Comparison Geometry for Ricci Curvature

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A Ricci curvature bound is weaker than a sectional curvature bound but stronger than a scalar curvature bound. Ricci curvature is also special that it occurs in the Einstein equation and in the Ricci flow. Comparison geometry plays a very important role in the study of manifolds with lower Ricci curvature bound, especially the Laplacian and the Bishop-Gromov volume comparisons. Many important tools and results for manifolds with Ricci curvature lower bound follow from or use these comparisons, e.g. Meyers' theorem, Cheeger-Gromoll's splitting theorem, Abresch-Gromoll's excess estimate, Cheng-Yau's gradient estimate, Milnor's result on fundamental group. We will present the Laplacian and the Bishop-Gromov volume comparison theorems in the first lecture, then discuss their generalizations to integral Ricci curvature, Bakry-Emery Ricci tensor and Ricci flow in the rest of lectures.

Contents

Chapter 1

Basic Tools and Characterizations of Ricci Curvature Lower Bound

The most basic tool in studying manifolds with Ricci curvature bound is the Bochner formula, which measures the non-commutativity of the covariant derivative and the connection Laplacian. Applying the Bochner formula to distance functions we get important tools like mean curvature and Laplacian comparison theorems, volume comparison theorem. Each of these tools can be used to give a characterization of the Ricci curvature lower bound. These tools have many applications, see next two chapters.

1.1 Bochner's formula

For a smooth function u on a Riemannian manifold (M^n, g), the gradient of u is the vector field ∇u such that $\langle \nabla u, X\rangle = X(u)$ for all vector fields X on M. The Hessian of u is the symmetric bilinear form

$$\mathrm{Hess}\,(u)(X,Y) = XY(u) - \nabla_X Y(u) = \langle \nabla_X \nabla u, Y\rangle,$$

and the Laplacian is the trace $\Delta u = \mathrm{tr}(\mathrm{Hess}\, u)$. For a bilinear form A, we denote $|A|^2 = \mathrm{tr}(AA^t)$.

The Bochner formula for functions is

Theorem 1.1.1 (Bochner's Formula) *For a smooth function u on a Riemannian manifold (M^n, g),*

$$\frac{1}{2}\Delta|\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{1.1.1}$$

Proof: We can derive the formula by using local geodesic frame and commuting the derivatives. Fix $x \in M$, let $\{e_i\}$ be an orthonormal frame in a neighborhood of x such that, at x, $\nabla_{e_i} e_j(x) = 0$ for all i, j. At x,

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\sum_i e_i e_i \langle \nabla u, \nabla u\rangle \\
&= \sum_i e_i \langle \nabla_{e_i}\nabla u, \nabla u\rangle = \sum_i e_i \mathrm{Hess}\, u(e_i, \nabla u) \\
&= \sum_i e_i \mathrm{Hess}\, u(\nabla u, e_i) = \sum_i e_i \langle \nabla_{\nabla u}\nabla u, e_i\rangle \\
&= \sum_i \langle \nabla_{e_i}\nabla_{\nabla u}\nabla u, e_i\rangle \\
&= \sum_i \left[\langle \nabla_{\nabla u}\nabla_{e_i}\nabla u, e_i\rangle + \langle \nabla_{[e_i,\nabla u]}\nabla u, e_i\rangle + \langle R(e_i,\nabla u)\nabla u, e_i\rangle\right] \qquad (1.1.2)
\end{aligned}$$

Now at x,

$$\begin{aligned}
\sum_i \langle \nabla_{\nabla u}\nabla_{e_i}\nabla u, e_i\rangle &= \sum_i \left[\nabla u\langle \nabla_{e_i}\nabla u, e_i\rangle - \langle \nabla_{e_i}\nabla u, \nabla_{\nabla u} e_i\rangle\right] \\
&= \nabla u(\Delta u) = \langle \nabla u, \nabla(\Delta u)\rangle, \qquad (1.1.3)
\end{aligned}$$

and

$$\begin{aligned}
\sum_i \langle \nabla_{[e_i,\nabla u]}\nabla u, e_i\rangle &= \sum_i \mathrm{Hess}\, u([e_i,\nabla u], e_i) \\
&= \sum_i \mathrm{Hess}\, u(e_i, \nabla_{e_i}\nabla u) \\
&= \sum_i \langle \nabla_{e_i}\nabla u, \nabla_{e_i}\nabla u\rangle = |\mathrm{Hess}\, u|^2. \qquad (1.1.4)
\end{aligned}$$

Combining (1.1.2), (1.1.3) and (1.1.4) gives (1.1.1). ■

Applying the Cauchy-Schwarz inequality $|\mathrm{Hess}\, u|^2 \geq \frac{(\Delta u)^2}{n}$ to (1.1.1) we obtain the following inequality

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u), \qquad (1.1.5)$$

with equality if and only if $\mathrm{Hess}\, u = hI_n$ for some $h \in C^\infty(M)$. If $\mathrm{Ric} \geq (n-1)H$, then

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + (n-1)H|\nabla u|^2. \qquad (1.1.6)$$

The Bochner formula simplifies whenever $|\nabla u|$ or Δu are simply. Hence it is natural to apply it to the distance functions, harmonic functions, and the eigenfunctions among others, getting many applications. The formula has a more general version (Weitzenböck type) for vector fields (1-forms).

1.2 Mean Curvature and Local Laplacian Comparison

Here we apply the Bochner formula to distance functions. We call $\rho : U \to \mathbb{R}$, where $U \subset M^n$ is open, is a distance function if $|\nabla \rho| \equiv 1$ on U.

Example 1.2.1 *Let $A \subset M$ be a submanifold, then $\rho(x) = d(x, A) = \inf\{d(x, y) | y \in A\}$ is a distance function on some open set $U \subset M$. When $A = q$ is a point, the distance function $r(x) = d(q, x)$ is smooth on $M \setminus \{q, C_q\}$, where C_q is the cut locus of q. When A is a hypersurface, $\rho(x)$ is smooth outside the focal points of A.*

For a smooth distance function $\rho(x)$, $\operatorname{Hess} \rho$ is the covariant derivative of the normal direction $\partial_r = \nabla \rho$. Hence $\operatorname{Hess} \rho = II$, the second fundamental form of the level sets $\rho^{-1}(r)$, and $\Delta \rho = m$, the mean curvature. For $r(x) = d(q, x)$, $m(r, \theta) \sim \frac{n-1}{r}$ as $r \to 0$; for $\rho(x) = d(x, A)$, where A is a hypersurface, $m(y, 0) = m_A$, the mean curvature of A, for $y \in A$.

Putting $u(x) = \rho(x)$ in (1.1.1), we obtain the Riccati equation along a radial geodesic,

$$0 = |II|^2 + m' + \operatorname{Ric}(\partial_r, \partial_r). \tag{1.2.1}$$

By the Cauchy-Schwarz inequality,

$$|II|^2 \geq \frac{m^2}{n-1}.$$

Thus we have the Riccati inequality

$$m' \leq -\frac{m^2}{n-1} - \operatorname{Ric}(\partial_r, \partial_r). \tag{1.2.2}$$

If $\operatorname{Ric}_{M^n} \geq (n-1)H$, then

$$m' \leq -\frac{m^2}{n-1} - (n-1)H. \tag{1.2.3}$$

From now on, unless specified otherwise, we assume $m = \Delta r$, the mean curvature of geodesic spheres. Let M_H^n denote the complete simply connected space of constant curvature H and m_H (or m_H^n when dimension is needed) the mean curvature of its geodesics sphere, then

$$m_H' = -\frac{m_H^2}{n-1} - (n-1)H. \tag{1.2.4}$$

Let $\operatorname{sn}_H(r)$ be the solution to

$$\operatorname{sn}_H'' + H \operatorname{sn}_H = 0$$

such that $\mathrm{sn}_H(0) = 0$ and $\mathrm{sn}'_H(0) = 1$, i.e. sn_H are the coefficients of the Jacobi fields of the model spaces M^n_H:

$$\mathrm{sn}_H(r) = \begin{cases} \frac{1}{\sqrt{H}} \sin \sqrt{H} r & H > 0 \\ r & H = 0 \\ \frac{1}{\sqrt{|H|}} \sinh \sqrt{|H|} r & H < 0 \end{cases} . \tag{1.2.5}$$

Then

$$m_H = (n-1) \frac{\mathrm{sn}'_H}{\mathrm{sn}_H}. \tag{1.2.6}$$

As $r \to 0$, $m_H \sim \frac{n-1}{r}$. The mean curvature comparison is

Theorem 1.2.2 (Mean Curvature Comparison) *If* $\mathrm{Ric}_{M^n} \geq (n-1)H$, *then along any minimal geodesic segment from* q,

$$m(r) \leq m_H(r). \tag{1.2.7}$$

Moreover, equality holds if and only if all radial sectional curvatures are equal to H.

Since $\lim_{r \to 0}(m - m_H) = 0$, this follows from the Riccati equation comparison. However, a direct proof using only the Riccati inequalities (1.2.3), (1.2.4) does not seem to be in the literature. From (1.2.3) and (1.2.4) we have

$$(m - m_H)' \leq -\frac{1}{n-1}(m^2 - m_H^2). \tag{1.2.8}$$

Here we present three somewhat different proofs. The first proof uses the continuity method, the second solving linear ODE, the third by considering $\mathrm{sn}^2_H(m - m_H)$ directly. The last two proofs are motived from generalizations of the mean curvature comparison to weaker Ricci curvature lower bounds [111, **?**], allowing natural extensions, see Chapters **??**.
Proof I: Let $m^H_+ = (m - m_H)_+ = \max\{m - m_H, 0\}$, amount of mean curvature comparison failed. By (1.2.8)

$$(m^H_+)' \leq -\frac{1}{n-1}(m + m_H) m^H_+.$$

If $m + m_H \geq 0$, then $(m^H_+)' \leq 0$. When r is small, m is close to m_H, so $m + m_H > 0$. Therefore $m^H_+ = 0$ for all r small. Let r_0 be the biggest number such that $m^H_+(r) = 0$ on $[0, r_0]$ and $m^H_+ > 0$ on $(r_0, r_0 + \epsilon_0]$ for some $\epsilon_0 > 0$. We have $r_0 > 0$. Claim: $r_0 =$ the maximum of r, where m, m_H are defined on $(0, r]$. Otherwise, we have on $(r_0, r_0 + \epsilon_0]$

$$\frac{(m^H_+)'}{m^H_+} \leq -\frac{1}{n-1}(m + m_H) \tag{1.2.9}$$

and m, m_H are bounded. Integrate (1.2.9) from $r_0+\epsilon$ to $r_0+\epsilon_0$ (where $0<\epsilon<\epsilon_0$) gives

$$\ln \frac{m_+^H(r_0+\epsilon_0)}{m_+^H(r_0+\epsilon)} \le \int_{r_0+\epsilon}^{r_0+\epsilon_0} -\frac{1}{n-1}(m+m_H)dr.$$

The right hand side is bounded by $C\epsilon_0$ since m, m_H are bounded on $(r_0, r_0+\epsilon_0]$. Therefore $m_+^H(r_0+\epsilon_0) \le m_+^H(r_0+\epsilon)e^{C\epsilon_0}$. Now let $\epsilon \to 0$ we get $m_+^H(r_0+\epsilon_0) \le 0$, which is a contradiction.

Proof II: We only need to work on the interval where $m - m_H \ge 0$. On this interval $-(m^2 - m_H^2) = -m_+^H(m - m_H + 2m_H) = -m_+^H(m_+^H + 2m_H)$. Thus (1.2.8) gives

$$(m_+^H)' \le -\frac{(m_+^H)^2}{n-1} - 2\frac{m_+^H \cdot m_H}{n-1} \le -2\frac{m_+^H \cdot m_H}{n-1} = -2\frac{sn_H'}{sn_H}m_+^H.$$

Hence $(sn_H^2 m_+^H)' \le 0$. Since $sn_H^2(0)m_+^H(0) = 0$, we have $sn_H^2 m_+^H \le 0$ and $m_+^H \le 0$. Namely $m \le m_H$.

Proof III: We have

$$\begin{aligned}
\left(\mathrm{sn}_H^2(m-m_H)\right)' &= 2\mathrm{sn}_H'\mathrm{sn}_H(m-m_H) + sn_H^2(m-m_H)' \\
&\le \frac{2}{n-1}\mathrm{sn}_H^2 m_H(m-m_H) - \frac{1}{n-1}\mathrm{sn}_H^2(m^2-m_H^2) \\
&= -\frac{\mathrm{sn}_H^2}{n-1}(m-m_H)^2 \le 0
\end{aligned}$$

Here in the 2nd line we have used (1.2.8) and (1.2.6).

Since $\lim_{r\to 0}\mathrm{sn}_H^2(m-m_H) = 0$, integrating from 0 to r yields

$$\mathrm{sn}_H^2(r)(m(r) - m_H(r)) \le 0,$$

which gives (6.3.3).

When equality occurs, the Cauchy- Schwarz inequality is an equality, which means $II = \frac{sn_H'}{sn_H} I_{n-1}$ along the minimal geodesic. Therefore all radial sectional curvatures are equal to H. ■

Recall that $m = \Delta r$. From (6.3.3), we get the local Laplacian comparison for distance functions

$$\Delta r \le \Delta_H r, \quad \text{for all } x \in M \setminus \{q, C_q\}. \tag{1.2.10}$$

The local Laplacian comparison immediately gives us Myers' theorem [97], a diameter comparison. Let S_H^n be the sphere with radius $1/\sqrt{H}$.

Theorem 1.2.3 (Myers, 1941) *If* $\mathrm{Ric}_M \ge (n-1)H > 0$, *then* $\mathrm{diam}(M) \le \mathrm{diam}(S_H^n) = \pi/\sqrt{H}$. *In particular,* $\pi_1(M)$ *is finite.*

Proof: If $\mathrm{diam}(M) > \pi/\sqrt{H}$, let $q, q' \in M$ such that $d(q,q') = \pi/\sqrt{H} + \epsilon$ for some $\epsilon > 0$, and γ be a minimal geodesic connecting q, q' with $\gamma(0) = q, \gamma(\pi/\sqrt{H}+\epsilon) = q'$. Then $\gamma(t) \notin C_q$ for all $0 < t \leq \pi/\sqrt{H}$. Let $r(x) = d(q,x)$, then r is smooth at $\gamma(\pi/\sqrt{H})$, therefore Δr is well defined at $\gamma(\pi/\sqrt{H})$. By (1.2.10) $\Delta r \leq \Delta_H r$ at all $\gamma(t)$ with $0 < t < \pi/\sqrt{H}$. Now $\lim_{r\to\pi/\sqrt{H}} \Delta_H r = -\infty$ so Δr is not defined at $\gamma(\pi/\sqrt{H})$. This is a contradiction. ■

Equation (1.2.4) also holds when $m_H = \Delta d(x, A_H)$, where $A_H \subset M^n_H$ is a hypersurface. Therefore the proof of Theorem 1.2.2 carries over, and we have a comparison of the mean curvature of level sets of $d(x,A)$ and $d(x,A_H)$ when A and A_H are hypersurfaces with $m_A \leq m_{A_H}$ and $\mathrm{Ric}_M \geq (n-1)H$. Equation (1.2.4) doesn't hold if A_H is a submanifold which is not a point or hypersurface, therefore one needs stronger curvature assumption to do comparison [**?**].

1.3 Global Laplacian Comparison

The Laplacian comparison (1.2.10) holds globally in various weak senses and the standard PDE theory carries over. As a result the Laplacian comparison is very powerful, see next Chapter for some crucial applications.

First we prove an important property about cut locus.

Lemma 1.3.1 *For each $q \in M$, the cut locus C_q has measure zero.*

One can show C_q has measure zero by observing that the region inside the cut locus is star-shaped [27, Page 112]. The author comes up with the following argument in proving that Perelman's l-cut locus [107] has measure zero.

since the $\mathcal{L}$-exponential map is smooth and the l-distance function is locally Lipschitz. **Proof:** Recall that if $x \in C_q$, then either x is a (first) conjugate point of q or there are two distinct minimal geodesics connecting q and x [39], so $x \in$\{conjugate locus of q\} $\cup$ \{the set where r is not differentiable\}. The conjugate locus of q consists of the critical values of $\exp_q$. Since $\exp_q$ is smooth, by Sard's theorem, the conjugate locus has measure zero. The set where r is not differentiable has measure zero since r is Lipschitz. Therefore the cut locus C_q has measure zero. ■

First we review the definitions (for simplicity we only do so for the Laplacian) and study the relationship between these different weak senses.

Definition 1.3.2 *For a continuous function f on $M, q \in M$, a function f_q defined in a neighborhood U of q, is an upper (lower) barrier of f at q if f_q is $C^2(U)$ and*

$$f_q(q) = f(q), \quad f_q(x) \geq f(x) \quad (f_q(x) \leq f(x)) \quad (x \in U). \tag{1.3.11}$$

Definition 1.3.3 *For a continuous function f on M, we say $\Delta f(q) \leq c$ $(\Delta f(q) \geq c)$ in the barrier sense (f is a barrier subsolution to the equation $\Delta f = c$ at q), if for all $\epsilon > 0$, there exists an upper (lower) barrier $f_{q,\epsilon}$ such that $\Delta f_{q,\epsilon}(q) \leq c + \epsilon$ $(\Delta f(q) \geq c - \epsilon)$.*

This notion was defined by Calabi [25] back in 1958 (he used the terminology "weak sense" rather than "barrier sense"). A weaker version is in the sense of viscosity, introduced by Crandall and Lions in [48].

Definition 1.3.4 *For a continuous function f on M, we say $\Delta f(q) \leq c$ in the viscosity sense (f is a viscosity subsolution of $\Delta f = c$ at q), if $\Delta\phi(q) \leq c$ whenever $\phi \in C^2(U)$ and $(f-\phi)(q) = \inf_U(f-\phi)$, where U is a neighborhood of q.*

Clearly barrier subsolutions are viscosity subsolutions.

Another very useful notion is subsolution in the sense of distributions.

Definition 1.3.5 *For continuous functions f, h on an open domain $\Omega \subset M$, we say $\Delta f \leq h$ in the distribution sense (f is a distribution subsolution of $\Delta f = h$) on Ω, if $\int_\Omega f\Delta\phi \leq \int_\Omega h\phi$ for all $\phi \geq 0$ in $C_0^\infty(\Omega)$.*

By [73] if f is a viscosity subsolution of $\Delta f = h$ on Ω, then it is also a distribution subsolution and vice verse, see also [83], [71, Theorem 3.2.11].

For geometric applications, the barrier and distribution sense are very useful and the barrier sense is often easy to check. Viscosity gives a bridge between them. As observed by Calabi [25] one can easily construct upper barriers for the distance function.

Lemma 1.3.6 *If γ is minimal from p to q, then for all $\epsilon > 0$, the function $r_{q,\epsilon}(x) = \epsilon + d(x, \gamma(\epsilon))$, is an upper barrier for the distance function $r(x) = d(p,x)$ at q.*

Since $r_{q,\epsilon}$ trivially satisfies (1.3.2) the lemma follows by observing that it is smooth in a neighborhood of q.

Upper barriers for Perelman's l-distance function can be constructed very similarly.

Therefore the Laplacian comparison (1.2.10) holds globally in all the weak senses above. Cheeger-Gromoll (unaware of Calabi's work at the time) had proved the Laplacian comparison in the distribution sense directly by observing the very useful fact that near the cut locus ∇r points towards the cut locus [40], see also [31]. (However it is not clear if this fact holds for Perelman's l-distance function.)

One reason why these weak subsolutions are so useful is that they still satisfy the following classical Hopf strong maximum principle, see [25], also e.g. [31] for the barrier sense, see [84, 75] for the distribution and viscosity senses, also [71, Theorem 3.2.11] in the Euclidean case.

Theorem 1.3.7 (Strong Maximum Principle) *If on a connected open set, $\Omega \subset M^n$, the function f has an interior minimum and $\Delta f \leq 0$ in any of the weak senses above, then f is constant on Ω.*

These weak solutions also enjoy the regularity (e.g. if f is a weak sub and sup solution of $\Delta f = 0$, then f is smooth), see e.g. [57].

