left[\psi'(s,t) + \psi^2(s,t) + \lambda_1\right]; \quad (1.12)\\ \psi(\cdot,0) &= \psi_0(\cdot);\\ \psi(0,t) &= 0, \end{aligned}$$

where $\psi' = \frac{\partial}{\partial s}\psi$ and $\psi'' = \frac{\partial^2}{\partial s^2}\psi$, then $\psi(\cdot,t) + \frac{n-1}{2}\operatorname{tn}_K$ is a modulus of concavity for $\log u(\cdot,t)$ for each $t \geq 0$.

Given Theorem 21, we now look into the existence and convergence of the differential inequality (1.12). Note $f = (\log\bar{\phi}_1)' = -\frac{\pi}{D}\tan\left(\frac{\pi s}{D}\right)$ satisfies

$$f' + f^2 + (\tfrac{\pi}{D})^2 = 0.$$

On the other hand, the stationary solutions of ψ satisfy

$$0 = (\psi'(s) + \psi^2(s) + \lambda_1)' - 2\operatorname{tn}_K(s)(\psi' + \psi^2(s) + \lambda_1).$$

Let $y = \psi' + \psi^2 + \lambda_1$. Solving the ODE $y' - 2\operatorname{tn}_K(s)y = 0$, we obtain $y = y(0)K\operatorname{cs}_K^{-2}(s)$. Hence an initial condition $y(0) = 0$ would imply the trivial solution in y, which is equivalent to $\psi'+\psi^2+\lambda_1 = 0$. The condition $y(0) = 0$ can be achieved by adding the condition $\psi'(0) = -\lambda_1$. In other words, a stationary solution ψ

satisfying the Dirichlet boundary condition, with an additional boundary condition $\psi'(0) = -\lambda_1$, will yield the model solution. Thus the task is then to set up a version of (1.12), solve it with the additional boundary condition $\psi'(0) = -\lambda_1$, and show that it converges to the model solution in the large time limit. Once again we refer to [19, 23] for more detail.

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