The following result is intuitively clear:

Proposition 1.1

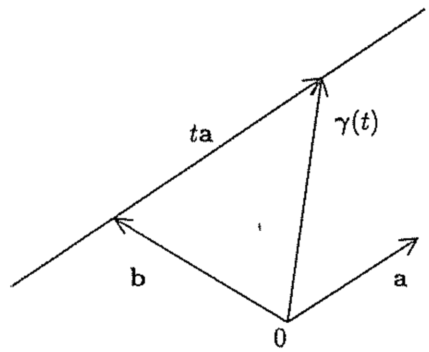
If the tangent vector of a parametrised curve is constant, the image of the curve is (part of) a straight line

Proof 1.1

If $\dot{\gamma}(t) = \mathbf{a}$ for all t, where $\mathbf{a}$ is a constant vector, we have, integrating componentwise,

$$\gamma(t) = \int \frac{d\gamma}{dt} dt = \int \mathbf{a} dt = t\mathbf{a} + \mathbf{b},$$

where $\mathbf{b}$ is another constant vector. If $\mathbf{a} \neq \mathbf{0}$, this is the parametric equation of the straight line parallel to $\mathbf{a}$ and passing through the point with position vector $\mathbf{b}$:



If $\mathbf{a} = \mathbf{0}$, the image of γ is a single point (namely, the point with position vector $\mathbf{b}$). □

EXERCISES

1.1 Is $\gamma(t) = (t^2, t^4)$ a parametrisation of the parabola $y = x^2$?

1.2 Find parametrisations of the following level curves:
(i) $y^2 - x^2 = 1$;
(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

1.3 Find the cartesian equations of the following parametrised curves:
(i) $\gamma(t) = (\cos^2 t, \sin^2 t)$;
(ii) $\gamma(t) = (e^t, t^2)$.

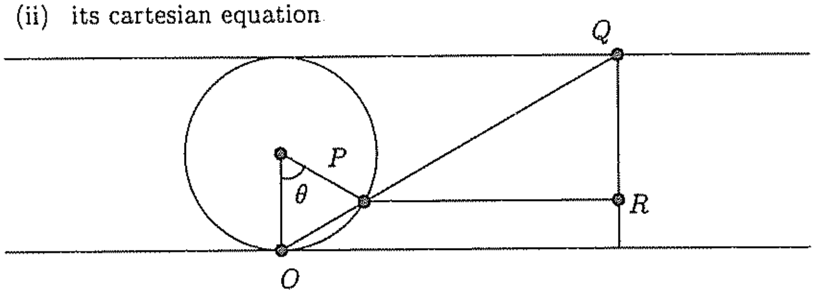
1.4 Calculate the tangent vectors of the curves in Exercise 1.3.

1.5 Sketch the astroid in Example 1.3. Calculate its tangent vector at each point. At which points is the tangent vector zero?

1.6 If P is any point on the circle $\mathcal{C}$ in the xy-plane of radius $a > 0$ and centre $(0, a)$, let the straight line through the origin and P intersect the line $y = 2a$ at Q, and let the line through P parallel to the x-axis intersect the line through Q parallel to the y-axis at R. As P moves around $\mathcal{C}$, R traces out a curve called the *witch of Agnesi*. For this curve, find

(i) a parametrisation;

(ii) its cartesian equation.

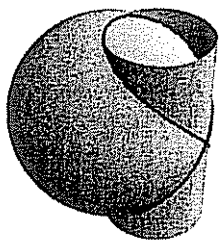


1.7 A *cycloid* is the plane curve traced out by a point on the circumference of a circle as it rolls without slipping along a straight line. Show that, if the straight line is the x-axis and the circle has radius $a > 0$, the cycloid can be parametrised as

$$\gamma(t) = a(t - \sin t, 1 - \cos t)$$

1.8 Generalise the previous exercise by finding parametrisations of an *epicycloid* (resp. *hypocycloid*), the curve traced out by a point on the circumference of a circle as it rolls without slipping around the outside (resp. inside) of a fixed circle.