The Laplacian comparison also works for radial functions (functions composed with the distance function). In geodesic polar coordinate, we have

$$\Delta f = \tilde{\Delta} f + m(r,\theta)\frac{\partial}{\partial r} f + \frac{\partial^2 f}{\partial r^2}, \tag{1.3.12}$$

where $\tilde{\Delta}$ is the induced Laplacian on the sphere and $m(r,\theta)$ is the mean curvature of the geodesic sphere in the inner normal direction. Therefore

Theorem 1.3.8 (Global Laplacian Comparison) *If* $\mathrm{Ric}_{M^n} \geq (n-1)H$, *in all the weak senses above, we have*

$$\Delta f(r) \leq \Delta_H f(r) \quad (\textit{if } f' \geq 0), \tag{1.3.13}$$
$$\Delta f(r) \geq \Delta_H f(r) \quad (\textit{if } f' \leq 0). \tag{1.3.14}$$

1.4 Volume Comparison

1.4.1 Volume of Riemannian Manifold

How do we compute the volume of Riemannian manifold? Recall that for a subset $U \subset \mathbb{R}^n$, we define

$$\mathrm{Vol}(U) = \int_U 1\, dvol = \int_U 1\, dx_1 \cdots dx_n,$$

where $x_1, \cdots, x_n$ are the standard coordinate. One can compute it with different coordinates by using the change of variable formula.

Lemma 1.4.1 (Change of Variables Formula) *Suppose* $U, V \subset \mathbb{R}^n$ *and that* $\psi : V \to U$ *is a diffeomorphism. Suppose* $\psi(x) = y$*. Then*

$$\int_U dvol = \int_U 1\, dy_1 \cdots dy_n = \int_V |Jac(\psi)|\, dx_1 \cdots dx_n.$$

For a general Riemannian manifold M^n, let

$$\psi_\alpha : U_\alpha \to \mathbb{R}^n$$

be a chart and set $E_{ip} = (\psi_\alpha^{-1})_*(\frac{\partial}{\partial x_i})$. In general, the E_{ip}'s are not orthonormal. Let $\{e_k\}$ be an orthonormal basis of T_pM. Then

$$E_{ip} = \sum_{k=1}^{n} a_{ik} e_k.$$

The volume of the parallelepiped spanned by $\{E_{ip}\}$ is $|\det(a_{ik})|$. Now

$$g_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk},$$

so $\det(g_{ij}) = (\det(a_{ij}))^2$. Thus

$$\mathrm{Vol}(U_\alpha) = \int_{\psi(U_\alpha)} \sqrt{|\det(g_{ij})|} \circ (\psi_\alpha^{-1})\, dx_1 \cdots dx_n.$$

By the change of variables formula in $\mathbb{R}^n$ (Lemma 1.4.1), this volume is well defined, namely it is independent of local coordinate charts.

Definition 1.4.2 (Volume Form) *Any term*

$$dvol = \sqrt{|\det(g_{ij})|} \circ (\psi_\alpha^{-1})\, dx_1 \cdots dx_n$$

is called a volume density element, or volume form, on M.

Now we can compute the volume of M by partition of unity,

$$\mathrm{Vol}(M) = \int_M 1\, dvol = \sum_\alpha \int_{\psi(U_\alpha)} f_\alpha\, dvol,$$

where $\{U_\alpha\}$ are coordinate charts covering M, $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$.

Since partitions of unity are not practically effective, we look for charts that cover all but a measure zero set. Since the cut locus has measure zero, the best is the exponential coordinate. For $q \in M^n$, let $D_q \subset T_qM$ be the segment disk. Then

$$\exp_q : D_q \to M \setminus C_q$$

is a diffeomorphism. We can either use Euclidean coordinates or polar coordinates on D_q. For balls it is convenient to use polar coordinate. From the diffeomorphism

$$\exp_q : D_q \setminus \{0\} \to M \setminus (C_q \cup \{q\}),$$

set

$$E_i = (\exp_q)_*(\frac{\partial}{\partial \theta_i})$$

and

$$E_n = (\exp_q)_*(\frac{\partial}{\partial r}).$$

To compute the g_{ij}'s, we want explicit expressions for E_i and E_n. Since $\exp_q$ is a radial isometry, $g_{nn} = 1$ and $g_{ni} = 0$ for $1 \le i < n$. Let $J_i(r, \theta)$ be the Jacobi field with $J_i(0) = 0$ and $J_i{}'(0) = \frac{\partial}{\partial \theta_i}$. Then

$$E_i(\exp_q(r, \theta)) = J_i(r, \theta).$$

If we write J_i and $\frac{\partial}{\partial r}$ in terms of an orthonormal basis $\{e_k\}$, we have $J_i = \sum_{k=1}^{n} a_{ik} e_k$. Thus

$$\sqrt{\det(g_{ij})(r,\theta)} = |\det(a_{ik})| = ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}||.$$

Let

$$\mathcal{A}(r,\theta) = ||J_1 \wedge \cdots \wedge J_{n-1} \wedge \frac{\partial}{\partial r}||, \tag{1.4.1}$$

the volume density, or volume element, of M is

$$dvol = \mathcal{A}(r,\theta)\, dr d\theta_{n-1}.$$

Example 1.4.3 *$\mathbb{R}^n$ has Jacobi equation $J'' = R(T,J)T$. Thus, if $J(0) = 0$ and $J'(0) = \frac{\partial}{\partial \theta_i}$,*

$$J(r) = r\frac{\partial}{\partial \theta_i}.$$

Hence the volume element is

$$dvol = r^{n-1}\, dr d\theta_{n-1}.$$

Example 1.4.4 *S^n has $J_i(r) = \sin(r)\frac{\partial}{\partial \theta_i}$. Hence*

$$dvol = \sin^{n-1}(r)\, dr d\theta_{n-1}.$$

Example 1.4.5 *$\mathbb{H}^n$ has $J_i(r) = \sinh(r)\frac{\partial}{\partial \theta_i}$. Hence*

$$dvol = \sinh^{n-1}(r)\, dr d\theta_{n-1}.$$

In fact for space form M_H^n, the volume element is $dvol = sn_H^{n-1}(r)\, dr d\theta_{n-1}$, where sn_H is defined in (1.2.5).

Example 1.4.6 *We can compute the volume of unit disk in $\mathbb{R}^n$.*

$$\omega_n = \int_{S^{n-1}} \int_0^1 r^{n-1}\, dr d\theta_{n-1} = \frac{1}{n}\int_{S^{n-1}} d\theta_{n-1},$$

noting that

$$\int_{S^{n-1}} d\theta_{n-1} = \frac{2(\pi)^{n/2}}{\Gamma(n/2)}.$$

1.4.2 Comparison of Volume Elements

Theorem 1.4.7 *Suppose M^n has $\mathrm{Ric}_M \geq (n-1)H$. Let $dvol = \mathcal{A}(r,\theta)\, drd\theta_{n-1}$ be the volume element of M in geodesic polar coordinate at q and let $dvol_H = \mathcal{A}_H(r,\theta)\, drd\theta_{n-1}$ be the volume element of the model space M^n_H. Then*

$$\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)} \text{ is nonincreasing along any minimal geodesic segment from } q. \tag{1.4.2}$$

This follows from the following lemma and the mean curvature comparison.

Lemma 1.4.8 *The relative rate of change of the volume element is given by the mean curvature,*

$$\frac{\mathcal{A}'}{\mathcal{A}}(r,\theta) = m(r,\theta). \tag{1.4.3}$$

Proof: Let γ be a unit speed geodesic with $\gamma(0) = q$, $J_i(r)$ be the Jacobi field along γ with $J_i(0) = 0$ and $J_i'(0) = \frac{\partial}{\partial\theta_i}$ for $i = 1 \cdots n-1$ and $J_n'(0) = \gamma'(0)$ where $\{\frac{\partial}{\partial\theta_1}, \frac{\partial}{\partial\theta_2}, \cdots, \gamma'(0)\}$ is an orthonormal basis of T_qM. Recall

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)} = \frac{||J_1 \wedge \cdots \wedge J_n||'}{||J_1 \wedge \cdots \wedge J_n||}.$$

For any $r = r_0$ such that $\gamma|_{[0,r_0+\epsilon)}$ is minimal, let $\{\bar{J}_i(r_0)\}$ be an orthonormal basis of $T_{\gamma(r_0)}M$ with $\bar{J}_n(r_0) = \gamma'(r_0)$. Since we are inside the cut locus, there are no conjugate points. Therefore, $\{J_i(r_0)\}$ is also a basis of $T_{\gamma(r_0)}M$. So we can write

$$\bar{J}_i(r_0) = \sum_{k=1}^n b_{ik} J_k(r_0).$$

For all $0 \leq r < r_0 + \epsilon$, define

$$\bar{J}_i(r) = \sum_{k=1}^n b_{ik} J_k(r).$$

Then $\{\bar{J}_i\}$ are Jacobi fields along γ which is an orthonormal basis at $\gamma(r_0)$. Since

$$||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n|| = \det(b_{ij})||J_1 \wedge \cdots \wedge J_n||$$

for all $r \in [0, r_0 + \epsilon)$, and $b_{ij}s$ are constant,

$$\frac{||J_1 \wedge \cdots \wedge J_n||'}{||J_1 \wedge \cdots \wedge J_n||}(r) = \frac{||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||'}{||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||}(r).$$

At r_0, $||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||(r_0) = 1$. Therefore

$$\begin{aligned} \frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)}(r_0) &= ||\bar{J}_1 \wedge \cdots \wedge \bar{J}_n||'(r_0) \\ &= \sum_{k=1}^n ||\bar{J}_1 \wedge \cdots \wedge \bar{J}_k' \wedge \cdots \wedge \bar{J}_n||. \end{aligned} \tag{1.4.4}$$

Since $\{\bar{J}_i(r_0)\}$ is an orthonormal basis of $T_{\gamma(r_0)}M$ we have that

$$\bar{J}_k'(r_0) = \sum_{l=1}^{n} \langle \bar{J}_k'(r_0), \bar{J}_l(r_0)\rangle \bar{J}_l(r_0).$$

Plug this into (1.4.4), we get

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)}(r_0) = \sum_{k=1}^{n} \langle \bar{J}_k'(r_0), \bar{J}_k(r_0)\rangle = \sum_{k=1}^{n-1} \langle \nabla_{\bar{J}_k}\gamma', \bar{J}_k\rangle(r_0) = m(r_0, \gamma'(0)).$$

■

Proof of Theorem 1.4.7: By (1.4.3),

$$\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right)' = \frac{m\mathcal{A}\mathcal{A}_H - \mathcal{A}m_H\mathcal{A}_H}{\mathcal{A}_H^2} = (m - m_H)\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}.$$

The mean curvature comparison (6.3.3) gives $m - m_H \le 0$, therefore $\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}$ is nonincreasing in r. ■

1.4.3 Volume Comparison

Integrating (1.4.2) along the sphere directions, and then the radial direction gives the relative volume comparison of geodesic spheres and balls. Let $\Gamma_r = \{\theta \in S^{n-1} |$ the normal geodesic γ with $\gamma(0) = x, \gamma'(0) = \theta$ has $d(\gamma(0), \gamma(r)) = r\}$. Then the volume of the geodesic sphere, $S(x,r) = \{y \in M | d(x,y) = r\}$, $A(x,r) = \int_{\Gamma_r} \mathcal{A}(r,\theta)d\theta_{n-1}$. Extend $\mathcal{A}(r,\theta)$ by zero to all S^{n-1}, we have $A(x,r) = \int_{S^{n-1}} \mathcal{A}(r,\theta)d\theta_{n-1}$. Let $A_H(r)$ be the volume of the geodesic sphere in the model space. If $\mathcal{A}(r_0,\theta) = 0$, then $\mathcal{A}(r_0,\theta) = 0$ for all $r \ge r_0$, so (1.4.2) also holds in the extended region.

Theorem 1.4.9 (Bishop-Gromov's Relative Volume Comparison) *Suppose M^n has $\mathrm{Ric}_M \ge (n-1)H$. Then*

$$\frac{A(x,r))}{A_H(r))} \text{ and } \frac{\mathrm{Vol}\,(B(x,r))}{\mathrm{Vol}_H(B(r))} \text{ are nonincreasing in } r. \tag{1.4.5}$$

In particular,

$$\mathrm{Vol}\,(B(x,r)) \le \mathrm{Vol}_H(B(r)) \quad \text{for all } \ r > 0, \tag{1.4.6}$$

$$\frac{\mathrm{Vol}\,(B(x,r))}{\mathrm{Vol}\,(B(x,R))} \ge \frac{\mathrm{Vol}_H(B(r))}{\mathrm{Vol}_H(B(R))} \quad \text{for all } \ 0 < r \le R, \tag{1.4.7}$$

and equality holds if and only if $B(x,r)$ is isometric to $B_H(r)$.

Proof:

$$\frac{d}{dr}\left(\frac{A(x,r)}{A_H(r)}\right) = \frac{1}{\mathrm{Vol}S^{n-1}}\int_{S^{n-1}}\frac{d}{dr}\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right)d\theta_{n-1} \le 0.$$

The monotonicity of the ratio of volume of balls follows from this and the lemma below since $\mathrm{Vol}(B(x,r)) = \int_0^r A(x,t)dt$. ∎

Lemma 1.4.10 *If $f(t)/g(t)$ is nonincreasing in t, with $g(t) > 0$, then*

$$H(r,R) = \frac{\int_r^R f(t)dt}{\int_r^R g(t)dt}$$

is nonincreasing in r and R.

Proof: We have

$$\frac{\partial}{\partial r}H(r,R) = \frac{-f(r)\int_r^R g(t)dt + g(r)\int_r^R f(r)dr}{\left(\int_r^R g(t)dt\right)^2}.$$

Now

$$\frac{f(t)}{g(t)} \le \frac{f(r)}{g(r)}$$

implies

$$g(r)f(t) \le f(r)g(t),$$

so

$$\int_r^R g(r)f(t)dt \le \int_r^R f(r)g(t)dt.$$

Thus $\frac{\partial}{\partial r}H(r,R) \le 0$. Similarly we have $\frac{\partial}{\partial R}H(r,R) \le 0$. ∎

Instead of integrating (1.4.2) along the whole unit sphere and/or radial direction, we can integrate along any sector of S^{n-1} and/or segment of the radial direction. Fix $x \in M^n$, for any measurable set $B \subset M$, connect every point of $y \in B$ to x with a minimal geodesic γ_y such that $\gamma_y(0) = x, \gamma_y(1) = y$. For $t \in [0,1]$, let $B_t = \{\gamma_y(t) | y \in B\}$. Since cut locus C_x has measure zero, B_t is uniquely determined up to a modification on a null measure set. Integrating (1.4.2) gives immediately the following localized Bishop-Gromov volume comparison [101], see also [130, 35].

Proposition 1.4.11 *If $\mathrm{Ric}_M \ge (n-1)H$, then for $0 \le t \le 1$*

$$\mathrm{Vol}(B_t) \ge t\int_B \frac{\mathcal{A}_H(td(x,y))}{\mathcal{A}_H(d(x,y))}dvol_y. \tag{1.4.8}$$

Proof: Since B_t is obtained from B by scaling the radial direction with t, $\mathrm{Vol}(B_t) = \int_{\exp_x^{-1}(B)\cap D_x} t\mathcal{A}(tr,\theta)drd\theta$, where D_x is the injectivity domain of x in T_xM. From (1.4.2) we have $\mathcal{A}(tr,\theta) \geq \frac{\mathcal{A}_H(tr)}{\mathcal{A}_H(r)}\mathcal{A}(r,\theta)$. Therefore

$$\mathrm{Vol}(B_t) \geq t\int_{\exp_x^{-1}(B)\cap D_x} \frac{\mathcal{A}_H(tr)}{\mathcal{A}_H(r)}\mathcal{A}(r,\theta)drd\theta = t\int_B \frac{\mathcal{A}_H(td(x,y))}{\mathcal{A}_H(d(x,y))}dvol_y.$$

■

In particular, we have volume comparison for annulus and star-shaped sets. All these comparisons need to have the same centers or it will depends on the distance between the centers. When $H = 0$, we have the following comparison for nonconcentric balls which does not depends on the distance between the centers.

Proposition 1.4.12 (Volume Comparison for Nonconcentric Balls) *Given $x, y \in M$, $0 < r \leq R$ with $r + d(x,y) \leq R$, if $\mathrm{Ric}_M \geq 0$, then*

$$\frac{\mathrm{Vol}B(y,r)}{\mathrm{Vol}B(x,R)} \geq \left(\frac{r}{R}\right)^{3n}. \tag{1.4.9}$$

Proof: We will show that for any $\frac{1}{2} \leq \alpha < 1$ (so $-\ln\alpha \leq 2\ln(2-\alpha)$),

$$\frac{\mathrm{Vol}B(y,r)}{\mathrm{Vol}B(x,R)} \geq \alpha^n\left(\frac{r}{R}\right)^{3n}.$$

Let $\gamma : [0, d(x,y)] \to M$ be a minimizing geodesic from x to y. We will construct a sequence of increasing balls centered on γ. Let $k = \lfloor\frac{\ln(1+\frac{d(x,y)}{r})}{\ln(2-\alpha)}\rfloor + 2$ and $y_i = \gamma(d(x,y)+r-(2-\alpha)^{i-1}r)$ for $1 \leq i \leq k-1$, $y_k = x$, $r_i = \alpha(2-\alpha)^{i-2}r$, $R_i = (2-\alpha)^{i-1}r$. We have $B(y_{i+1}, r_{i+1}) \subset B(y_i, R_i) \subset B(x,R)$ for any $1 \leq i \leq k-1$ and $r_i/R_i = \alpha/(2-\alpha)$. Now

$$\begin{aligned}\frac{\mathrm{Vol}B(y_i,R_i)}{\mathrm{Vol}B(x,R)} &\geq \frac{\mathrm{Vol}B(y_{i+1},r_{i+1})}{\mathrm{Vol}B(x,R)}\\ &\geq \left(\frac{\alpha}{2-\alpha}\right)^n\frac{\mathrm{Vol}B(y_{i+1},R_{i+1})}{\mathrm{Vol}B(x,R)}.\end{aligned}$$

Since $y_1 = y, R_1 = r, y_k = x$, by iteration,

$$\begin{aligned}\frac{\mathrm{Vol}B(y,r)}{\mathrm{Vol}B(x,R)} &\geq \left(\frac{\alpha}{2-\alpha}\right)^{n(k-2)}\frac{\mathrm{Vol}B(y_{k-1},R_{k-1})}{\mathrm{Vol}B(x,R)}\\ &\geq \left(\frac{\alpha}{2-\alpha}\right)^{n(k-2)}\frac{\mathrm{Vol}B(y_k,r_k)}{\mathrm{Vol}B(x,R)}\\ &\geq \left(\frac{\alpha}{2-\alpha}\right)^{n(k-2)}\left(\frac{r_k}{R}\right)^n\\ &= \alpha^n\alpha^{n(k-2)}\left(\frac{r}{R}\right)^n.\end{aligned}$$

By the definition of k, $\alpha^{n(k-2)} \geq \alpha^{n\frac{\ln(1+\frac{d(x,y)}{r})}{\ln(2-\alpha)}} = \left(\frac{r}{r+d(x,y)}\right)^{\frac{-n\ln\alpha}{\ln(2-\alpha)}} \geq \left(\frac{r}{R}\right)^{2n}$. ■

The Bishop-Gromov volume comparison (Theorem 1.4.9) is a powerful result because it is a global comparison. The volume of any ball is bounded above by the volume of the corresponding ball in the model, and if the volume of a big ball has a lower bound, then all smaller balls also have lower bounds. Hence the volume comparison has many geometric and topological applications. Here we give two nice simple applications, see Chapter 3 for topological applications.

By Myers's theorem (Theorem 1.2.3) $\operatorname{diam}_M \leq \operatorname{diam}(S^n_H) = \pi/\sqrt{H}$ if $\operatorname{Ric}_M \geq (n-1)H > 0$. Cheng [45] gives a characterization of the equality case — a rigidity result.