1.9 Show that $\gamma(t) = (\cos^2 t - \frac{1}{2}, \sin t \cos t, \sin t)$ is a parametrisation of the curve of intersection of the circular cylinder of radius $\frac{1}{2}$ and axis the z-axis with the sphere of radius 1 and centre $(-\frac{1}{2}, 0, 0)$. (This is called *Viviani's Curve*).



Definition 1.4

If $\gamma : (\alpha, \beta) \to \mathbf{R}^n$ is a parametrised curve, its *speed* at the point $\gamma(t)$ is $\| \dot{\gamma}(t) \|$, and γ is said to be a *unit-speed curve* if $\dot{\gamma}(t)$ is a unit vector for all $t \in (\alpha, \beta)$.

We shall see many examples of formulas and results relating to curves that take on a much simpler form when the curve is unit-speed. The reason for this simplification is given in the next proposition. Although this admittedly looks uninteresting at first sight, it will be extremely useful for what follows.

Proposition 1.2

Let $\mathbf{n}(t)$ *be a* unit vector *that is a smooth function of a parameter* t. *Then, the dot product*

$$\dot{\mathbf{n}}(t).\mathbf{n}(t) = 0$$

for all t, *i.e.* $\dot{\mathbf{n}}(t)$ *is zero or perpendicular to* $\mathbf{n}(t)$ *for all* t.

In particular, if γ *is a unit-speed curve, then* $\ddot{\gamma}$ *is zero or perpendicular to* $\dot{\gamma}$.

Proof 1.2

We use the 'product formula' for differentiating dot products of vector-valued functions $\mathbf{a}(t)$ and $\mathbf{b}(t)$:

$$\frac{d}{dt}(\mathbf{a}.\mathbf{b}) = \frac{d\mathbf{a}}{dt}.\mathbf{b} + \mathbf{a}.\frac{d\mathbf{b}}{dt}.$$

Using this to differentiate both sides of the equation $\mathbf{n}.\mathbf{n} = 1$ with respect to t gives

$$\dot{\mathbf{n}}.\mathbf{n} + \mathbf{n}.\dot{\mathbf{n}} = 0,$$

so $2\dot{\mathbf{n}}.\mathbf{n} = 0$.

The last part follows by taking $\mathbf{n} = \dot{\gamma}$. □

EXERCISES

1.11 Calculate the arc-length of the *catenary* $\gamma(t) = (t, \cosh t)$ starting at the point $(0, 1)$.

1.12 Show that the following curves are unit-speed:
(i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$;
(ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1 - \sin t, -\frac{3}{5}\cos t\right)$.

rting at (1, 0) was found
speed reparametrisation

in $\left(\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)\right)$.

ng at $\gamma(0) = \mathbf{0}$ is

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example of something

Our final example shows that a given level curve can have both regular and non-regular parametrisations.

Example 1 8

For the parametrisation

$$\gamma(t) = (t, t^2)$$

of the parabola $y = x^2$, $\dot{\gamma}(t) = (1, 2t)$ is obviously never zero, so γ is regular.

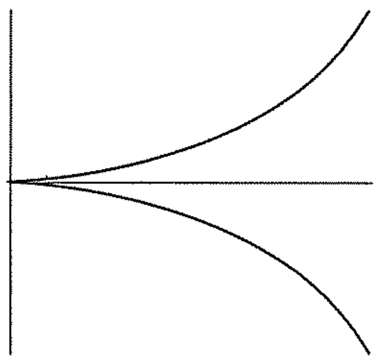
But

$$\tilde{\gamma}(t) = (t^3, t^6)$$

is also a parametrisation of the same parabola. This time, $\dot{\tilde{\gamma}} = (3t^2, 6t^5)$, and this is zero when $t = 0$, so $\tilde{\gamma}$ is *not* regular.

EXERCISES

1.14 Which of the following curves are regular?
(i) $\gamma(t) = (\cos^2 t, \sin^2 t)$ for $-\infty < t < \infty$;
(ii) the same curve as in (i), but with $0 < t < \pi/2$;
(iii) $\gamma(t) = (t, \cosh t)$ for $-\infty < t < \infty$.
Find unit-speed reparametrisations of the regular curve(s).



1.15 The *cissoid of Diocles* (see above) is the curve whose equation in terms of polar coordinates (r, θ) is

$$r = \sin\theta \tan\theta, \quad -\pi/2 < \theta < \pi/2.$$

Write down a parametrisation of the cissoid using θ as a parameter and show that

$$\gamma(t) = \left(t^2, \frac{t^3}{\sqrt{1-t^2}}\right), \quad -1 < t < 1,$$

is a reparametrisation of it.

1.16 Let γ be a curve in $\mathbb{R}^n$ and let $\tilde{\gamma}$ be a reparametrisation of γ with reparametrisation map ϕ (so that $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$). Let $\tilde{t}_0$ be a fixed value of $\tilde{t}$ and let $t_0 = \phi(\tilde{t}_0)$. Let s and $\tilde{s}$ be the arc-lengths of γ and $\tilde{\gamma}$ starting at the point $\gamma(t_0) = \tilde{\gamma}(\tilde{t}_0)$. Prove that $\tilde{s} = s$ if $d\phi/d\tilde{t} > 0$ for all $\tilde{t}$, and $\tilde{s} = -s$ if $d\phi/d\tilde{t} < 0$ for all $\tilde{t}$.

1.4. Level Curves vs. Parametrised Curves

We shall now try to clarify the precise relation between the two types of curve we have considered in previous sections.

Level curves in the generality we have defined them are not always the kind of objects we would want to call curves. For example, the level 'curve' $x^2 + y^2 = 0$ is a single point. The correct conditions to impose on a function $f(x, y)$ in order that $f(x, y) = c$, where c is a constant, will be an acceptable level curve in the plane are contained in the following theorem, which shows that such level curves can be parametrised. Note that we might as well assume that $c = 0$ (since we can replace f by $f - c$).

Theorem 1.1

Let $f(x, y)$ be a smooth function of two variables (which means that all the partial derivatives of f, of all orders, exist and are continuous functions). Assume that, at every point of the level curve

$$C = \{(x, y) \in \mathbf{R}^2 \mid f(x, y) = 0\},$$

$\partial f/\partial x$ and $\partial f/\partial y$ are not both zero. If P is a point of C, with coordinates (x_0, y_0), say, there is a regular parametrised curve $\gamma(t)$, defined on an open interval containing 0, such that γ passes through P when $t = 0$ and $\gamma(t)$ is contained in C for all t.

The proof of this theorem makes use of the inverse function theorem (one version of which has already been used in the proof of Proposition 1.5). For the moment, we shall only try to convince the reader of the truth of this theorem. The proof will be given in a later exercise (Exercise 4.31) after the inverse for some value of θ, then $x^2+y^2=a^2$, showing that the helix lies on the cylinder with axis the z-axis and radius $|a|$; the positive number $|a|$ is called the *radius* of the helix. As θ increases by 2π, the point $(a\cos\theta, a\sin\theta, b\theta)$ rotates once round the z-axis and moves up the z-axis by $2\pi b$; the positive number $2\pi|b|$ is called the *pitch* of the helix (we take absolute values since we did not assume that a or b is positive).

Let us compute the curvature of the helix using the formula in Proposition 2.1. Denoting $d/d\theta$ by a dot, we have

$$\dot{\gamma}(\theta) = (-a\sin\theta, a\cos\theta, b),$$
$$\therefore \parallel \dot{\gamma}(\theta) \parallel = \sqrt{a^2+b^2}.$$

This shows that $\dot{\gamma}(\theta)$ is never zero, so γ is regular (unless $a=b=0$, in which case the image of the helix is a single point). Hence, the formula in Proposition 2.1 applies, and we have

$$\ddot{\gamma} = (-a\cos\theta, -a\sin\theta, 0),$$
$$\ddot{\gamma}\times\dot{\gamma} = (-ab\sin\theta, ab\cos\theta, -a^2),$$
$$\therefore \quad \kappa = \frac{\parallel (-ab\sin\theta, ab\cos\theta, -a^2)\parallel}{\parallel (-a\sin\theta, a\cos\theta, b)\parallel^3} = \frac{(a^2b^2+a^4)^{1/2}}{(a^2+b^2)^{3/2}} = \frac{|a|}{a^2+b^2}. \qquad (3)$$

Thus, the curvature of the helix is constant.