Theorem 1.4.13 (Cheng's Maximal Diameter Theorem 1975) *Suppose M^n has $\operatorname{Ric}_M \geq (n-1)H > 0$. If $\operatorname{diam}_M = \pi/\sqrt{H}$, then M is isometric to the sphere S^n_H with radius $1/\sqrt{H}$.*

Proof: Cheng's original proof uses the rigidity of first eigenvalue comparsion (see Section 2.4.1). The following proof uses the rigidity of volume comparison, due to Shiohama [122]. Let $p, q \in M$ have $d(p,q) = \pi/\sqrt{H}$. By volume comparison (1.4.7)

$$\begin{aligned}\frac{\operatorname{Vol} B(p, \pi/(2\sqrt{H}))}{\operatorname{Vol} M} &= \frac{\operatorname{Vol} B(p, \pi/(2\sqrt{H}))}{\operatorname{Vol} B(p, \pi/\sqrt{H})} \\ &\geq \frac{\operatorname{Vol}_H B(\pi/(2\sqrt{H}))}{\operatorname{Vol}_H B(\pi/\sqrt{H})} = 1/2\end{aligned}$$

Thus $\operatorname{Vol} B(p, \pi/2\sqrt{H}) \geq (\operatorname{Vol} M)/2$. Similarly $\operatorname{Vol} B(q, \pi/2\sqrt{H}) \geq (\operatorname{Vol} M)/2$. Since the balls $B(p, \pi/2\sqrt{H})$ and $B(q, \pi/2\sqrt{H})$ are disjoint, we have

$$\operatorname{Vol} B(p, \pi/(2\sqrt{H})) = (\operatorname{Vol} M)/2, \tag{1.4.10}$$

so we have equality in the volume comparison. By rigidity, $B(p, \pi/(2\sqrt{H}))$ is isometric to the upper hemisphere of S^n_H. Then by (1.4.10) $\operatorname{Vol} M = \operatorname{Vol} S^n_H$. Hence M is isometric to the sphere S^n_H. ■

Given a rigidity result, one naturally asks if it is stable. Namely what happens if diameter is close to be maximal? In the sectional curvature case, we have the beautiful Grove-Shiohama diameter sphere theorem [65], that if M^n has sectional curvature $K_M \geq 1$ and $\operatorname{diam}_M > \pi/2$ then M is homeomorphic to S^n. For Ricci curvature there are manifolds $M^n (n \geq 4)$ with $\operatorname{Ric} \geq n-1$ and diameter arbitrarily close to π which are not homotopic to sphere [3, 102]. On the other hand, these spaces still have very nice structure. If M^n has $\operatorname{Ric}_M \geq n-1$, and for $\epsilon > 0$ small, there are k pairs of points p_i, q_i with $d(p_i, q_i) \geq \pi - \epsilon, i = 1, \cdots, k$, and $|d(p_i, p_j) - \pi/2| \leq \epsilon$ for $i \neq j$, then M^n looks like (in Gromov-Hausdorff sense) a k-fold spherical suspension of a compact geodesic space. Moreover when k, M^n is diffeomorphic to S^n [34, 35, **?**].

1.5 The Jacobian Determinant of the Exponential Map

Besides the Bochner's formula, another basic entity that encodes Ricci curvature is the Jacobian determinant of the exponential map. The two are in fact equivalent. However, we will derive the equation about the Jacobian determinant of the exponential map directly since it is very geometric and elegant. We will show the equivalence with the Bochner's formula at the end of the section.

Let ξ be a vector field on a Riemannian manifold M^n. Consider the family of maps $T_t(x) = T(t,x) = \exp_x(t\xi(x)) : M^n \to M^n$, which moves x along the geodesic in the direction of $\xi(x)$. Then its differential is given in terms of Jacobi fields. Indeed, let $\{e_i\}$ be an orthonormal basis of T_xM^n. Then

$$\begin{array}{rccl} dT_t|_x : & T_xM^n & \to & T_{T(t,x)}M^n \\ & e_i & \mapsto & dT_t(e_i) = \frac{d}{ds}|_{s=0} T_t(\gamma_{e_i}(s)), \end{array}$$

and $J_i(t) = dT_t(e_i)$ is the Jacobi field along the geodesic $\gamma(t) = \exp_x(t\xi(x))$ with $J_i(0) = e_i$, $J_i'(0) = (\nabla_{e_i}\xi)(x)$. Denote by $\{e_i(t)\}$ the orthonormal basis at $\gamma(t)$ by parallel translating $\{e_i\}$ from x along γ. Then $J(t) = (J_{ij}(t))_{n\times n}$, with $J_{ij} = \langle J_i, e_j\rangle$, is the Jacobian matrix of T, and it satisfies the Jacobi equation

$$J''(t) + K(t)J(t) = 0, \tag{1.5.1}$$

where $K(t) = (K_{ij}(t))_{n\times n}$ with $K_{ij}(t) = \langle R(e_i(t), \dot\gamma(t))\dot\gamma(t), e_j(t)\rangle_{\gamma(t)}$, and $J(0) = I_n$, $J'(0) = \nabla\xi(x)$.

Set $\mathcal{J}(t,x) = \det J(t,x)$. Then

$$\dot{\mathcal{J}}(t,x) = \mathcal{J}(t,x)\left(\mathrm{tr}(\dot J(t,x)J^{-1}(t,x))\right) = \mathcal{J}(t,x)\left(\mathrm{tr}U(t,x)\right),$$

where we have denoted by $U(t,x) = \dot J\cdot J^{-1}$. Then $\dot U = \ddot J\cdot J^{-1} - \dot J\cdot J^{-1}\cdot \dot J\cdot J^{-1} = -K - U^2$. With this change of variable, (1.5.1) becomes the first order Ricatti type equation

$$\dot U + K + U^2 = 0. \tag{1.5.2}$$

By taking trace, we arrive at another important equation involving Ricci curvature:

$$\frac{d}{dt}(\mathrm{tr}\, U) + \mathrm{tr}\,(U^2) + \mathrm{Ric}(\dot\gamma,\dot\gamma) = 0. \tag{1.5.3}$$

When $\xi = \nabla\psi$ is a gradient vector field, $\nabla\xi = \mathrm{Hess}\,\psi$ is symmetric, which implies $U(t,x)$ is symmetric. Therefore we can use the Cauchy-Schwartz inequality, $\mathrm{tr}(U^2) \geq \frac{(\mathrm{tr}U)^2}{n}$, to deduce

$$\frac{d}{dt}(\mathrm{tr}\, U) + \frac{(\mathrm{tr}\, U)^2}{n} + \mathrm{Ric}(\dot\gamma,\dot\gamma) \leq 0. \tag{1.5.4}$$

Using $\operatorname{tr} U = \frac{\dot{\mathcal{J}}}{\mathcal{J}}$, (1.5.4) becomes

$$\frac{\ddot{\mathcal{J}}}{\mathcal{J}} - (1-\frac{1}{n})(\frac{\dot{\mathcal{J}}}{\mathcal{J}})^2 \leq -\mathrm{Ric}(\dot\gamma,\dot\gamma). \tag{1.5.5}$$

This can be better formulated using the following notations. Let $\mathcal{D}(t) = (\mathcal{J}(t))^{\frac{1}{n}}$, $l(t) = -\log \mathcal{J}(t)$. Then the above can be rewritten into the following two important inequalities:

$$\frac{\ddot{\mathcal{D}}}{\mathcal{D}} \leq -\frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n}, \tag{1.5.6}$$

$$\ddot{l}(t) \geq \frac{\dot{l}(t)^2}{n} + \mathrm{Ric}(\dot\gamma,\dot\gamma). \tag{1.5.7}$$

By using the fact that $K(t)$ is effectively an $(n-1)\times(n-1)$ matrix, we can refine the above estimate by a factor of $(n-1)/n$, which will be useful later. The intuitive idea is that one does not feel the effect of curvature along the direction of motion (i.e. along geodesics). Choose $e_1(0) = \dot\gamma(0)/|\dot\gamma(0)|$, then

$$K_{1j} = K_{j1} = 0.$$

Let U_{11} be the $(1,1)$ entry of U and $U_\perp$ be the $(n-1)\times(n-1)$ matrix obtained by removing the first row and first column in U. Then

$$\operatorname{tr} U = U_{11} + \operatorname{tr} U_\perp.$$

Write $\mathcal{J} = \mathcal{J}_{11}\cdot\mathcal{J}_\perp$ where $\mathcal{J}_{11}$ is obtained by solving the equation

$$\frac{\dot{\mathcal{J}}_{11}}{\mathcal{J}_{11}} = U_{11}, \quad \mathcal{J}_{11}(0) = 1,$$

i.e. $\mathcal{J}_{11} = e^{\int_0^t U_{11}(s)ds}$. Also let $\mathcal{D}_{11} = \mathcal{J}_{11}$, $\mathcal{D}_\perp = \mathcal{J}_\perp^{\frac{1}{n-1}}, l_{11} = -\log\mathcal{J}_{11}$ and $l_\perp = -\log\mathcal{J}_\perp$. Since the first row of $K(t)$ is zero, by (1.5.2), $\dot U_{11} = -\sum_j U_{1j}^2 \leq U_{11}^2$. Hence

$$\ddot{\mathcal{D}}_{11} = \ddot{\mathcal{J}}_{11} \leq 0, \qquad \ddot{l}_{11} \geq \dot{l}_{11}^2. \tag{1.5.8}$$

For the orthogonal part, note that

$$\begin{aligned} \operatorname{tr}(U_\perp^2) &= \operatorname{tr}(U^2) - 2\sum_j U_{1j}^2 + U_{11}^2 \leq \operatorname{tr}(U^2) - \sum_j U_{1j}^2, \\ \operatorname{tr}\dot U_\perp &= \operatorname{tr}\dot U - \dot U_{11} = \operatorname{tr}\dot U + \sum_j U_{1j}^2. \end{aligned}$$

Hence

$$\operatorname{tr}\dot U_\perp + \operatorname{tr} U_\perp^2 + \mathrm{Ric}(\dot\gamma,\dot\gamma) \leq 0.$$

From

$$U_{11} + \operatorname{tr} U_\perp = \operatorname{tr} U = (\log\mathcal{J})' = (\log\mathcal{J}_{11})' + (\log\mathcal{J}_\perp)'$$

and the choice of U_{11} we obtain $\mathrm{tr}U_\perp = \dot{\mathcal{J}}_\perp/\mathcal{J}_\perp$. Thus the same argument as before gives

$$\frac{\ddot{\mathcal{D}}_\perp}{\mathcal{D}_\perp} \le -\frac{\mathrm{Ric}(\dot\gamma,\dot\gamma)}{n-1}, \tag{1.5.9}$$

$$\ddot{l}_\perp \ge \frac{(\dot{l}_\perp)^2}{n-1} + \mathrm{Ric}(\dot\gamma,\dot\gamma). \tag{1.5.10}$$

As before, the above are equivalent to

$$\frac{\ddot{\mathcal{J}}_\perp}{\mathcal{J}_\perp} - (1-\frac{1}{n-1})(\frac{\dot{\mathcal{J}}_\perp}{\mathcal{J}_\perp})^2 \le -\mathrm{Ric}(\dot\gamma,\dot\gamma). \tag{1.5.11}$$

The fundamental formula (1.5.3) is actually equivalent to the Bochner formula. Philosophically, (1.5.3) is a equation along a geodesic (the Lagrangian point of view), while the Bochner formula involves its velocity field (the Eulerian point of view); hence they are dual to each other. To see the equivalence directly, put $\gamma(t,x) = \exp_x t\xi(x)$, and let $\xi(t,x)$ be its velocity vector field, i.e. $\dot\gamma(t,x) = \xi(t,\gamma(t,x))$. Since γ is a geodesic, $\ddot\gamma = \frac{\partial\xi}{\partial t} + \nabla_\xi\xi = 0$. On the other hand, the Jacobian map

$$J(t,x) = dT_t(x):\ T_xM \longrightarrow T_{T_t(x)}M \tag{1.5.12}$$

satisfies $\dot J(t,x) = \nabla\xi(t,\gamma(t,x))\cdot J(t,x)$. Hence $U(t,x) = \dot J J^{-1} = \nabla\xi(t,\gamma(t,x))$.

$$\begin{aligned}
\frac{d}{dt}(\mathrm{tr}\,U)(t,x) &= \frac{d}{dt}(\mathrm{div}\,\xi)(t,\gamma(t,x))\\
&= \mathrm{div}\left(\frac{\partial\xi}{\partial t}(t,\gamma(t,x))\right) + \dot\gamma(t,x)\cdot\nabla(\mathrm{div}\,\xi)(t,\gamma(t,x))\\
&= \left(-\mathrm{div}(\nabla_\xi(\xi) + \xi\cdot\nabla(\mathrm{div}\xi)\right)(t,\gamma(t,x)).
\end{aligned}$$

Consequently, (1.5.3) implies that

$$\left(-\mathrm{div}(\nabla_\xi\xi) + \xi\cdot\nabla(\mathrm{div}\xi) + \mathrm{tr}\,(\nabla\xi)^2 + \mathrm{Ric}(\xi,\xi)\right)(t,\gamma(t,x)) = 0.$$

When $t=0$, this is the Bochner formula for a general vector field. In particular, when $\xi = \nabla\psi$, we have

$$-\frac{1}{2}\Delta|\nabla\psi|^2 + \nabla\psi\cdot\nabla(\Delta\psi) + |\mathrm{Hess}\psi|^2 + \mathrm{Ric}(\nabla\psi,\nabla\psi) = 0,$$

which is exactly the Bochner formula (1.1.1).

1.6 Characterizations of Ricci Curvature Lower Bound

Proposition 1.6.1 *For a Riemannian manifold (M^n,g), the following are equivalent:*

a) $\mathrm{Ric}_M \ge (n-1)H$*;*

b) The inequality (1.1.6) holds for all $u \in C^3(M)$;

c) For every $x \in M$, the mean curvature (Laplacian) comparison (6.3.3) holds in a neighborhood of x;

d) For every $x \in M$, the volume element comparison (1.4.2) holds in a neighborhood of x;

e) For every $x \in M$, the local Bishop-Gromov volume comparison (1.4.8) holds in a neighborhood of x;

f) For $\mathcal{D}$ in (1.5.6), $\frac{\mathcal{D}''}{\mathcal{D}} \le -\frac{(n-1)H}{n}$.

g) For all $f \in C_c^\infty(M), t > 0, x \in M$,

$$|\nabla(E_t f)|^2(x) \le e^{-2(n-1)Ht} E_t(|\nabla f|^2)(x). \tag{1.6.1}$$

Proof: In the previous sections we proved a) implies b), c), d), e) and f). We will show each b), c), d) $\Rightarrow$ a), e) $\Rightarrow$ d) and a) $\Rightarrow$ g).

b) $\Rightarrow$ a): Given any $x_0 \in M$ and $v_0 \in T_{x_0}M$, let u be a C^3 function such that $\nabla u(x_0) = v_0$ and $\operatorname{Hess} u(x_0) = \lambda_0 I_n$. For example, in terms of a geodesics frame at x_0 with coordinate $\{x_i\}$, write $v_0 = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$. Then $u = \frac{\lambda_0}{2}(x_1^2 + \cdots + x_n^2) + \sum_{i=1}^n a_i x_i$ satisfies the conditions. Plug u into (1.1.1) and (1.1.6), we have $\mathrm{Ric}(v_0, v_0) \ge (n-1)H|v_0|^2$, so $\mathrm{Ric} \ge (n-1)H$.

d) $\Rightarrow$ a): The Jacobi fields J_i has the Taylor expansion

$$J_i(r) = r\frac{\partial}{\partial \theta_i} + \frac{r^3}{3!} R(\partial_r, \frac{\partial}{\partial \theta_i})\partial_r + \cdots,$$

plug this into (1.4.1), we have the following Taylor expansion for $\mathcal{A}$,

$$\mathcal{A}(r,\theta) = r^{n-1} - \frac{r^{n+1}}{6}\mathrm{Ric}(\partial_r, \partial_r) + \cdots. \tag{1.6.2}$$

We have same expansion for $\mathcal{A}_H$. Since $\mathcal{A} \le \mathcal{A}_H$ for all r small, we have $\mathrm{Ric} \ge (n-1)H$.

c) $\Rightarrow$ a): From (1.4.3) and (1.6.2) one gets the following Taylor expansion for the mean curvature,

$$m(r,\theta) = \frac{n-1}{r} - \frac{r}{3}\mathrm{Ric}(\partial_r, \partial_r) + \cdots. \tag{1.6.3}$$

Same argument as above gives $\mathrm{Ric} \ge (n-1)H$.

e) $\Rightarrow$ *d)*: Given any point $x_0 = (r_0, \theta_0) \in M$. Let B^ϵ be the ϵ-ball at x_0. By (1.4.8)

$$\frac{\mathrm{Vol}(B_t^\epsilon)}{\mathrm{Vol}(B^\epsilon)} \ge \frac{t}{\mathrm{Vol}(B^\epsilon)} \int_{B^\epsilon} \frac{\mathcal{A}_H(td(x,y))}{\mathcal{A}_H(d(x,y))} dvol_y.$$

When $\epsilon \to 0, B^\epsilon \to (r_0, \theta_0)$, we have

$$t\frac{\mathcal{A}(tr_0,\theta_0)}{\mathcal{A}(r_0,\theta_0)} \ge t\frac{\mathcal{A}_H(tr_0)}{\mathcal{A}_H(r_0)},$$

which is (1.4.2).

a) $\Rightarrow$ g): Let $\psi(s) = e^{-2(n-1)Hs}E_s\left(|\nabla E_{t-s}f|^2\right)$. Then $\psi(0) = |\nabla E_t f|^2, \psi(t) = e^{-2(n-1)Ht}E_t(|\nabla f|^2)$. It is enough to show ψ is increasing in s.

$$\begin{aligned}\psi'(s) &= -2(n-1)He^{-2(n-1)Hs}E_s\left(|\nabla E_{t-s}f|^2\right) \\ &+e^{-2(n-1)Hs}\left[\Delta E_s\left(|\nabla E_{t-s}f|^2\right) + 2E_s\langle\nabla(-\Delta E_{t-s}f), \nabla E_{t-s}f\rangle\right].\end{aligned}$$

By Bochner formula (1.1.1) and a),

$$\begin{aligned}&2\langle\nabla(-\Delta E_{t-s}f), \nabla E_{t-s}f\rangle \\ &= 2|\mathrm{Hess}\, E_{t-s}f|^2 + 2\mathrm{Ric}(\nabla E_{t-s}f, \nabla E_{t-s}f) - \Delta|\nabla E_{t-s}f|^2 \\ &\geq 2(n-1)H|\nabla E_{t-s}f|^2 - \Delta|\nabla E_{t-s}f|^2.\end{aligned}$$

Therefore $\psi'(s) \geq 0$. ■

and Cheeger-Colding's segment inequality [34, Theorem 2.11], see also [31]. Given a function $g \geq 0$ on M^n, put

$$\mathcal{F}_g(x_1, x_2) = \inf_\gamma \int_0^l g(\gamma(s))ds,$$

where the inf is taken over all minimal geodesics γ from x_1 to x_2 and s denotes the arclength.

Theorem 1.6.2 (Segment Inequality, Cheeger-Colding 1996) *Let* $\mathrm{Ric}_{M^n} \geq -(n-1)$, $A_1, A_2 \subset B(p,r)$, *and* $r \leq R$. *Then*

$$\int_{A_1\times A_2} \mathcal{F}_g(x_1, x_2) \leq c(n,R)\cdot r\cdot(\mathrm{Vol}(A_1) + \mathrm{Vol}(A_2))\cdot\int_{B(p,2R)} g, \quad (1.6.4)$$

where $c(n,R) = 2\sup_{0<\frac{s}{2}\leq u\leq s, 0<s<R}\frac{\mathrm{Vol}_{-1}(\partial B(s))}{\mathrm{Vol}_{-1}(\partial B(u))}$.

The segment inequality shows that if the integral of g on a ball is small then the integral of g along almost all segments is small. It also implies a Poincaré inequality of type $(1,p)$ for all $p \geq 1$ for manifolds with lower Ricci curvature bound [24]. In particular it gives a lower bound on the first eigenvalue of the Laplacian for the Dirichlet problem on a metric ball; compare [79].

1.7 Characterization of Warped Product

Definition 1.7.1 *Given two Riemannian manifolds* $(B^n, g_M), (F^m, g_F)$ *and a positive smooth function* f *on* B, *the warped product metric on* $B \times F$ *is defined by*

$$g = g_B + f^2 g_F. \quad (1.7.1)$$

We denote it as $B \times_f F$.

Warped product metrics are useful in many constructions since quantities like curvature are easy to compute. Many interesting examples are constructed by multiple warped product, see e.g. [16, Chapter 9, J], [108], [134, **?**]. One especially important and simple case is when B is one dimensional. One has the following nice characterization about these warped product. It goes back to Brinkmann [22], see also [16, 9.117], [34].

Theorem 1.7.2 *A Riemannian manifold (M, g) is a warped product $((a, b) \times_f N, dr^2 + f^2(r)g_N)$ if and only if there is a nontrivial smooth function u on M such that*

$$\nabla u \neq 0, \quad \operatorname{Hess} u = hg$$

for some function $h : M \longrightarrow \mathbb{R}$. ($f = u'$ up to a multiplicative constant)

Proof: If $g = dr^2 + f^2(r)g_0$ simply take $u(r) = \int f(t)dt$. Then $u'(r) = \frac{\partial}{\partial r}u = f(r) > 0$, $\operatorname{Hess} u = u''(r)g = f'(r)g$.

Conversely, if u satisfies $\operatorname{Hess} u = hg$, then for any vector field X,

$$X\left(\frac{1}{2}|\nabla u|^2\right) = \operatorname{Hess} u(X, \nabla u) = h \cdot g(X, \nabla u)$$

which shows that $|\nabla u|$ is constant on level sets of u. Let $N = u^{-1}(c)$, a level set of u, g_N the metric restricted to this level set, and r the signed distance to N defined by requiring that ∇r and ∇u point in the same direction. Then it is easy to see that u is a function of r: $u = u(r)$. Hence,

$$\nabla u = u'\nabla r, \quad \operatorname{Hess} u = u''dr^2 + u'\operatorname{Hess} r.$$

Comparing with the equation $\operatorname{Hess} u = hg$ shows that $h = u''$ and that

$$\operatorname{Hess} r = \frac{u''}{u'}g$$

on the orthogonal complement of ∇r. On the other hand, $g = dr^2 + g_r$ with g_r the restriction of g on the level set of r. Since

$$L_{\nabla r}g_r = 2\operatorname{Hess} r = 2\frac{u''}{u'}g_r.$$

Again the Hessian here is restricted to the orthogonal complement of ∇r. Thus, $g = dr^2 + (ku')^2 g_N$ where $ku'(0) = 1$. ■

Corollary 1.7.3 *(M, g) is a Riemannian product $((a, b) \times N, dr^2 + g_N)$ if and only if there is a nontrivial smooth function u on M such that $\nabla u \neq 0$, $\operatorname{Hess} u = 0$.*

Of course this follows from the de Rham decomposition theorem since $\operatorname{Hess} u = 0$ is equivalent to ∇u is a parallel vector field. This characterization is useful in splitting and almost splitting results.

If a is finite and $\lim_{r\to a} f = 0$, then the warped product metric $((a,b)\times_f F, dr^2 + f^2(r)g_F)$ is smooth at $r=a$ if and only if $F=\mathbb{S}^{n-1}, f'(a)=1$ and f is odd at a. If $\lim_{r\to a} f > 0$ (and also at b if b is finite), in order to be smooth, the identification on the boundary of $[a,b]\times F$ or $[0,\infty)\times F$ can be: either an isometry of order two of $\{a\}\times F$ with itself acting freely (and similarly at b if b is finite), and then f has to be even at a (and similarly at b); or an isometry of $\{a\}\times F$ with $\{b\}\times F$ and then f has to be the restriction of a smooth function on $\mathbb{R}$ satisfying $f(a)f(r+b-a) = f(b)f(t)$.

Example 1.7.4 *Spaces with constant sectional curvature can be viewed as various warped product:*
$\mathbb{R}^n = \mathbb{R}\times_1 \mathbb{R}^{n-1}$ *with* $u = x$ *and* $\operatorname{Hess} u = 0$*;*
$\mathbb{R}^n\setminus 0 = (0,\infty)\times_r \mathbb{S}^{n-1}$ *with* $u=\frac{1}{2}r^2$ *and* $\operatorname{Hess} u = g$*;*
$\mathbb{S}^n\setminus\{x,-x\} = (0,\pi)\times_{\sin r}\mathbb{S}^{n-1}$ *with* $u=-\cos r$ *and* $\operatorname{Hess} u = \cos r g$*;*
$\mathbb{R}P^n\setminus\{x, C_x\} = (0,\frac{\pi}{2})\times_{\sin r}\mathbb{S}^{n-1}$ *with* $u=-\cos r$ *and* $\operatorname{Hess} u = \cos r g$*, where* C_x *is the cut-locus of* x*;*
$\mathbb{H}^n = \mathbb{R}\times_{e^t}\mathbb{R}^{n-1}$ *with* $u=e^t$ *and* $\operatorname{Hess} u = e^t g$*;*
$\mathbb{H}^n\setminus 0 = (0,\infty)\times_{\sinh r}\mathbb{S}^{n-1}$ *with* $u=\cosh r$ *and* $\operatorname{Hess} u = \cosh r g$*.*

Corollary 1.7.5 (M,g) *has constant sectional curvature* H *if and only if there is a nontrivial smooth function* u *on* M *such that* $\operatorname{Hess} u = -Hug$*.*

Chapter 2

Geometry of Manifolds with Ricci Curvature Lower Bound

2.1 Cheeger-Gromoll's Splitting Theorem

The Cheeger-Gromoll's splitting theorem [40] is the most important rigidity result for manifolds with nonnegative Ricci curvature. It plays a very fundamental and important role in studying manifolds with nonnegative Ricci curvature and later, manifolds with general Ricci lower bound. In a certain sense, the splitting theorem is an analog of the maximal diameter theorem (Theorem 1.4.13) in the case of nonnegative Ricci curvature and noncompact spaces.

Definition 2.1.1 *A normalized geodesic $\gamma : [0, \infty) \to M$ is called a ray if $d(\gamma(0), \gamma(t)) = t$ for all t. A normalized geodesic $\gamma : (-\infty, \infty)$ is called a line if $d(\gamma(t), \gamma(s)) = s - t$ for all $s \geq t$.*

Example 2.1.2 *A paraboloid has rays but no lines. A cylinder has lines. A surface of revolution has lines.*

Definition 2.1.3 *M is called connected at infinity if for all $K \subset M$, K compact, there is a compact set $\tilde{K} \supset K$ such that every two points in $M - \tilde{K}$ can be connected in $M - K$.*

A well known fact about existence of rays and lines is the following.

Lemma 2.1.4 *If M is noncompact then for each $p \in M$ there is a ray γ with $\gamma(0) = p$. If M is disconnected at infinity then M has a line.*

When a noncompact manifold has a line, it has "maximal diameter", so it has strong rigidity when Ricci curvature is nonnegative.

Theorem 2.1.5 (Splitting Theorem, Cheeger-Gromoll 1971) *Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_M \geq 0$. If M has a line, then M is isometric to the product $\mathbb{R} \times N^{n-1}$, where N is an $n-1$ dimensional manifold with $\mathrm{Ric}_N \geq 0$.*

To use the "big" diameter, it is natural to consider the distance function from infinity, of course we need to renormalize it so it is finite. This is the so called the Busemann functions.

Definition 2.1.6 *If $\gamma : [0, \infty) \to M$ is a ray, set $b_t^\gamma(x) = t - d(x, \gamma(t))$.*

Lemma 2.1.7 *We have*

1. $|b_t^\gamma(x)| \leq d(x, \gamma(0))$.
2. *For x fixed, $b_t^\gamma(x)$ is nondecreasing in t.*
3. $|b_t^\gamma(x) - b_t^\gamma(y)| \leq d(x, y)$.

Proof: (1) and (3) are the triangle inequality. For (2), suppose $s < t$. Then

$$\begin{aligned} b_s^\gamma(x) - b_t^\gamma(x) &= (s-t) - d(x,\gamma(s)) + d(x,\gamma(t)) \\ &= d(x,\gamma(t)) - d(x,\gamma(s)) - d(\gamma(s),\gamma(t)) \leq 0 \end{aligned}$$

■

Definition 2.1.8 *If $\gamma : [0, \infty) \to M$ is a ray, the Busemann function associated to γ is*

$$b^\gamma(x) = \lim_{t \to \infty} b_t^\gamma(x) = \lim_{t \to \infty} (t - d(x, \gamma(t))).$$

By the above, Busemann functions are well defined and Lipschitz continuous. Intuitively, $b^\gamma(x)$ is the distance from $\gamma(\infty)$. Also, since

$$\begin{aligned} b^\gamma(\gamma(s)) &= \lim_{t\to\infty} t - d(\gamma(s), \gamma(t)) \\ &= \lim_{t\to\infty} t - (t-s) = s, \end{aligned}$$

$b^\gamma(x)$ is linear along $\gamma(t)$.

Example 2.1.9 *In $\mathbb{R}^n$, the rays are $\gamma(t) = \gamma(0) + \gamma'(0)t$. In this case, $b^\gamma(x) = \langle x - \gamma(0), \gamma'(0)\rangle$. The level sets of b^γ are hyperplanes.*

The local Laplacian comparison (1.2.10) gives the following key estimate.

Proposition 2.1.10 *If M has $\mathrm{Ric}_M \geq 0$ and γ is a ray on M then $\Delta(b^\gamma) \geq 0$ in the barrier sense (see Definition 1.3.3).*

Proof: For each $q \in M$, we construct a family of lower barrier functions of b^γ at q as follows.

Pick any $t_i \to \infty$. For each i, connect q and $\gamma(t_i)$ by a minimal geodesic σ_i. Then $\{\sigma_i'(0)\} \subset S^{n-1}$, so there is a subsequential limit $v_0 \in T_qM$. We call the geodesic $\tilde{\gamma}$ with $\tilde{\gamma}'(0) = v_0$ an asymptotic ray of γ at q; note that in general different sequence of $t_i \to \infty$ might give different asymptotic rays.

Define the function $h_t(x) = t - d(x, \tilde{\gamma}(t)) + b^\gamma(q)$. Since $\tilde{\gamma}$ is a ray, the points $q = \tilde{\gamma}(0)$ and $\tilde{\gamma}(t)$ are not cut points to each other. Hence the function $d(x, \tilde{\gamma}(t))$ is smooth in a neighborhood of q and thus so is h_t. Clearly $h_t(q) = b^\gamma(q)$, thus to show h_t is a lower barrier for b^γ we only need to show $h_t(x) \leq b^\gamma(x)$. To see this, first note that for any s,

$$\begin{aligned} -d(x, \tilde{\gamma}(t)) &\leq -d(x, \gamma(s)) + d(\tilde{\gamma}(t), \gamma(s)) \\ &= s - d(x, \gamma(s)) - s + d(\tilde{\gamma}(t), \gamma(s)). \end{aligned}$$

Taking $s \to \infty$, this gives

$$-d(x, \tilde{\gamma}(t)) \leq b^\gamma(x) - b^\gamma(\tilde{\gamma}(t)). \tag{2.1.1}$$

Also

$$\begin{aligned} b^\gamma(q) &= \lim_{i \to \infty} (t_i - d(q, \gamma(t_i))) \\ &= \lim_{i \to \infty} (t_i - d(q, \sigma_i(t)) - d(\sigma_i(t), \gamma(t_i))) \\ &= -d(q, \tilde{\gamma}(t)) + \lim_{i \to \infty} (t_i - d(\sigma_i(t), \gamma(t_i))) \\ &= -t + b^\gamma(\tilde{\gamma}(t)). \end{aligned} \tag{2.1.2}$$

Combining (2.1.1) and (2.1.2) gives

$$h_t(x) \leq b^\gamma(x), \tag{2.1.3}$$

so h_t is a lower barrier function for b^γ at q.

Finally, since $\mathrm{Ric}_M \geq 0$, by the local Laplacian comparison (1.2.10)

$$\Delta(h_t(x)) = -\Delta(d(x, \tilde{\gamma}(t))) \geq -\frac{n-1}{d(x, \tilde{\gamma}(t))},$$

which tends to 0 as $t \to \infty$. Thus $\Delta(b^\gamma) \geq 0$ in the barrier sense. ■

With this estimate, the maximal principle (Theorem 1.3.7) and the regularity result for harmonic functions immediately gives us that the Busemann functions are smooth and harmonic. Furthermore we show the norm of its gradient is constant 1; this together with Bochner formula finishes the proof of the splitting theorem.

Proof of Theorem 2.1.5. Denote by γ^+ and γ^- the two rays which form the line γ and let b^+ and b^- denote their Busemann functions.

Observe that

$$\begin{aligned} b^+(x) + b^-(x) &= \lim_{t \to \infty} (t - d(x, \gamma^+(t))) + \lim_{t \to \infty} (t - d(x, \gamma^-(t))) \\ &= \lim_{t \to \infty} 2t - (d(x, \gamma^+(t)) + d(x, \gamma^-(t))) \\ &\leq 2t - d(\gamma^+(t), \gamma^-(t)) = 0. \end{aligned}$$

Since $b^+(\gamma(0)) + b^-(\gamma(0)) = 0$, $b^+ + b^-$ has a maximal at $\gamma(0)$. Also Proposition 2.1.10 gives

$$\Delta(b^+ + b^-) = \Delta b^+ + \Delta b^- \geq 0.$$

By the strong maximal principle (Theorem 1.3.7) $b^+ + b^- \equiv 0$. But then $b^+ = -b^-$. Thus

$$0 \leq \Delta b^+ = -\Delta b^- \leq 0,$$

so both b^+ and b^- are smooth by elliptic regularity.

Moreover, for any point $q \in M$ we can consider asymptotic rays $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$ to γ^+ and γ^- and denote their Busemann functions by $\tilde{b}^+$ and $\tilde{b}^-$. From (2.1.3) it follows that

$$\tilde{b}^+(x) + b^+(q) \leq b^+(x). \tag{2.1.4}$$

The next step is to show that this inequality is in fact an equality when γ^+ extends to the line.

First we show that the two asymptotic rays $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$ form a line. By triangle inequality, for any t

$$\begin{aligned} d(\tilde{\gamma}^+(s_1), \tilde{\gamma}^-(s_2)) &\geq d(\tilde{\gamma}^-(s_2), \gamma^+(t)) - d(\tilde{\gamma}^+(s_1), \gamma^+(t)) \\ &= (t - d(\tilde{\gamma}^+(s_1), \gamma^+(t))) - (t - d(\tilde{\gamma}^-(s_2), \gamma^+(t))), \end{aligned}$$

so by taking $t \to \infty$ we have

$$\begin{aligned} d(\tilde{\gamma}^+(s_1), \tilde{\gamma}^-(s_2)) &\geq b^+(\tilde{\gamma}^+(s_1)) - b^+(\tilde{\gamma}^-(s_2)) \\ &= b^+(\tilde{\gamma}^+(s_1)) + b^-(\tilde{\gamma}^-(s_2)) \\ &\geq \tilde{b}^+(\tilde{\gamma}^+(s_1)) + b^+(q) + \tilde{b}^-(\tilde{\gamma}^-(s_2)) + b^-(q) \\ &= s_1 + s_2. \end{aligned}$$

Thus any asymptotic ray to γ^+ forms a line with any asymptotic ray to γ^-. Applying the same argument given above for b^+ and b^- we see that $\tilde{b}^+ = -\tilde{b}^-$. By applying (2.1.4) to b^-

$$-\tilde{b}^-(x) - b^-(q) \leq -b^-(x).$$

Substituting $b^+ = -b^-$ and $\tilde{b}^+ = -\tilde{b}^-$ we have

$$\tilde{b}^+(x) + b^+(q) \geq b^+(x).$$

This along with (2.1.4) gives

$$\tilde{b}^+(x) + b^+(q) = b^+(x).$$

Thus, $\tilde{b}^+$ and b^+ differ only by a constant. Clearly, at q the derivative of $\tilde{b}^+$ in the direction of $(\tilde{\gamma}^+)'(0)$ is 1. Since $\tilde{b}^+$ has Lipschitz constant 1, this implies that $\nabla b^+(q) = (\tilde{\gamma}^+)'(0)$ and $|\nabla b^+| \equiv 1$. Apply the Bochner formula (1.1.1) to b^+ we have

$$0 = |\operatorname{Hess} b^+|^2 + \langle \nabla b^+, \nabla(\Delta b^+) \rangle + \operatorname{Ric}(\nabla b^+, \nabla b^+).$$

Since $\Delta b^+ = 0$ and $\mathrm{Ric} \geq 0$ we then have that $\mathrm{Hess}\, b^+ = 0$ which, along with the fact that $|\nabla b^+| = 1$ implies that M splits isometrically in the direction of ∇b^+. ■

Cheeger-Gromoll's splitting theorem has many geometric and topological applications. It also has several generalizations. See Sections 3.2.2, **??**.

2.2 Gradient Estimate

2.2.1 Harmonic Functions

For a complete manifold M^n with $\mathrm{Ric}_M \geq (n-1)H > 0$, there are no (non-trivial) harmonic functions since it's closed. For noncompact manifolds, there are in general positive harmonic functions. Using Bochner's formula, maximal principle, cut-off function and Laplacian comparison Yau proved a gradient estimate for $\log u$, where u is a positive harmonic function [146]. The following version is from an improved estimate given by Li-Wang [80].

Theorem 2.2.1 (Gradient Estimate, Yau 1975, Li-Wang 2002) *Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_{M^n} \geq -(n-1)H^2 (H \geq 0)$. If u is a positive harmonic function defined on the closed ball $\overline{B(q, 2R)} \subset M$. Then*

$$|\nabla(\log u)(x)| \leq (n-1)H + c(n,H)R^{-1} \quad \text{on } \overline{B(q,R)}. \tag{2.2.5}$$

In particular, for positive harmonic function u defined on M,

$$|\nabla(\log u)(x)| \leq (n-1)H. \tag{2.2.6}$$

Estimate (2.2.6) is optimal. When equality occurs, it implies strong rigidity: the manifold is a warped product, see [80]. When $H = 0$, (2.2.6) gives the following Loiuville type result.

Corollary 2.2.2 *Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_{M^n} \geq 0$, then all positive harmonic functions are constant. In particular, all bounded harmonic functions are constant.*

Proof of Theorem 2.2.1: Let $h = \log u$. Then $\nabla h = \frac{\nabla u}{u}, \Delta h = \frac{\Delta u}{u} - \left|\frac{\nabla u}{u}\right|^2 = -|\nabla h|^2$. Apply the Bochner formula (1.1.1) to the function h we have

$$\begin{aligned} \frac{1}{2}\Delta|\nabla h|^2 &= |\mathrm{Hess}\, h|^2 + \langle \nabla h, \nabla(\Delta h)\rangle + \mathrm{Ric}(\nabla h, \nabla h) \\ &\geq |\mathrm{Hess}\, h|^2 - \langle \nabla h, \nabla(|\nabla h|^2)\rangle - (n-1)H^2|\nabla h|^2. \end{aligned} \tag{2.2.7}$$

For the Hessian term one could use the Schwarz inequality

$$|\mathrm{Hess}\, h|^2 \geq \frac{(\Delta h)^2}{n}. \tag{2.2.8}$$

Indeed, when $H = 0$, this estimate is enough. For $H > 0$, using this estimate one would get $(n(n-1))^{\frac{1}{2}}$ instead of $n-1$ in (2.2.6). To get the best constant for $H > 0$, note that (2.2.8) is only optimal when Hessian at all directions are same. Harmonic functions in the model spaces are radial functions, so their Hessian along the radial direction would be different from the spherical directions. Therefore one computes the norm by separating the radial direction. Let $\{e_i\}$ be an orthonormal basis with $e_1 = \frac{\nabla h}{|\nabla h|}$, the potential radial direction, denote $h_{ij} = \operatorname{Hess} h(e_i, e_j)$. We compute

$$\begin{aligned}
|\operatorname{Hess} h|^2 &= \sum_{i,j} h_{ij}^2 \\
&\geq h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \sum_{j=2}^n h_{jj}^2 \\
&\geq h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{(\Delta h - h_{11})^2}{n-1} \\
&= h_{11}^2 + 2\sum_{j=2}^n h_{1j}^2 + \frac{(|\nabla h|^2 + h_{11})^2}{n-1} \\
&\geq \frac{n}{n-1}\sum_{j=1}^n h_{1j}^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}|\nabla h|^2 h_{11}.
\end{aligned}$$

Now $h_{11} = \frac{1}{|\nabla h|^2}\langle \nabla_{\nabla h}\nabla h, \nabla h\rangle = \frac{1}{2|\nabla h|^2}\langle \nabla|\nabla h|^2, \nabla h\rangle$, $h_{1j} = \frac{1}{|\nabla h|}\langle \nabla_{e_j}\nabla h, \nabla h\rangle = \frac{1}{2|\nabla h|}e_j(|\nabla h|^2)$. Therefore

$$|\operatorname{Hess} h|^2 \geq \frac{n}{4(n-1)}\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{|\nabla h|^4 + \langle \nabla|\nabla h|^2, \nabla h\rangle}{n-1}, \tag{2.2.9}$$

and equality holds if and only if $\operatorname{Hess} h$ are same on the level set of h, i.e.

$$\operatorname{Hess} h = -\frac{|\nabla h|^2}{n-1}\left(g - \frac{1}{|\nabla h|^2}dh \otimes dh\right).$$

Plug (2.2.9) into (2.2.7) we get

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla h|^2 \geq\ & \frac{n}{4(n-1)}\frac{|\nabla|\nabla h|^2|^2}{|\nabla h|^2} + \frac{|\nabla h|^4}{n-1} - \frac{n-2}{n-1}\langle \nabla|\nabla h|^2, \nabla h\rangle \\
& -(n-1)H^2|\nabla h|^2. \qquad (2.2.10)
\end{aligned}$$

If $|\nabla h|^2$ achieves a maximum inside $B(q, 2R)$ then we are done. Assume $|\nabla h|(q_0)$ is the maximum for some $q_0 \in B(q, 2R)$, then $\nabla|\nabla h|^2(q_0) = 0, \Delta|\nabla h|^2(q_0) \leq 0$. Plug these into (2.2.10) gives

$$0 \geq \frac{|\nabla h|^4}{n-1} - (n-1)H^2|\nabla h|^2.$$

Hence $|\nabla h| \leq (n-1)H$.