Let us examine some limiting cases to see if this result agrees with what we already know. First, suppose that $b=0$ (but $a\neq 0$). Then, the helix is simply a circle in the xy-plane of radius $|a|$, so by the calculation following Definition 1.1 its curvature is $1/|a|$. On the other hand, the formula (3) gives the curvature as

$$\frac{|a|}{a^2+0^2} = \frac{|a|}{a^2} = \frac{|a|}{|a|^2} = \frac{1}{|a|}.$$

Next, suppose that $a=0$ (but $b\neq 0$). Then, the image of the helix is just the z-axis, a straight line, so the curvature is zero. And (3) gives zero when $a=0$ too.

EXERCISES

2.1 Compute the curvature of the following curves:

(i) $\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right)$;

(ii) $\gamma(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$;

(iii) $\gamma(t) = (t, \cosh t)$;

(iv) $\gamma(t) = (\cos^3 t, \sin^3 t)$.

For the astroid in (iv), show that the curvature tends to ∞ as we approach one of the points $(\pm 1, 0)$, $(0, \pm 1)$. Compare with the sketch found in Exercise 1.5.

2.2 Show that, if the curvature $\kappa(t)$ of a regular curve $\gamma(t)$ is > 0 everywhere, then $\kappa(t)$ is a smooth function of t. Give an example to show that this may not be the case without the assumption that $\kappa > 0$.

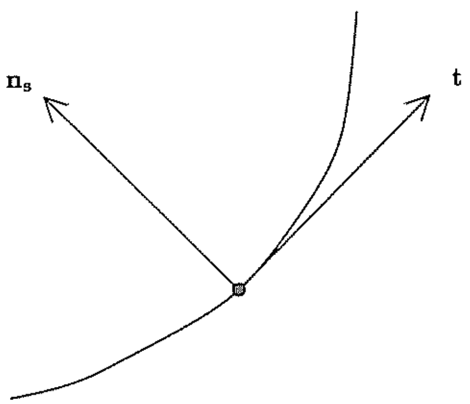
2.2. Plane Curves

For plane curves, it is possible to refine the definition of curvature slightly and give it an appealing geometric interpretation.

Suppose that $\gamma(s)$ is a unit-speed curve in $\mathbf{R}^2$. Denoting d/ds by a dot, let

$$\mathbf{t} = \dot{\gamma}$$

be the tangent vector of γ; note that $\mathbf{t}$ is a unit vector. There are two unit vectors perpendicular to $\mathbf{t}$; we make a choice by defining $\mathbf{n}_s$, the *signed unit normal of* γ, to be the unit vector obtained by rotating $\mathbf{t}$ anti-clockwise by $\pi/2$.



By Proposition 1.2, $\dot{\mathbf{t}} = \ddot{\gamma}$ is perpendicular to $\mathbf{t}$, and hence parallel to $\mathbf{n}_s$. Thus, there is a number κ_s such that

$$\ddot{\gamma} = \kappa_s \mathbf{n}_s.$$

The scalar κ_s is called the *signed curvature* of γ (it can be positive, negative or zero). Note that, since $\| \mathbf{n}_s \| = 1$, we have

$$\kappa = \| \ddot{\gamma} \| = \| \kappa_s \mathbf{n}_s \| = |\kappa_s|, \qquad (4)$$

so the curvature of γ is the absolute value of its signed curvature. The following diagrams show how the sign of the signed curvature is determined (in each case, the arrow on the curve indicates the direction of increasing s).