In general the maximum could occur at the boundary and one has to use a cut-off function to force the maximum is achieved in the interior. Let $f : [0, 2R] \to [0, 1]$ be a smooth function with

$$f|_{[0,R]} \equiv 1, \ \mathrm{supp} f \subset [0, 2R), \tag{2.2.11}$$

$$-cR^{-1} f^{1/2} \le f' \le 0, \tag{2.2.12}$$

$$|f''| \le cR^{-2}, \tag{2.2.13}$$

where $c > 0$ is a universal constant. Let $\phi : \overline{B(q, 2R)} \to [0, 1]$ with $\phi(x) = f(r(x))$, where $r(x) = d(x, q)$ is the distance function. Set $G = \phi|\nabla h|^2$. Then G is nonnegative on M and has compact support in $B(q, 2R)$. Therefore it achieves its maximum at some point $q_0 \in B(q, 2R)$. We can assume $r(q_0) \in [R, 2R)$ since if $q_0 \in B(q, R)$, then $|\nabla h|^2$ achieves maximal at q_0 on $B(q, R)$ and previous argument applies.

If q_0 is not a cut point of q then ϕ is smooth at q_0 and we have

$$\Delta G(q_0) \le 0, \quad \nabla G(q_0) = 0. \tag{2.2.14}$$

At the smooth point of r, $\nabla G = \nabla\phi|\nabla h|^2 + \phi\nabla|\nabla h|^2$,

$$\Delta G = \Delta\phi|\nabla h|^2 + 2\langle\nabla\phi, \nabla|\nabla h|^2\rangle + \phi\Delta|\nabla h|^2.$$

Using (2.2.10), (2.2.14) and express $|\nabla h|^2, \nabla|\nabla h|^2$ in terms of $G, \nabla G$ we get, At the maximal point q_0 of G,

$$\begin{aligned} 0 \ge \Delta G(q_0) \quad \ge \quad & \frac{\Delta\phi}{\phi} G - 2\frac{|\nabla\phi|^2}{\phi^2} G + \frac{n}{2(n-1)} \frac{|\nabla\phi|^2}{\phi^2} G + \frac{2}{n-1} \frac{G^2}{\phi} \\ & + \frac{2(n-2)}{n-1} \langle\nabla h, \nabla\phi\rangle \frac{G}{\phi} - 2(n-1)H^2 G. \end{aligned} \tag{2.2.15}$$

Since $\phi(x) = f(r(x))$ is a radial function, $|\nabla\phi| = |f'|$, by (2.2.12),

$$\langle\nabla h, \nabla\phi\rangle \ge -|\nabla h||\nabla\phi| = -G^{\frac{1}{2}} \frac{|\nabla\phi|}{\phi^{1/2}} \ge -G^{\frac{1}{2}} \frac{c}{R}.$$

Also $\Delta\phi = f'\Delta r + f''$. Since $f' \le 0$, by the Laplacian comparison (1.2.10) and (2.2.13)

$$\Delta\phi \ge f'\Delta_H r - cR^{-2},$$

where $\Delta_H r = (n-1)H\coth(Hr)$, which is $\le (n-1)H\coth(HR)$ on $[R, 2R]$. Hence

$$\begin{aligned} \Delta\phi \quad & \ge \quad -cR^{-1}\left((n-1)H\coth(HR) + R^{-1}\right) \\ & \ge \quad -cR^{-1}\left[(n-1)(2R^{-1} + 4H) + R^{-1}\right] \end{aligned}$$

Multiply (2.2.15) by $\frac{(n-1)\phi}{G}$ and plug these in, we get

$$\begin{aligned} 0 \quad \ge \quad & 2G - 2(n-2)\frac{c}{R} G^{\frac{1}{2}} \\ & - \frac{c}{R}\left(\frac{(n-1)(2n-1)}{R} + 4(n-1)^2 H + \left(\frac{3n}{2} - 2\right)\right) - 2(n-1)^2 H^2. \end{aligned}$$

Solving this quadratic inequality gives

$$(G(q_0))^{\frac{1}{2}} \leq (n-1)H + c(n,H)R^{-1}. \tag{2.2.16}$$

Therefore

$$\sup_{B(q,R)} |\nabla h| = \sup_{B(q,R)} G^{1/2} \leq \sup_{B(q,2R)} G^{1/2} \leq (n-1)H + c(n,H)R^{-1}.$$

If q_0 is in the cut locus of q, use the upper barrier $r_{q_0,\epsilon}(x)$ for $r(x)$ and let $\epsilon \to 0$ gives the same estimate. ■

Similar estimate also holds for positive functions u with $\Delta u = K(u)$, see e.g. [31]. See [**?**, 21, 96] for generalizations.

2.2.2 Heat Kernel

2.3 First Eigenvalue and Heat Kernel Comparison

Given a compact Riemannian manifold M^n, let

$$L^2(M) = \{\phi \text{ measurable function on } M \mid \int_M |\phi|^2 d\mathrm{vol} < +\infty\}.$$

With the inner product and induced norm given by

$$(\phi, \psi) = \int_M \phi\psi \, d\mathrm{vol}, \quad \|\phi\|^2 = (\phi, \phi),$$

$L^2(M)$ is a Hilbert space.

When M is connected and closed, $\lambda \in \mathbb{R}$ is an eigenvalue of Δ if there is a nontrivial $\phi \in C^2(M)$ such that

$$\Delta\phi = -\lambda\phi. \tag{2.3.1}$$

When M has boundary, $\partial M \neq 0$, $\bar{M}$ is compact and connected, λ is a Dirichlet eigenvalue if there is a nontrivial $\phi \in C^2(M) \cap C^0(\bar{M})$ satisfying (2.3.1) and $\phi|_{\partial M} = 0$; λ is a Neumann eigenvalue if there is a nontrivial $\phi \in C^2(M) \cap C^1(\bar{M})$ satisfying (2.3.1) and $\frac{\partial}{\partial n}\phi|_{\partial M} = 0$, where n is the outward unit normal vector field on ∂M.

From (2.3.1) (and with these boundary conditions)

$$\lambda = \|\phi\|^{-2} \int_M |\nabla\phi|^2 d\mathrm{vol} \geq 0.$$

Hence $\lambda = 0$ if and only if the eigenfuctions are constant functions.

The set of eigenvalues, listed with multiplicity, consists of a sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \uparrow +\infty,$$

which represent the energy levels, so the metaphysical principle is that smaller spaces correspond to larger eigenvalues.

By the mini-max principle, the first eigenvalue has the following upper bound,

$$\lambda_1 \leq \|\phi\|^{-2} \int_M |\nabla \phi|^2 dvol \tag{2.3.2}$$

for all $\phi \in H_2^1(M)$, $\phi \neq 0$. Here $H_2^1(M)$ denotes the completion of $C_0^\infty(M)$ with respect to the norm

$$\int_M (|\phi|^2 + |\nabla\phi|^2) d\text{vol}.$$

Lower bound on the first eigenvalue is especially useful since it gives Poincaré inequality.

2.3.1 First Nonzero Eigenvalue of Closed Manifolds

For closed manifold, $\lambda = 0$ is alway an eigenvalue (with multiplicity one). In this case we denote λ_1 the first nonzero eigenvalue. An immediate consequence of the Bochner inequality (1.1.5) is the Lichnerowicz first eigenvalue comparison [82], with equality characterized by Obata [100].

Theorem 2.3.1 *Let M^n be a complete Riemannian manifold with* $\text{Ric} \geq (n-1)H > 0$. *Then*

$$\lambda_1(M^n) \geq \lambda_1(M_H^n) = nH. \tag{2.3.3}$$

Moreover, equality holds if and only if M is isometric to M_H^n.

Proof: By Myers' theorem (Theorem 1.2.3) M is compact. Applying (1.1.6) to an eigenfunction u of λ_1 gives

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{\lambda_1^2 u^2}{n} + ((n-1)H - \lambda_1)\,|\nabla u|^2. \tag{2.3.4}$$

Integrating over M, we have

$$0 \geq \int_M \left(\frac{\lambda_1^2 u^2}{n} + ((n-1)H - \lambda_1)\,|\nabla u|^2 \right) dvol.$$

Since $\int_M |\nabla u|^2 dvol = \lambda_1 \int_M u^2 dvol$ and $u \not\equiv$ constant, we obtain (2.3.3).

Equality in (2.3.3) implies equality in (2.3.4) on all of M. Therefore $\text{Hess}\, u = hI_n$ for some $h \in C^\infty(M)$. Now $nh = \text{trHess}\, u = \Delta u = -\lambda_1 u$, so $h = -Hu$. By Theorem 1.7.2, M is isometric to M_H^n. ∎

Stability case!

The estimate (2.3.3) does not give any information when $H \leq 0$. In this case Li-Yau's method of getting a gradient estimate on the first eigenfunction by using the Bochner formula and maximal principle gives estimates on λ_1.

When $H = 0$, Zhong and Yang [148], improving an earlier estimate of Li and Yau [81], obtained a sharp estimate. The equality case is characterized by Hang and Wang [69]. From our slightly modified proof below, the equality case follows quickly.

Theorem 2.3.2 *If M^n is a closed compact Riemannian manifold with* $\mathrm{Ric} \geq 0$, *then*

$$\lambda_1(M^n) \geq \frac{\pi^2}{d_M^2}, \tag{2.3.5}$$

where d_M is the diameter of M. Moreover, equality holds if and only if M is isometric to S^1 with radius $\frac{d_M}{\pi}$.

Proof: Let u be an eigenfunction with $\Delta u = -\lambda_1 u$. Since $\lambda_1 > 0$, $\int_M u \, d\mathrm{vol} = 0$. We can assume that

$$1 = \max_M u > \min_M u = -k, \quad 0 < k \leq 1.$$

For small $\epsilon > 0$, let (the oscillation of u from the midpoint of its range, suitably normalized)

$$v_\epsilon = \frac{u - \frac{1-k}{2}}{(1+\epsilon)\frac{1+k}{2}},$$

then

$$\begin{cases} \Delta v_\epsilon = -\lambda_1(v_\epsilon + a_\epsilon), \quad a_\epsilon = \frac{1-k}{1+k} \cdot \frac{1}{1+\epsilon}, \\ \max_M v_\epsilon = \frac{1}{1+\epsilon}, \quad \min_M v_\epsilon = -\frac{1}{1+\epsilon}. \end{cases} \tag{2.3.6}$$

Set $v_\epsilon = \sin\theta_\epsilon$. The function $\theta_\epsilon = \sin^{-1} v_\epsilon$ has its range in $[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta]$, where δ is specified by

$$\sin\left(\frac{\pi}{2} - \delta\right) = \frac{1}{1+\epsilon}.$$

Then $\nabla\theta_\epsilon = \dfrac{\nabla v_\epsilon}{\sqrt{1-v_\epsilon^2}}$ and

$$\begin{aligned} \Delta\theta_\epsilon &= \frac{\Delta v_\epsilon}{\sqrt{1-v_\epsilon^2}} + \frac{v_\epsilon|\nabla v_\epsilon|^2}{(\sqrt{1-v_\epsilon^2})^3} = \frac{-\lambda_1(v_\epsilon + a_\epsilon) + v_\epsilon|\nabla\theta_\epsilon|^2}{\sqrt{1-v_\epsilon^2}} \\ &= -\lambda_1(\tan\theta_\epsilon + a_\epsilon\sec\theta_\epsilon) + \tan\theta_\epsilon|\nabla\theta_\epsilon|^2. \end{aligned} \tag{2.3.7}$$

Now, by the Bochner's formula (1.1.1), we have

$$\frac{1}{2}\Delta|\nabla\theta_\epsilon|^2 = |\mathrm{Hess}\,\theta_\epsilon|^2 + \langle\nabla\theta_\epsilon, \nabla\Delta\theta_\epsilon\rangle + \mathrm{Ric}(\nabla\theta_\epsilon, \nabla\theta_\epsilon) \geq \langle\nabla\theta_\epsilon, \nabla\Delta\theta_\epsilon\rangle. \tag{2.3.8}$$

From (2.3.7),

$$\nabla\Delta\theta_\epsilon = \sec^2\theta_\epsilon[-\lambda_1(1 + a_\epsilon\sin\theta_\epsilon) + |\nabla\theta_\epsilon|^2]\nabla\theta_\epsilon + \tan\theta_\epsilon\nabla|\nabla\theta_\epsilon|^2. \tag{2.3.9}$$

At the maximal point of $|\nabla\theta_\epsilon|^2$, $\Delta|\nabla\theta_\epsilon|^2 \le 0, \nabla|\nabla\theta_\epsilon|^2 = 0$. Hence evaluate (2.3.8) at the maximal point of $|\nabla\theta_\epsilon|^2$ gives

$$0 \ge \sec^2\theta_\epsilon[-\lambda_1(1+a_\epsilon\sin\theta_\epsilon)+|\nabla\theta_\epsilon|^2]|\nabla\theta_\epsilon|^2.$$

Namely

$$|\nabla\theta_\epsilon|^2 \le \lambda_1(1-a_\epsilon\sin\theta_\epsilon) \le \lambda_1(1+a_\epsilon) \tag{2.3.10}$$

at the maximal point of $|\nabla\theta_\epsilon|^2$, hence everywhere (for the second inequality).

Now, let γ be the shortest geodesic from the minimal point of θ_ϵ to the maximal point. Then

$$\begin{aligned} d_M\sqrt{\lambda_1(1+a_\epsilon)} &\ge \int_\gamma \sqrt{\lambda_1(1+a_\epsilon)}dt \ge \int_\gamma |\nabla\theta_\epsilon|dt \\ &\ge \int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} d\theta_\epsilon = \pi - 2\delta. \end{aligned} \tag{2.3.11}$$

Taking ϵ to zero, we derive that

$$d_M\sqrt{\lambda_1(1+a)} \ge \pi,$$

with $a = \frac{1-k}{1+k}$. This gives us the optimal estimate when $a=0$, but weaker than the desired estimate by a factor of $\frac{1}{1+a}$ when $a \ne 0$. From now on, we assume $0 < a < 1$.

The estimate (2.3.10) can not be improved when the righthand side is independent of the points. To get a more precise estimate for $|\nabla\theta_\epsilon|^2$, the estimate will have to be a function. Motivated from (2.3.10), we look for estimate in the form of $\lambda_1(1+a_\epsilon\phi(\theta))$. For this purpose, we consider

$$h(\theta) = \max_{x\in M, \theta_\epsilon(x)=\theta} |\nabla\theta_\epsilon|^2.$$

Then $h(\theta) \in C^0\left(\left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right]\right)$ and $h(-\frac{\pi}{2}+\delta) = h(\frac{\pi}{2}-\delta) = 0$ since $-\frac{\pi}{2}+\delta$ and $\frac{\pi}{2}-\delta$ are minimum and maximum points of θ_ϵ respectively. Now define $\phi(\theta) \in C^0\left(\left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right]\right)$ so that

$$h(\theta) = \lambda_1(1+a_\epsilon\phi(\theta)), \tag{2.3.12}$$

and the goal is to get an estimate on $\phi(\theta)$.

By (2.3.10), $h(\theta) \le \lambda_1(1+a_\epsilon)$, so $\phi(\theta) \le 1$. Since $h(\theta)$ vanishes at the end points of the interval $\left[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right]$, $\phi(-\frac{\pi}{2}+\delta) = \phi(\frac{\pi}{2}-\delta) < -1$. Since ϕ is only continuous, we will get an estimate on its upper barrier (see Definition 1.3.2).

Using the Bochner formula and maximal principle again we get the following key estimate for upper barrier of ϕ.

Lemma 2.3.3 *Let $y(\theta)$ be an upper barrier function of $\phi(\theta)$ at $\theta_0 \in \left(-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta\right)$. Moreover, $y'(\theta_0) \ge 0$ and $|y(\theta_0)| \le 1$. Then*

$$y''(\theta_0) - 2\tan\theta_0 \cdot y'(\theta_0) - 2\sec^2\theta_0 \cdot y(\theta_0) \ge -2\tan\theta_0 \cdot \sec\theta_0. \tag{2.3.13}$$

The proof of the lemma will be deferred at the end. The lemma motivates us to consider the solution of the ODE

$$y''(\theta) - 2\tan\theta \cdot y'(\theta) - 2\sec^2\theta \cdot y(\theta) = -2\tan\theta \cdot \sec\theta. \tag{2.3.14}$$

Rewrite this as

$$(y' - 2\tan\theta y)' = -2\tan\theta \cdot \sec\theta,$$

we find the general solutions are

$$y = -2\tan\theta \cdot \sec\theta + c_1(\theta\sec^2\theta + \tan\theta) + c_2\sec^2\theta.$$

In order for the limits of the solutions to exist at $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$, we have to let $c_2 = 0, c_1 = \frac{4}{\pi}$. This leads us to $\psi : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$,

$$\begin{cases} \psi(\theta) = \frac{4}{\pi}(\theta\sec^2\theta + \tan\theta) - 2\tan\theta \cdot \sec\theta, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \psi\left(\frac{\pi}{2}\right) = 1, \quad \psi\left(-\frac{\pi}{2}\right) = -1 \end{cases}$$

So ψ satisfies the ODE (2.3.14) and is an odd function which is continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We computer $\psi'(\theta) = 2\sec^3\theta\left[\frac{4}{\pi}(\cos\theta + \theta\sin\theta) - 1 - \sin^2\theta\right] > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Hence $|\psi(\theta)| \le 1$.

Now we show

$$\phi(\theta) \le \psi(\theta) \quad \text{on } [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]. \tag{2.3.15}$$

Otherwise,

$$\max_{\theta\in[-\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta]} (\phi(\theta) - \psi(\theta)) = b > 0.$$

Since $\phi(-\frac{\pi}{2} + \delta) = \phi(\frac{\pi}{2} - \delta) < -1$ and $|\psi(\theta)| \le 1$, the maximum is achieved at an interior point $\theta_0 \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$. Then $y(\theta) = \psi(\theta) + b$ is an upper barrier of $\phi(\theta)$ at θ_0. Clearly $y'(\theta_0) \ge 0$. Moreover

$$-1 \le \psi(\theta_0) \le y(\theta_0) = \psi(\theta_0) + b = \phi(\theta_0) \le 1.$$

Hence Lemma 2.3.3 yields

$$\begin{aligned} \phi(\theta_0) = y(\theta_0) &\le \sin\theta_0 - \sin\theta_0\cos\theta_0\, y'(\theta_0) + \frac{1}{2}\cos^2\theta_0\, y''(\theta_0) \\ &= \sin\theta_0 - \sin\theta_0\cos\theta_0\, \psi'(\theta_0) + \frac{1}{2}\cos^2\theta_0\, \psi''(\theta_0) \\ &= \psi(\theta_0) \end{aligned}$$

which contradicts to $b > 0$.

Now we can finish the proof of the theorem. By (2.3.12), (2.3.15), we have

$$|\nabla\theta_\epsilon|^2 \le \lambda_1(1 + a_\epsilon\psi(\theta_\epsilon)).$$

Choosing γ as before, we obtain

$$\begin{aligned} d_M\sqrt{\lambda_1} &\geq \int_\gamma \sqrt{\lambda_1}dt \geq \int_{-\frac{\pi}{2}+\delta}^{\frac{\pi}{2}-\delta} \frac{d\theta}{\sqrt{1+a_\epsilon\psi(\theta)}} \\ &= \int_0^{\frac{\pi}{2}-\delta}\left(\frac{1}{\sqrt{1+a_\epsilon\psi(\theta)}}+\frac{1}{\sqrt{1-a_\epsilon\psi(\theta)}}\right)d\theta \\ &= 2\int_0^{\frac{\pi}{2}-\delta}\left(1+\sum_{k=1}^{\infty}\frac{(4k-1)!!}{(4k)!!}a_\epsilon^{2k}\psi^{2k}(\theta)\right)d\theta \\ &\geq \pi-2\delta+\frac{3}{4}a_\epsilon^2\int_0^{\frac{\pi}{2}}\psi^2(\theta)d\theta. \end{aligned}$$

Letting $\epsilon\to 0^+$, hence $\delta\to 0$, we get

$$d_M\sqrt{\lambda_1}\geq \pi+\frac{3}{4}a^2\int_0^{\frac{\pi}{2}}\psi^2(\theta)d\theta.$$

Therefore, $\lambda_1 \geq \frac{\pi^2}{d_M^2}$ and the inequality is strict unless $a=0$.

Now when $a=0$ and $\lambda_1=\frac{\pi^2}{d_M^2}$, the inequalities (2.3.11) (2.3.10) (2.3.8) all become equalities as $\epsilon\to 0$. Therefore we have $\lambda_1=|\nabla\theta|^2$, $\operatorname{Hess}\theta=0$ and $M=S^1\times N$ splits isometrically. ■

Proof of Lemma 2.3.3. Consider the function

$$G(x)=|\nabla\theta_\epsilon(x)|^2-\lambda_1(1+a_\epsilon y(\theta_\epsilon(x))),$$

then $G(x)\leq 0, G(x_0)=0$, where $\theta_0=\theta_\epsilon(x_0)$, i.e. $G(x)$ achieves the maximum value at x_0. By maximum principle,

$$\nabla G(x_0)=0,\quad \Delta G(x_0)\leq 0.$$

I.e.

$$\begin{aligned} |\nabla\theta_\epsilon|^2(x_0) &= \lambda_1(1+a_\epsilon y(\theta_\epsilon(x_0))), \\ \nabla|\nabla\theta_\epsilon|^2(x_0) &= \lambda_1 a_\epsilon y'(\theta_\epsilon(x_0))\cdot(\nabla\theta_\epsilon)(x_0), \\ \Delta|\nabla\theta_\epsilon|^2(x_0) &\leq \lambda_1 a_\epsilon(\Delta[y(\theta_\epsilon)])(x_0). \end{aligned}$$

By (2.3.8) and (2.3.9),

$$\Delta|\nabla\theta_\epsilon|^2\geq 2\sec^2\theta_\epsilon[-\lambda_1(1+a_\epsilon\sin\theta_\epsilon)+|\nabla\theta_\epsilon|^2]|\nabla\theta_\epsilon|^2+2\tan\theta_\epsilon\langle\nabla|\nabla\theta_\epsilon|^2,\nabla\theta_\epsilon\rangle.$$

By (2.3.7),

$$\begin{aligned} \Delta[y(\theta_\epsilon(x))] &= y''|\nabla\theta_\epsilon(x)|^2+y'\Delta\theta_\epsilon(x) \\ &= (y''+y'\tan\theta_\epsilon)|\nabla\theta_\epsilon|^2-\lambda_1(\tan\theta_\epsilon+a_\epsilon\sec\theta_\epsilon)y'. \end{aligned}$$

Combining these together and dividing by $\lambda_1 a_\epsilon|\nabla\theta_\epsilon|^2$, we have at x_0,

$$2\sec^2\theta_\epsilon\left[-\sin\theta_\epsilon+y\right]+2\tan\theta_\epsilon y'\leq y''+y'\tan\theta_\epsilon-\frac{\tan\theta_\epsilon+a_\epsilon\sec\theta_\epsilon}{1+a_\epsilon y}y'.\tag{2.3.16}$$

Since $|y(\theta_0)| \leq 1,\ 0 < a_\epsilon < 1$, we have

$$\tan\theta_\epsilon - \frac{\tan\theta_\epsilon + a_\epsilon \sec\theta_\epsilon}{1 + a_\epsilon y} = \frac{a_\epsilon}{\cos\theta_\epsilon} \cdot \frac{y\sin\theta_\epsilon - 1}{1 + a_\epsilon y} \leq 0.$$

Also $y'(\theta_0) \geq 0$, therefore (2.3.16) gives

$$2\sec^2\theta_\epsilon\left[-\sin\theta_\epsilon + y\right] + 2\tan\theta_\epsilon y' \leq y''$$

at x_0, which is (2.3.13). ■

For compact manifolds M^n with $\mathrm{Ric} \geq -(n-1)H^2$, Li-Yau [81] also obtained the lower bound estimate $\lambda_1 \geq \frac{C_1(n)}{d_M^2}\exp(-C_2(n)d_M H)$, see [116, 79].

For noncompact manifolds

2.3.2 Dirichlet and Neumann Eigenvalue Comparison

As another application of Laplacian comparison, one has the following Dirichlet eigenvalue comparison theorem of Cheng [45].

Theorem 2.3.4 *Let M be a complete Riemannian manifold with* $\mathrm{Ric} \geq (n-1)H$. *For any $x \in M, R > 0$, if $\partial B(x,R) \neq$, then*

$$\lambda_1(B(x,R)) \leq \lambda_1^H(R), \tag{2.3.17}$$

where $\lambda_1^H(R$ is the first Dirichlet eigenvalue of R balls in the model space M_H^n. Equality holds if and only if $B(x,R)$ is isometric to $B_H(R)$.

(contradiction to metaphysics?, understand in barrier sense, $B(x,R)$ may not be a regular domain.)

Before we give the proof, let us look at the eigenvalues of $B_H(R)$. Write the metric of M_H^n in polar coordinates $g = dr^2 + \mathrm{sn}_H^2(r)g_{S^{n-1}}$. Then

$$\Delta_H = \mathrm{sn}_H^{1-n}\frac{\partial}{\partial r}\left(\mathrm{sn}_H^{n-1}\frac{\partial}{\partial r}\right) + \mathrm{sn}_H^{-2}\Delta_{S^{n-1}}.$$

Using separation of variables, one finds that eigenfunctions ϕ of eigenvalue λ are of the form $\phi = T(r)G(\theta)$, with $G(\theta)$ eigenfunctions of S^{n-1}:

$$\Delta_{S^{n-1}}G + \nu G = 0,$$

while $T(r)$ satisfies the ODE

$$T'' + (n-1)\frac{\mathrm{cn}_H}{\mathrm{sn}_H}T' + (\lambda - \frac{\nu}{\mathrm{sn}_H^2})T = 0.$$

For Dirichlet boundary condition on $B_H(R)$, T is also required to satisfy $T(R) = 0$. There is an additional requirement for T at $r = 0$ coming out of the smoothness of ϕ at the origin $r = 0$.

For the eigenfunction ϕ associated with the first eigenvalue $\lambda_1^H(R)$, we must have $G \equiv 1$ and $\nu = 0$ (since ϕ does not change sign hence nor can G). Therefore $\phi = T(r)$ with

$$T'' + (n-1)\frac{\mathrm{cn}_H}{\mathrm{sn}_H}T' + \lambda T = 0, \qquad \lambda = \lambda_1^H(R). \tag{2.3.18}$$

We can now turn to the proof of the theorem.

Proof: Consider the test function $\phi(y) = T(d(x,y))$. Obviously it satisfies the Dirichlet boundary condition on $B(x,R)$. Also, since $T > 0$, (2.3.18) gives

$$\mathrm{sn}_H^{n-1}(r)T'(r) = -\lambda \int_0^r \mathrm{sn}_H^{n-1}(s)T(s)ds \leq 0.$$

Hence, by the global Laplacian comparison, Theorem 1.2.10,

$$\Delta T(r) \geq \Delta_H T(r) = -\lambda\, T(r).$$

Multiplying this inequality by $T(r)$ and integrating over $B(x,R)$, we obtain

$$\lambda_1(B(x,R)) \leq \|\phi\|^{-2}\int_{B(x,R)} |\nabla\phi|^2 d\mathrm{vol} \leq \lambda = \lambda_1^H(R),$$

with the equality holds iff we have equality in the Laplacian comparison. Hence $B(x,R)$ must be isometric to $B_H(R)$ in that case. ■

2.3.3 Heat Kernel Comparison

The volume element comparison (1.4.2) can also be used to prove a heat kernel comparison [43].

2.4 Isoperimetric Inequality

2.5 Abresch-Gromoll's Excess Estimate

Abresch-Gromoll's excess estimate [1] gives the first distance estimate in terms of a lower Ricci curvature bound, giving a new tool for studying manifolds with Ricci curvature lower bound. It is in the spirit of splitting theorem, instead of having a line, one only has a long segment.

Definition 2.5.1 *Given $p, q \in M$, the excess function associated to p and q is*

$$e(x) = d(p,x) + d(q,x) - d(p,q).$$

Clearly, e is a nonnegative Lipschitz function with Lipschitz constant ≤ 2.

The key tool is the following estimate for Lipschitz functions whose Laplacian is bounded from above. It uses the maximal principle and Laplacian comparison.

Theorem 2.5.2 (Abresch-Gromoll, 1990) *If* $\mathrm{Ric}_M \geq (n-1)H(H \leq 0)$ *and* $U : \overline{B(y,R)} \subset M^n \to \mathbb{R}$ *is a Lipschitz function with*

1. $U \geq 0$,
2. $Lip\,(U) \leq a$,
3. $U(y_0) = 0$ *for some* $y_0 \in B(y,R)$ *and*
4. $\Delta U \leq b$ *in the barrier sense.*

Then $U(y) \leq ac + G(c)$ *for all* $0 < c < R$*, where* $G(r(x))$ *is the smallest function on the model space* M^n_H *such that:*

1. $G(r) > 0$ *for* $0 < r < R$.
2. $G'(r) < 0$ *for* $0 < r < R$.
3. $G(R) = 0$.
4. $\Delta_H G \equiv b$.

Proof: For late use we construct G explicitly. Since $\Delta_H = \frac{\partial^2}{\partial r^2} + m_H(r)\frac{\partial}{\partial r} + \tilde{\Delta}$ (see (1.3.12) and (1.2.6)), we would like to solve the ODE

$$G'' + m_H(r)G' = b.$$

For $H = 0$, this is

$$\begin{aligned} G'' + (n-1)G'/r &= b, \\ \text{or}\quad G''r^2 + (n-1)G'r &= br^2, \end{aligned}$$

which is an Euler type O.D.E. For $n \geq 3$, the solutions are $G = G_p + G_h$, where $G_p = \frac{b}{2n}r^2$ and $G_h = c_1 + c_2 r^{-(n-2)}$.

Now $G(R) = 0$ gives

$$\frac{b}{2n}R^2 + c_1 + c_2 R^{-(n-2)} = 0,$$

while $G' < 0$ gives

$$\frac{b}{n}r - (n-2)c_2 r^{-(n-1)} < 0$$

for all $0 < r < R$. Thus $c_2 \geq \frac{b}{n(n-2)}R^n$.

Hence $G(r) = \frac{b}{2n}\left(r^2 + \frac{2R^n}{n-2}r^{-(n-2)} - \frac{n}{n-2}R^2\right)$. Note that $G > 0$ follows from $G(R) = 0$ and $G' < 0$.

For $H < 0$,

$$G(r) = b\int_r^R \int_r^t \left(\frac{\sinh\sqrt{-H}t}{\sinh\sqrt{-H}s}\right)^{n-1} dsdt.$$

By the Laplacian comparison (Theorem 1.3.8) we have $\Delta G \geq \Delta_H G = b$. Consider $V = G - U$. Then $\Delta V = \Delta G - \Delta U \geq 0$. Fix $0 < c < R$. Apply the maximal principle to $V : \overline{A(y,c,R)} \to \mathbb{R}$ gives

$$V(x) \leq \max\{V|_{\partial B(R)}, V|_{\partial B(c)}\}$$

for all $x \in \overline{A(y,c,R)}$. By assumption $V|_{\partial B(R)} \leq 0$ and $V(y_0) > 0$. If $y_0 \in A(y,c,R)$, then $\max V|_{\partial B(c)} > 0$, so $V(y') > 0$ for some $y' \in \partial B(y,c)$. Since

$$U(y) - U(y') \leq ad(y,y') = ac$$

and

$$G(c) - U(y') = V(y') > 0,$$

we have

$$U(y) \leq ac + U(y') < ac + G(c).$$

Now if $d(y,y_0) \leq c$,

$$\begin{aligned} U(y) &= U(y) - U(y_0) \leq ad(y,y_0) \\ &\leq ac \leq ac + G(c). \end{aligned}$$

This finishes the proof. ■

By applying this result to $e(x)$ Abresch-Gromoll obtained an estimate for thin triangles. First we fix some notations. If γ is a minimal geodesic connecting p and q with $\gamma(0) = p$ and $\gamma(1) = q$, let $h(x) = \min_{0 \leq t \leq 1} d(x, \gamma(t))$ be the height. Then

$$0 \leq e(x) \leq 2h(x).$$

Let y be the point along γ between p and q with $d(x,y) = h(x)$. Set

$$\begin{aligned} s_1 = d(p,x), \quad t_1 = d(p,y) \\ s_2 = d(q,x), \quad t_2 = d(q,y). \end{aligned}$$

DRAW A PICTURE!!!

Example 2.5.3 *In* $\mathbb{R}^n$,

$$s_1 = \sqrt{h^2 + t_1^2} = t_1\sqrt{1 + (h/t_1)^2} \leq t_1(1 + (h/t_1)^2).$$

Thus

$$\begin{aligned} e(x) &= s_1 + s_2 - t_1 - t_2 \\ &\leq h^2/t_1 + h^2/t_2 \\ &= h(h/t_1 + h/t_2) \\ &\leq 2h(h/t), \end{aligned}$$

where $t = \min\{t_1, t_2\}$. *Thus* $e(x)$ *is small if* h^2/t *is small.*

If M has nonnegative sectional curvature then the Toponogov comparison shows that $s_1 \le \sqrt{h^2+t_1^2}$, so the same estimate holds. For manifolds with nonnegative Ricci curvature obtained an estimate along the same line.

Theorem 2.5.4 (Abresch-Gromoll 1990) *If $M^n (n \ge 3)$ has $\mathrm{Ric}_M \ge 0$, $h(x) \le \frac{s(x)}{2}$, where $s = \min\{s_1, s_2\}$, then*

$$e(x) \le 8\left(\frac{h^n}{s}\right)^{\frac{1}{n-1}}. \tag{2.5.19}$$

Proof: Recall $e(x) \ge 0$, $\mathrm{Lip}\, e \le 2$. Let $R = h(x)$. By Laplacian comparison (Theorem 1.3.8), on $B(x,R)$,

$$\Delta e \le \frac{n-1}{s_1-h} + \frac{n-1}{s_2-h} \le \frac{4(n-1)}{s(x)}.$$

Thus by Theorem 2.5.2, with $a=2, b=\frac{4(n-1)}{s(x)}$,

$$e(x) \le 2c + G(c)$$

for all $0<c<h$.

The function $ac+G(c)$, $0<c<h$ is convex and the infimum is assumed at the unique $c_0 \in (0,h)$ with $a + G'(c_0) = 0$ or more explicitly,

$$c_0^{n-1} = \frac{b}{an}(h^n - c_0^n) \le \frac{b}{an}h^n.$$

Therefore

$$\begin{aligned} e(x) &\le 2c_0 + G(c_0) = 2c_0 + \frac{b}{2n}\left(c_0^2 + \frac{2}{n-2}h^n c_0^{2-n} - \frac{n}{n-2}h^2\right) \\ &= 2\frac{n-1}{n-2}c_0 + \frac{b}{2n}\cdot\frac{n}{n-2}(c_0^2-h^2) \\ &\le 2\frac{n-1}{n-2}\left(\frac{b}{an}h^n\right)^{\frac{1}{n-1}} \le 8\left(\frac{h^n}{s}\right)^{\frac{1}{n-1}}. \end{aligned}$$

■

Remark There is similar estimate for $\mathrm{Ric}_M \ge (n-1)H$: $e(x) \le hF(\frac{h}{s})$ for some continuous F satisfying $F(0)=0$ [1].

Using the above estimate Abresch-Gromoll proved that

2.6 Almost Splitting Theorem

Chapter 3

Topology of Manifolds with Ricci Curvature Lower Bound

3.1 First Betti Number Estimate

For a manifold M, its first Betti number is the dimension of the first cohomology,

$$b_1(M) = \dim H_1(M, \mathbb{R}).$$

By Hodge theorem $H_1(M, \mathbb{R})$ is isomorphic to $\mathcal{H}^1(M)$, the space of harmonic 1-forms. Using this and Bochner's formula we can give a quick estimate on $b_1(M)$ for manifolds with nonnegative Ricci curvature [17].

Theorem 3.1.1 *If M^n is a compact Riemannian manifold with $\mathrm{Ric} \geq 0$, then $b_1(M) \leq n$, with equality holds iff M isometric to the flat torus T^n.*

$b_1(M)$ is closely related to the fundamental group $\pi_1(M)$. In fact $H_1(M, \mathbb{Z}) = \pi_1(M)/[\pi_1(M), \pi_1(M)]$, the abelianization of $\pi_1(M)$. Since $H(M, \mathbb{Z})$ is abelian, the set of its torsion elements, T, is a normal subgroup. Hence $\Gamma = H_1(M, \mathbb{Z})/T$ is a free abelian group. Moreover,

$$b_1(M) = \mathrm{rank}(\Gamma) = \mathrm{rank}(\Gamma'),$$

where Γ' is any subgroup of Γ with finite index.

Let $\bar{M} = \tilde{M}/[\pi_1(M), \pi_1(M)]/T$ be the covering space of M. Then Γ acts isometrically as deck transformations on $\bar{M}$. To estimate $b_1(M)$ one needs to choose a good finite index subgroup of Γ. This is possible when M is compact.

Lemma 3.1.2 (Gromov, 1980) *For fixed $x \in \bar{M}$ there is a subgroup $\Gamma' \leq \Gamma$, $[\Gamma : \Gamma']$ finite, such that $\Gamma' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$, and*

1. $d(x, \gamma_k(x)) \leq 2\mathrm{diam}_M$
2. *For any* $\gamma \in \Gamma' - \{e\}, d(x, \gamma(x)) > \mathrm{diam}_M$.

Proof: For each $\varepsilon \geq 0$ let $\Gamma_\varepsilon \leq \Gamma$ be generated by

$$\{\gamma \in \Gamma : d(x, \gamma(x)) < 2\mathrm{diam}_M + \varepsilon\}.$$

We will show that $\mathrm{diam}(\bar{M}/\Gamma_\varepsilon) < 2\,\mathrm{diam}_M + 2\varepsilon$ so $\bar{M}/\Gamma_\varepsilon$ is compact and Γ_ε has finite index.

Let $\pi_\varepsilon : \bar{M} \to \bar{M}/\Gamma_\varepsilon$ be the covering projection. If $\mathrm{diam}(\bar{M}/\Gamma_\varepsilon) \geq 2\mathrm{diam}_M + 2\varepsilon$, we can find $z \in \bar{M}$ such that $d(x, z) = d(\pi_\epsilon(x), \pi_\epsilon(z)) = \mathrm{diam}_M + \varepsilon$. Now there exists $\gamma \in \Gamma$ such that $d(\gamma(x), z) \leq \mathrm{diam}_M$. Then

$$\begin{aligned} d(\pi_\varepsilon(x), \pi_\varepsilon(\gamma(x))) &\geq d(\pi_\varepsilon(x), \pi_\varepsilon(z)) - d(\pi_\varepsilon(z), \pi_\varepsilon(\gamma(x))) \geq \varepsilon, \\ d(x, \gamma(x)) &\leq d(x, z) + d(z, \gamma(x)) \leq 2\mathrm{diam}_M + \varepsilon. \end{aligned}$$

We have $\gamma \notin \Gamma_\epsilon$ and $\gamma \in \Gamma_\epsilon$, a contradiction.

There are at most finite many elements in $\{\gamma \in \Gamma : d(x, \gamma(x)) \leq 3\mathrm{diam}_M\}$. Hence there is ε small such that

$$\{\gamma \in \Gamma : d(x, \gamma(x)) \leq 2\mathrm{diam}_M\} = \{\gamma \in \Gamma : d(x, \gamma(x)) \leq 2\mathrm{diam}_M + \varepsilon\}.$$

Pick such an $\varepsilon > 0$, we have $\Gamma_\varepsilon \leq \Gamma$ with finite index, and

$$\Gamma_\varepsilon = \langle \gamma_1, \ldots, \gamma_m \rangle, \quad d(x, \gamma_k(x)) \leq 2\,\mathrm{diam}_M.$$

Now we modify these generators (by choosing the longer ones) to get Γ'. First, since $\mathrm{rank}(\Gamma_\epsilon) = b_1(M)$, we can pick a subset of linearly independent generators $\gamma_1, \ldots, \gamma_{b_1}$ so that $\Gamma'' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ has finite index in Γ_ε.

Let $\Gamma' = \langle \tilde{\gamma}_1, \ldots, \tilde{\gamma}_{b_1} \rangle$, where $\tilde{\gamma}_k = \ell_{k1}\gamma_1 + \cdots + \ell_{kk}\gamma_k$ and the coefficients ℓ_{ki} are chosen so that ℓ_{kk} is maximal with respect to the constraints:

1. $d(x, \tilde{\gamma}_k(x)) \leq 2\mathrm{diam}_M$ and
2. $\mathrm{span}\{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_k\} \leq \mathrm{span}\{\gamma_1, \ldots, \gamma_k\}$ with finite index

for all $k = 1, \cdots, b_1$.

Then $\Gamma' \leq \Gamma''$ has finite index and first property of the Lemma is satisfied. To verify the second property, suppose there exists $\gamma \in \Gamma' - \{e\}$ with $d(x, \gamma(x)) \leq \mathrm{diam}_M$, write

$$\gamma = m_1\tilde{\gamma}_1 + \cdots + m_k\tilde{\gamma}_k,$$

with $m_k \neq 0$. Then $d(x, \gamma^2(x)) \leq 2d(x, \gamma(x)) \leq 2\mathrm{diam}_M$, but

$$\begin{aligned} \gamma^2 &= 2m_1\tilde{\gamma}_1 + \cdots + 2m_k\tilde{\gamma}_k \\ &= (\text{terms involving } \gamma_i, i < k) + 2m_k\ell_{kk}\gamma_k, \end{aligned}$$

which contradicts the maximality of ℓ_{kk}. ∎

Theorem 3.1.3 (Gromov, Gallot) *Suppose M^n is a compact manifold with $\mathrm{Ric}_M \geq (n-1)H$ and $\mathrm{diam}_M \leq D$. There is a function $C(n, HD^2)$ such that $b_1(M) \leq C(n, HD^2)$ and $\lim_{x\to 0^-} C(n,x) = n$ and $C(n,x) = 0$ for $x > 0$. In particular, if HD^2 is small, $b_1(M) \leq n$.*

Proof: When $H > 0$, by Myers' theorem (Theorem 1.2.3), $b_1(M) = 0$. So we can assume $H \leq 0$.

Let $\Gamma' = \langle \gamma_1, \ldots, \gamma_{b_1} \rangle$ be as in the lemma. Then, for $i \neq j$, $d(\gamma_i(x), \gamma_j(x)) = d(x, \gamma_i^{-1}\gamma_j(x)) > \mathrm{diam}_M$. Thus

$$B(\gamma_i(x), \mathrm{diam}_M/2) \cap B(\gamma_j(x), \mathrm{diam}_M/2) = \emptyset$$

for $i \neq j$. Also

$$B(\gamma_i(x), \mathrm{diam}_M/2) \subset B(x, 2\mathrm{diam}_M + \mathrm{diam}_M/2)$$

for all i, so that

$$\bigcup_{i=1}^{b_1} B(\gamma_i(x), \mathrm{diam}_M/2) \subset B(x, 2\mathrm{diam}_M + \mathrm{diam}_M/2).$$

Hence

$$b_1 \leq \frac{\mathrm{Vol}B(x, 5\mathrm{diam}_M/2)}{\mathrm{Vol}B(x, \mathrm{diam}_M/2)}.$$

By the relative volume comparison (1.4.7),

$$\frac{\mathrm{Vol}B(x, 5\mathrm{diam}_M/2)}{\mathrm{Vol}B(x, \mathrm{diam}_M/2)} \leq \frac{\mathrm{Vol}_H B(5\mathrm{diam}_M/2)}{\mathrm{Vol}_H B(\mathrm{diam}_M/2)}.$$

Since $\mathrm{diam}_M \leq D$, when $H \leq 0$ we have

$$\frac{\mathrm{Vol}_H B(5\mathrm{diam}_M/2)}{\mathrm{Vol}_H B(\mathrm{diam}_M/2)} \leq \frac{\mathrm{Vol}_H B(5D/2)}{\mathrm{Vol}_H B(D/2)} = \begin{cases} 5^n, & H = 0 \\ \frac{\int_0^{5D\sqrt{-H}/2} (\sinh r)^{n-1} dr}{\int_0^{D\sqrt{-H}/2} (\sinh r)^{n-1} dr}, & H < 0 \end{cases}.$$

Therefore $b_1 \leq C(n, HD^2)$.

To get a better estimate when HD^2 is small, consider

$$U(s) = \{\gamma \in \Gamma' : \gamma = l_1\gamma_1 + \cdots + l_{b_1}\gamma_{b_1}, \quad |l_1| + \cdots + |l_{b_1}| \leq s\}.$$

Then

$$\bigcup_{\gamma \in U(s)} B(\gamma(x), D/2) \subset B(x, 2Ds + D/2),$$

whence

$$\begin{aligned} \#U(s) &\leq \frac{\mathrm{Vol}B(x, 2Ds + D/2)}{\mathrm{Vol}B(x, D/2)} \\ &\leq \frac{\mathrm{Vol}_H B(2Ds + D/2)}{\mathrm{Vol}_H B(D/2)} = \begin{cases} (2s + \frac{1}{2})^n, & H = 0 \\ \frac{\int_0^{(2s+\frac{1}{2})D\sqrt{-H}} (\sinh r)^{n-1} dr}{\int_0^{D\sqrt{-H}/2} (\sinh r)^{n-1} dr}, & H < 0 \end{cases} \end{aligned}$$

If $b_1 \geq n+1$, then $U(s) \geq \left(\frac{2}{n+1}(s-n)\right)^{n+1}$ (see Example 3.2.3). For $s = n+1+3^n(n+1)^{n+1}$, we can find $\epsilon(n) > 0$ small such that when $HD^2 \geq -\epsilon(n)$,

$$\frac{\int_0^{(2s+\frac{1}{2})D\sqrt{-H}}(\sinh r)^{n-1}dr}{\int_0^{D\sqrt{-H}/2}(\sinh r)^{n-1}dr} \leq \frac{2(2s+\frac{1}{2})^n(D\sqrt{-H})^n}{(\frac{1}{2})^n(D\sqrt{-H})^n} \leq 2^{n+1}(3s)^n.$$

We get

$$\left(\frac{2}{n+1}(s-n)\right)^{n+1} \leq 2^{n+1}(3s)^n,$$

which is impossible when $s > n+3^n(n+1)^{n+1}$. Therefore we have $b_1(M) \leq n$ when $HD^2 \geq -\epsilon(n)$. ■

Gallot proved this using Bochner technique [55].

The celebrated Betti number estimate of Gromov [61] shows that all higher Betti numbers can be bounded by sectional curvature lower bound and diameter upper bound. Sha-Yang first constructed examples showing this is not true for Ricci curvature. They constructed metrics of positive Ricci curvature on the connected sum of k copies of $S^2 \times S^2$ for all $k \geq 1$ using semi-local surgery [119]. Recently using Seifert bundles over orbifolds with a Kähler Einstein metric Kollar showed that there are Einstein metrics with positive Ricci curvature on the connected sums of arbitrary number of copies of $S^2 \times S^3$ [76].

For M^n with $\mathrm{Ric}_M \geq (n-1)H$ and $\mathrm{diam}_M \leq D$, the number of generators of $\pi_1(M)$ is uniformly bounded by $C(n,H,D)$.

3.2 Fundamental Groups

In lower dimensions ($n \leq 3$) a Ricci curvature lower bound has strong topological implications. R. Hamilton [70] proved that compact manifolds M^3 with positive Ricci curvature are space forms. Schoen-Yau [115] proved that any complete open manifold M^3 with positive Ricci curvature must be diffeomorphic to $\mathbb{R}^3$ using minimal surfaces. In general Ricci curvature lower gives very good control on the fundamental group.

For manifolds with uniform positive Ricci curvature lower bound, by Myers' theorem (Theorem 1.2.3), the fundamental group is finite. Any finite group can be realized as the fundamental group of a compact manifold with positive Ricci curvature since any finite group is a subgroup of $SU(n)$ (for n sufficiently big) and $SU(n)$ has a metric with positive Ricci curvature (in fact Einstein).

For manifolds with nonnegative Ricci curvature, the volume comparison and Cheeger-Gromoll's splitting theorem give many structures on the fundamental group.

We first recall a rough measurement on the size of a group introduced by Milnor [94].

3.2.1 Growth of Groups

Suppose Γ is a finitely generated group, say $\Gamma = \langle g_1, \ldots, g_k \rangle$. Any $g \in \Gamma$ can be written as a word $g = \prod_i g_{k_i}^{n_i}$, where $k_i \in \{1, \ldots, k\}$. Define the length of this word to be $\sum_i |n_i|$, and let $|g|$ be the mininimum of the lengths of all word representations of g. Note that $|\cdot|$ depends on the choice of generators.

Fix a set of generators for Γ. The growth function of Γ is

$$\Gamma(s) = \#\{g \in \Gamma : |g| \leq s\}.$$

Example 3.2.1 *If Γ is a finite group then $\Gamma(s) \leq |\Gamma|$.*

Example 3.2.2 *If $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$, then $\Gamma = \langle g_1, g_2 \rangle$, where $g_1 = (1,0)$ and $g_2 = (0,1)$. Any $g \in \Gamma$ can be written as $g = s_1 g_1 + s_2 g_2$. To find $\Gamma(s)$, we need to count the number of elements with $|s_1| + |s_2| \leq s$. By counting the numbers with $|s_1| = 0, 1, 2, \cdots, s$ we have*

$$\begin{aligned} \Gamma(s) &= 2s + 1 + \sum_{t=1}^{s} 2(2(s-t)+1) \\ &= 2s^2 + 2s + 1, \end{aligned}$$

a polynomial of degree 2.

Example 3.2.3 *Let $\Gamma = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{k \geq 1}$. With respect to the standard generators, $\Gamma(s)$ corresponds to the number of elements $(s_1, \cdots, s_k)$ with $|s_1| + \cdots + |s_k| \leq s$. Let i be the number of s_i's which are not zero. Then $\Gamma(s) = \sum_{i=0}^{k} 2^i \binom{k}{i} \binom{s}{i}$.*

Example 3.2.4 *For the free group $\Gamma = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{k \geq 2}$, with respect to the standard generators, the growth function is given by*

$$\Gamma(s) = \frac{k(2k-1)^s - 1}{k-1}.$$

Definition 3.2.5 *Γ is said to have polynomial growth of degree $\leq n$ if for each set of generators the growth function $\Gamma(s) \leq as^n$ for some $a > 0$.*

Γ is said to have exponential growth if for each set of generators the growth function $\Gamma(s) \geq e^{as}$ for some $a > 0$.

Note that for each finitely generated group Γ there always exists $a > 0$ so that $\Gamma(s) \leq e^{as}$.

Lemma 3.2.6 *The type of growth of a group γ is independent of the choice of generators. If for some set of generators, $\Gamma(s) \leq as^n$ for some $a > 0$, then Γ has polynomial growth of degree $\leq n$. If for some set of generators, $\Gamma(s) \geq e^{as}$ for some $a > 0$, then Γ has exponential growth.*

Proof: If $\{g_1, \cdots, g_k\}$ and $\{h_1, \cdots, h_l\}$ are two sets of generators, let $\Gamma_g(s), \Gamma_h(s)$ be the corresponding growth functions. Write each generator h_i in terms of $g_1, \cdots, g_k$ and g_i in terms of $h_1, \cdots, h_l$. There are s_0, t_0 such that all the length are bounded as follows,

$$|h_i|_g \le s_0, \quad |g_i|_h \le t_0.$$

Therefore $\Gamma_h(s) \le \Gamma_g(s_0 s)$ and $\Gamma_g(s) \le \Gamma_h(t_0 s)$, which gives

$$\Gamma_h(\frac{s}{s_0}) \le \Gamma_g(s) \le \Gamma_h(t_0 s).$$

The functions have the same type. ■

From this and Examples 3.2.3 and 3.2.4 we have the abelian group $\mathbb{Z}^k$ has polynomial growth of degree k and the free group $\mathbb{Z} * \mathbb{Z}$ has exponential growth.

Gromov gives the following beautiful characterization of groups with polynomial growth, they are not too far away from abelian groups [67].

Theorem 3.2.7 (Gromov, 1981) *A finitely generated group Γ has polynomial growth iff Γ is almost nilpotent, i.e. it contains a nilpotent subgroup of finite index.*

3.2.2 Fundamental Group of Manifolds with Nonnegative Ricci Curvature

First as an application of their splitting theorem (Theorem 2.1.5) Cheeger-Gromoll showed that for a compact manifold M with nonnegative Ricci curvature, $\pi_1(M)$ has an abelian subgroup of finite index [40].

Theorem 3.2.8 (Cheeger-Gromoll 1971) *If M^n is compact with $\mathrm{Ric}_m \ge 0$, then its universal cover $\widetilde{M} \overset{iso}{\simeq} N \times \mathbb{R}^k$, where N is a compact $(n-k)$ dimensional manifold. Thus there is exact sequence*

$$0 \to F \to \pi_1(M) \to B_k \to 0,$$

where F is a finite group and B_k is the fundamental group of some compact k-dimensional flat manifold — Bieberbach group.

This immediately implies Theorem 3.1.1. Also it gives

Corollary 3.2.9 *If M^n is compact with $\mathrm{Ric}_M \ge 0$ and $\mathrm{Ric}_M > 0$ at one point, then $\pi_1(M)$ is finite.*

Remark This corollary improves the theorem of Bonnet-Myers. The corollary can also be proven using the Bochner technique. In fact, Aubin's deformation gives another metric that has $\mathrm{Ric}_M > 0$ everywhere.
Proof of Theorem 3.2.8. By Theorem 2.1.5, $\widetilde{M} = N \times \mathbb{R}^k$, where N contains no line. We show N is compact.

First note that isometries map lines to lines. Thus, if $\psi \in \mathrm{Iso}(\widetilde{M})$, then $\psi = (\psi_1, \psi_2)$, where $\psi_1 : N \to N$ and $\psi_2 : \mathbb{R}^k \to \mathbb{R}^k$ are isometries. Suppose N is not compact, so N contains a ray $\gamma : [0, \infty) \to M$. Let $\tilde{F} \subset \widetilde{M}$ be a fundamental domain of M, so its closure $\bar{\tilde{F}}$ is compact, and let p_1 be the projection $\widetilde{M} \to N$. Pick any $t_i \to \infty$. For each i there is $g_i \in \pi_1(M)$ such that $g_i(\gamma(t_i)) \in p_1(\tilde{F})$. But $p_1(\bar{\tilde{F}})$ is compact, so we may assume $g_i(\gamma(t_i)) \to p \in N$. Set $\gamma_i(t) = g_i(\gamma(t+t_i))$. Then $\gamma_i : [-t_i, \infty) \to N$ is minimal, and $\{\gamma_i\}$ converges to a line σ in N. Thus N is compact.

To see the exact sequence, let $p_2 : \pi_1(M) \to \mathrm{Iso}(\mathbb{R}^k)$ be the map $\psi = (\psi_1, \psi_2) \mapsto \psi_2$. Then

$$0 \to \ker(p_2) \to \pi_1(M) \to \mathrm{Im}(p_2) \to 0$$

is exact. Now $\ker(p_2) = \{(\psi_1, 0)\}$, while $\mathrm{Im}(\psi_2) = \{(0, \psi_2)\}$. Since $\ker(p_2)$ gives a properly discontinuous group action on a compact manifold, $\ker(p_2)$ is finite. On the other hand, $\mathrm{Im}(p_2)$ is an isometry group on $\mathbb{R}^k$, so $\mathrm{Im}(p_2)$ is a Bieberbach group. ■

Given a compact Riemannian manifold M, Milnor [94] and Svarc [**?**] discovered that the growth rate of $\pi_1(M)$ is the same as the volume growth of the universal cover of M, $\tilde{M}$. This is conceivable since $\tilde{M}$ is the number of $\pi_1(M)$ copies of M glued together. Hence control on the volume growth of the universal cover gives control on the growth of $\pi_1(M)$. M. Anderson generalizes this relation to give a sharper estimate when M is noncompact [4].

Theorem 3.2.10 (Anderson, 1990) *If M^n is a complete Riemannian manifold with $Ric_M \geq 0$ and there is $x \in M, \alpha > 0$ such that $\mathrm{Vol}(B(x,r)) \geq \alpha r^\beta$ for all r big, where $0 \leq \beta \leq n$, then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq n - \beta$. Moreover, when $\beta = n$ (Euclidean volume growth) $\pi_1(M)$ is finite and $|\pi_1(M)| \leq \frac{\omega_n}{\alpha}$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$.*

Any Riemannian manifold satisfies the volume growth condition for $\beta = 0$, so this recovers Milnor's result.

Theorem 3.2.11 (Milnor, 1968) *If M^n is complete with $\mathrm{Ric}_M \geq 0$, then any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of degree $\leq n$.*

When $\beta = n$, Peter Li had proved $|\pi_1(M)| \leq \frac{\omega_n}{\alpha}$ using a heat kernel estimate [78].

Proof of Theorem 3.2.10: Let $\pi : (\tilde{M}, \tilde{x}) \to (M, x)$ be the universal cover of M with the induced metric. Then $\mathrm{Ric}_{\tilde{M}} \geq 0$, and $\pi_1(M)$ acts isometrically on $\tilde{M}$. In order to take account of the volume of M, one covers $\tilde{M}$ with fundamental domains. For each $g \in \pi_1(M)$, let $\tilde{F}_g = \{\tilde{y} \in \tilde{M} \mid d(\tilde{y}, \tilde{x}) \leq d(\tilde{y}, g\tilde{x})\}$, the half space closer to $\tilde{x}$ than $g\tilde{x}$. Then $\tilde{F} = \cap_{g \in \pi_1(M)} \tilde{F}_g$ is a Dirichlet fundamental domain, so that the translates $g\tilde{F}$ cover $\tilde{M}$ and have pairwise disjoint interiors.

Let $\Gamma = \langle g_1, \ldots, g_k \rangle$ be a finitely generated subgroup of $\pi_1(M)$. Set $l = \max_i d(g_i \tilde{x}, \tilde{x})$. If $g \in \Gamma$ satisfies $|g| \leq s$, then

$$g(B(\tilde{x}, s) \cap \tilde{F}) \subseteq B(\tilde{x}, (l+1)s).$$

Therefore

$$\Gamma(s)\mathrm{Vol}(B(\tilde{x}, s) \cap \tilde{F}) \leq \mathrm{Vol}B(\tilde{x}, (l+1)s).$$

Since $\mathrm{Ric}_{\tilde{M}} \geq 0$, the Bishop volume comparison (1.4.6) gives us that

$$\mathrm{Vol}B(\tilde{x}, (l+1)s) \leq \omega_n (l+1)^n s^n.$$

$\mathrm{Vol}(B(\tilde{x}, s) \cap \tilde{F}) = \mathrm{Vol}B(x, s)$ since $\pi : B(\tilde{x}, s) \cap \tilde{F} \to B(x, s)$ is a bijection. Using the assumption on the volume growth we have

$$\Gamma(s) \leq \frac{\omega_n (l+1)^n s^n}{\alpha s^\beta} = \frac{\omega_n (l+1)^n}{\alpha} s^{n-\beta}$$

for all s large. Since $\frac{\omega_n (l+1)^n}{\alpha}$ is a positive constant, we have shown that Γ has polynomial growth of degree at most $n - \beta$.

When $\beta = n$, let Γ be a finite subgroup of $\pi_1(M)$. Since Γ is finite, there is l such that $d(g\tilde{x}, \tilde{x}) \leq l$ for all $g \in \Gamma$. Then we have for all $g \in \Gamma$,

$$g(B(\tilde{x}, s) \cap \tilde{F}) \subseteq B(\tilde{x}, l + s).$$

As above we get

$$|\Gamma| \leq \frac{\mathrm{Vol}B(\tilde{x}, l+s)}{\mathrm{Vol}B(x, s)} \leq \frac{\omega_n (l+s)^n}{\alpha s^n}.$$

Let $s \to \infty$, this gives $|\Gamma| \leq \frac{\omega_n}{\alpha}$. We have the order of all finite subgroup of $\pi_1(M)$ is uniformly bounded by $\frac{\omega_n}{\alpha}$. Hence $\pi_1(M)$ is finite and $|\pi_1(M)| \leq \frac{\omega_n}{\alpha}$. ∎

Example 3.2.12 *Let H be the Heisenberg group*

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, x \in \mathbb{R} \right\},$$

and let

$$H_{\mathbb{Z}} = \left\{ \begin{pmatrix} 1 & n_1 & n_2 \\ 0 & 1 & n_3 \\ 0 & 0 & 1 \end{pmatrix} : n_i \in \mathbb{Z} \right\}.$$

Then $H/H_{\mathbb{Z}}$ is a compact 3-manifold with $\pi_1(H/H_{\mathbb{Z}}) = H_{\mathbb{Z}}$. The growth of $H_{\mathbb{Z}}$ is polynomial of degree 4, so $H/H_{\mathbb{Z}}$ has no metric with $\mathrm{Ric} \geq 0$.

If M^n is a complete noncompact manifold with positive sectional curvature, then M^n is diffeomorphic to $\mathbb{R}^n$ [60]. For positive Ricci curvature, the manifold could have infinite topological type [118]. Sormani [123] found the following nice property about fundamental groups of noncompact manifolds with positive Ricci curvature, namely one can see the fundamental group at infinity.

Definition 3.2.13 *Suppose M^n is noncompact. Then M is said to have the geodesic loops to infinity property if for any ray γ in M, any $g \in \pi_1(M, \gamma(0))$ and any compact $K \subset M$ there is a geodesic loop $\bar{\alpha}$ at γ_{t_0} in $M \setminus K$ such that $(\gamma|_0^{t_0})^{-1} \circ \bar{\alpha} \circ \gamma|_0^{t_0}$ is homotopic to a loop α at $\gamma(0)$ with $g = [\alpha]$.(We say $\bar{\alpha}$ is homotopic to α along γ)*

Example 3.2.14 *Let $M = N \times \mathbb{R}$. If the ray γ is in the splitting direction, then any $g \in \pi_1(M, \gamma(0))$ is homotopic to a geodesic loop at infinity along γ.*

Theorem 3.2.15 (Sormani, 2001) *If M^n is complete and noncompact with $\mathrm{Ric}_M > 0$ then M has the geodesic loops to infinity property.*

This result follows from Cheeger and Gromoll's splitting theorem (Theorem 2.1.5) and the following construction of a line.

Theorem 3.2.16 (Line Theorem) *If M^n does not have the geodesic loops to infinity property then there is a line in the universal cover $\tilde{M}$.*

Proof: Let γ be a ray, $g \in \pi_1(M, \gamma(0))$ such that g does not satisfy the geodesic loops to infinity property. Let α be a representative of g based at $\gamma(0)$. Then there is a compact set K such that there is no closed geodesic contain in $M \setminus K$ which is homotopic to α along γ. Let $R_0 > 0$ such that $K \subset B(\gamma(0), R_0)$, and $r_i > R_0 \to \infty$. Then any loop based at $\gamma(r_i)$ which is homotopic to α along γ must pass through K. Let $\tilde{\alpha}$ be a lift of α to the universal cover $\tilde{M}$ connecting $\tilde{\gamma}(0)$ to $g\tilde{\gamma}(0)$, $\tilde{\gamma}$ the lift of γ starting at $\tilde{\gamma}(0)$, $g\tilde{\gamma}$ the lift of γ starting at $g\tilde{\gamma}(0)$, $\tilde{\alpha}_i$ a minimal geodesic connecting $\tilde{\gamma}(r_i)$ and $g\tilde{\gamma}(r_i)$. Since $\alpha_i = \pi \circ \tilde{\alpha}_i$ is homotopic to α along γ, there is t_i such that $\alpha_i(t_i) \in K$. By triangle inequality $t_i \geq r_i - R_0$. Set $l_i = L(\alpha_i)$, the length of α_i. Then $l_i - t_i \geq d(\alpha_i(t_i), \gamma(r_i)) \geq r_i - R_0$. Therefore as $i \to \infty$, the geodesic segments $\tilde{\alpha}_i$ extend to infinity from both sides of $\tilde{\alpha}_i(t_i)$. The projections of $\tilde{\alpha}_i(t_i)$ all lie in K, so a subsequence has a limit. Hence a subsequence of $\tilde{\alpha}_i$ converges, which gives a line.

GRAPH ■

Theorem 3.2.15 gives

Corollary 3.2.17 *A noncompact manifold with positive Ricci curvature is simply connected if it is simply connected at infinity.*

Using the loops-to-infinity property Shen-Sorman proved that if M^n is noncompact with $\mathrm{Ric}_M > 0$ then $H_{n-1}(M, \mathbb{Z}) = 0$ [120]. See [**?**] for more applications of the loops-to-infinity property.

Combining Theorem 3.2.11 with Theorem 3.2.7, we know that any finitely generated subgroup of $\pi_1(M)$ of manifolds with nonnegative Ricci curvature is almost nilpotent.

From the above one naturally wonders if all nilpotent groups occur as the fundamental group of a complete non-compact manifold with nonnegative Ricci curvature. Indeed, extending the warping product constructions in [98, 15], Wei showed [134] that any finitely generated torsion free nilpotent group could occur as fundamental group of a manifold with positive Ricci curvature. Wilking [138] extended this to any finitely generated almost nilpotent group. This gives a very good understanding of the fundamental group of a manifold with nonnegative Ricci curvature except the following long standing problem regarding the finiteness of generators [94].

Conjecture 3.2.18 (Milnor, 1968) *The fundamental group of a manifold with nonnegative Ricci curvature is finitely generated.*

There is some very good progress in this direction. Using short generators and a uniform cut lemma based on the excess estimate of Abresch and Gromoll [1] (see (2.5.19)) Sormani [125] proved that if $\mathrm{Ric}_M \geq 0$ and M^n has small linear diameter growth, then $\pi_1(M)$ is finitely generated. More precisely the small linear growth condition is:

$$\limsup_{r\to\infty} \frac{\mathrm{diam}\partial B(p,r)}{r} < s_n = \frac{n}{(n-1)3^n}\left(\frac{n-1}{n-2}\right)^{n-1}.$$

The constant s_n was improved in [143]. Then in [141] Wylie proved that in this case $\pi_1(M) = G(r)$ for r big, where $G(r)$ is the image of $\pi_1(B(p,r))$ in $\pi_1(B(p,2r))$. In an earlier paper [124], Sormani proved that all manifolds with nonnegative Ricci curvature and linear volume growth have sublinear diamter growth, so manifolds with linear volume growth are covered by these results.

In a very different direction Wilking [138], using algebraic methods, showed that if $\mathrm{Ric}_M \geq 0$ then $\pi_1(M)$ is finitely generated iff any abelian subgroup of $\pi_1(M)$ is finitely generated, effectively reducing the Milnor conjecture to the study of manifolds with abelian fundamental groups.

Kapovitch-Wilking [74] recently announced a proof of the compact analog of Milnor's conjecture that the fundamental group of a manifold satisfying (**??**) has a presentation with a universally bounded number of generators (as conjectured by this author), and that a manifold which admits almost nonnegative Ricci curvature has a virtually nilpotent fundamental group. The second result would greatly generalize Fukaya-Yamaguchi's work on almost nonnegative sectional curvature [54]. See [135, 136] for earlier partial results.

3.2.3 Finiteness of Fundamental Groups

When the volume is also bounded from below, by using a clever covering argument M. Anderson [5] showed that the number of the short homotopically nontrivial closed geodesics can be controlled and for the class of manifolds M

with $\mathrm{Ric}_M \geq (n-1)H$, $\mathrm{Vol}_M \geq V$ and $\mathrm{diam}_M \leq D$ there are only finitely many isomorphism types of $\pi_1(M)$. Again if the Ricci curvature is replaced by sectional curvature then much more can be said. Namely there are only finitely many homeomorphism types of the manifolds with sectional curvature and volume bounded from below and diameter bounded from above [64, 103]. By [106] this is not true for Ricci curvature unless the dimension is 3 [149].

Lemma 3.2.19 (Gromov, 1980) *For any compact M^n and each $\tilde{x} \in \tilde{M}$ there are generators $\gamma_1, \ldots, \gamma_k$ of $\pi_1(M)$ such that $d(\tilde{x}, \gamma_i \tilde{x}) \leq 2\mathrm{diam}_M$ and all relations of $\pi_1(M)$ are of the form $\gamma_i \gamma_j = \gamma_\ell$.*

Proof. Let $0 < \varepsilon <$ injectivty radius. Triangulate M so that the length of each adjacent edge is less than ε. Let $x_1, \ldots, x_k$ be the vertices of the triangulation, and let e_{ij} be minimal geodesics connecting x_i and x_j.

Connect x to each x_i by a minimal geodesic σ_i, and set $\sigma_{ij} = \sigma_j^{-1} e_{ij} \sigma_i$. Then $\ell(\sigma_{ij}) < 2\mathrm{diam}_M + \varepsilon$, so $d(\tilde{x}, \sigma_{ij}\tilde{x}) < 2\mathrm{diam}_M + \varepsilon$.

We claim that $\{\sigma_{ij}\}$ generates $\pi_1(M)$. For any loop at x is homotopic to a 1-skeleton, while $\sigma_{jk}\sigma_{ij} = \sigma_{ik}$ as adjacent vertices span a 2-simplex. In addition, if $1 = \sigma \in \pi_1(M)$, σ is trivial in some 2-simplex. Thus $\sigma = 1$ can be expressed as a product of the above relations.

Theorem 3.2.20 (Anderson, 1990)) *In the class of manifolds M with $\mathrm{Ric}_M \geq (n-1)H$, $\mathrm{Vol}_M \geq V$ and $\mathrm{diam}_M \leq D$ there are only finitely many isomorphism types of $\pi_1(M)$.*

Remark: The volume condition is necessary. For example, $S^3/\mathbb{Z}_n$ has $K \equiv 1$ and diam $= \pi/2$, but $\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$. In this case, $\mathrm{Vol}(S^3/\mathbb{Z}_n) \to 0$ as $n \to \infty$.

Proof of Theorem. Choose generators for $\pi_1(M)$ as in the lemma; it is sufficient to bound the number of generators.

Let F be a fundamental domain in $\tilde{M}$ that contains $\tilde{x}$. Then

$$\bigcup_{i=1}^{k} \gamma_i(F) \subset B(\tilde{x}, 3D).$$

Also, $\mathrm{Vol}(F) = \mathrm{Vol}(M)$, so

$$k \leq \frac{\mathrm{Vol}\, B(\tilde{x}, 3D)}{\mathrm{Vol} M} \leq \frac{\mathrm{Vol} B^H(3D)}{V}.$$

This is a uniform bound depending on H, D and V.

Theorem 3.2.21 (Anderson, 1990) *For the class of manifolds M with $\mathrm{Ric}_M \geq (n-1)H$, $\mathrm{Vol}_M \geq V$ and $\mathrm{diam}_M \leq D$ there are $L = L(n, H, V, D)$ and $N = N(n, H, V, D)$ such that if $\Gamma \subset \pi_1(M)$ is generated by $\{\gamma_i\}$ with each $\ell(\gamma_i) \leq L$ then the order of Γ is at most N.*

Proof. Let $\Gamma = \langle \gamma_1, \ldots, \gamma_k \rangle \subset \pi_1(M)$, where each $\ell(\gamma_i) \leq L$. Set

$$U(s) = \{\gamma \in \Gamma : |\gamma| \leq s\},$$

and let $F \subset \tilde{M}$ be a fundamental domain of M. Then $\gamma_i(F) \cap \gamma_j(F)$ has measure zero for $i \neq j$. Now

$$\bigcup_{\gamma \in U(s)} \gamma(F) \subset B(\tilde{x}, sL + D),$$

so

$$\#U(s) \leq \frac{\mathrm{Vol}B^H(sL + D)}{V}.$$

Note that if $U(s) = U(s+1)$, then $U(s) = \Gamma$. Also, $U(1) \geq 1$. Thus, if Γ has order greater than N, then $U(N) \geq N$.

Set $L = D/N$ and $s = N$. Then

$$N \leq U(N) \leq \frac{\mathrm{Vol}B^H(2D)}{V}.$$

Hence $|\Gamma| \leq N = \dfrac{\mathrm{Vol}B^H(2D)}{V} + 1$, so Γ is finite.

3.3 Volume entropy and simplicial volume

Given a compact Riemannian manifold (M^n, g), let $\tilde{M}$ be its universal cover and $\tilde{x} \in \tilde{M}$. The volume entropy measure the exponential growth rate of the volume in the universal cover. It is related to the growth of fundamental group and the topological entropy.

Definition 3.3.1 *The volume entropy of M is*

$$h(M, g) = \lim_{r \to \infty} \frac{\ln \mathrm{Vol}B(\tilde{x}, r)}{r}. \tag{3.3.1}$$

Proposition 3.3.2 (Manning, 1979) *[?] The limit in (3.3.1) exists and is independent of the center $\tilde{x} \in \tilde{M}$.*

Proof: Given any two points $\tilde{x}, \tilde{y} \in \tilde{M}$, there is deck transformation $g \in \pi_1(M)$ such that $d(\tilde{x}, g\tilde{y}) \leq d$, the diameter of M. Since $\mathrm{Vol}B(\tilde{y}, r) = \mathrm{Vol}B(g\tilde{y}, r)$, we have

$$\mathrm{Vol}B(\tilde{x}, r - d) \leq \mathrm{Vol}B(\tilde{y}, r) \leq \mathrm{Vol}B(\tilde{x}, r + d). \tag{3.3.2}$$

For any $b > 0$, let $c_b = \inf_{\tilde{z} \in \tilde{M}} \mathrm{Vol}B(\tilde{z}, b/2)$. Since M is compact, $c_b > 0$. We may assume $\mathrm{Vol}B(\tilde{x}, r)$ is unbounded, so there is b such that $c_b \geq 1$. Note that

$$B(\tilde{x}, r + s) \subset \bigcup_{\tilde{y} \in B(\tilde{x}, r - \frac{b}{2})} B(\tilde{y}, s + \frac{b}{2}) \subset \bigcup_{\tilde{y} \in Y} B(\tilde{y}, s + \frac{b}{2} + b),$$

where Y is a maximal subset of $B(\tilde{x}, r - \frac{b}{2})$ whose points are pairwise b apart. The cardinality of Y, $\#Y \leq c_b^{-1} \mathrm{Vol}B(\tilde{x}, r) \leq \mathrm{Vol}B(\tilde{x}, r)$. Therefore

$$\mathrm{Vol}B(\tilde{x}, r + s) \leq \mathrm{Vol}B(\tilde{x}, r) \cdot \mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d). \tag{3.3.3}$$

Now if $ks \le r < (k+1)s$, then

$$\begin{aligned}
\mathrm{Vol}B(\tilde{x}, r) &\le \mathrm{Vol}B(\tilde{x}, (k+1)s) \le \mathrm{Vol}B(\tilde{x}, ks)\mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d) \\
&\le \mathrm{Vol}B(\tilde{x}, s)\left(\mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d)\right)^k,
\end{aligned}$$

$$\begin{aligned}
r^{-1}\ln\mathrm{Vol}B(\tilde{x}, r) &\le r^{-1}\ln\mathrm{Vol}B(\tilde{x}, s) + kr^{-1}\ln\mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d) \\
&\le r^{-1}\ln\mathrm{Vol}B(\tilde{x}, s) + s^{-1}\ln\mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d),
\end{aligned}$$

$$\limsup_{r\to\infty} r^{-1}\ln\mathrm{Vol}B(\tilde{x}, r) \le s^{-1}\ln\mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d) \quad \text{for all } s$$

and so

$$\begin{aligned}
\limsup_{r\to\infty} r^{-1}\ln\mathrm{Vol}B(\tilde{x}, r) &\le \liminf_{s\to\infty} s^{-1}\ln\mathrm{Vol}B(\tilde{x}, s + \frac{3b}{2} + d) \\
&= \liminf_{s\to\infty} s^{-1}\ln\mathrm{Vol}B(\tilde{x}, s).
\end{aligned}$$

This shows the that $\lim_{r\to\infty} r^{-1}\ln\mathrm{Vol}B(\tilde{x}, r)$ exists. By (3.3.2) the limit is independent of $\tilde{x}$. ■

3.4 Examples and Questions

Many examples of manifolds with nonnegative Ricci curvature have been constructed, which contribute greatly to the study of manifolds with lower Ricci curvature bound. We only discuss the examples related to the basic methods here, therefore many specific examples are unfortunately omitted (some are mentioned in the previous sections). There are mainly three methods: fiber bundle construction, special surgery, and group quotient, all combined with warped products. These method are also very useful in constructing Einstein manifolds. A large class of Einstein manifolds is also provided by Yau's solution of Calabi conjecture.

Note that if two compact Riemannian manifolds $M^m, N^n (n, m \ge 2)$ have positive Ricci curvature, then their product has positive Ricci curvature, which is not true for sectional curvature but only needs one factor positive for scalar curvature. Therefore it is natural to look at the fiber bundle case. Using Riemannian submersions with totally geodesic fibers J. C. Nash [99], W. A. Poor [113], and Berard-Bergery [14] showed that the compact total space of a fiber bundle admits a metric of positive Ricci curvature if the base and the fiber admit metrics with positive Ricci curvature and if the structure group acts by isometries. Furthermore, any vector bundle of rank ≥ 2 over a compact manifold with $\mathrm{Ric} > 0$ carries a complete metric with positive Ricci curvature. In [58] Gilkey-Park-Tuschmann showed that a principal bundle P over a compact manifold with Ric > 0 and compact connected structure group G admits a G invariant metric with positive Ricci curvature if and only if $\pi_1(P)$ is finite. Unlike the product case, the corresponding statements for Ric ≥ 0 are not true in all these cases, e.g. the nilmanifold $S^1 \to N^3 \to T^2$ does not admit a metric with Ric ≥ 0. On the other hand Belegradek-Wei [13] showed that it is true in the stable sense. Namely, if E is the total space of a bundle over a compact base with Ric ≥ 0, and either a compact Ric ≥ 0 fiber or vector space as fibers, with compact structure group acting by isometry, then $E \times \mathbb{R}^p$ admits a complete metric with positive Ricci curvature for all sufficiently large p. See [139] for an estimate of p.

Surgery constructions are very successful in constructing manifolds with positive scalar curvature, see Rothenberg's article in this volume. Sha-Yang [118, 119] showed that this is also a useful method for constructing manifolds with positive Ricci curvature in special cases. In particular they showed that if M^{m+1} has a complete metric with Ric > 0, and $n, m \geq 2$, then $S^{n-1} \times \left(M^{m+1} \setminus \coprod_{i=0}^{k} D_i^{m+1}\right) \bigcup_{Id} D^n \times \coprod_{i=0}^{k} S_i^m$, which is diffeomorphic to $\left(S^{n-1} \times M^{m+1}\right) \# \left(\#_{i=1}^{k} S^n \times S^m\right)$, carries a complete metric with Ric > 0 for all k, showing that the total Betti number of a compact Riemannian n-manifold ($n \geq 4$) with positive Ricci curvature could be arbitrarily large. See also [6], and [140] when the gluing map is not the identity.

Note that a compact homogeneous space admits an invariant metric with positive Ricci curvature if and only if the fundamental group is finite [99, Proposition 3.4]. This is extended greatly by Grove-Ziller [66] showing that any cohomogeneity one manifold M admits a complete invariant metric with nonnegative Ricci curvature and if M is compact then it has positive Ricci curvature if and only if its fundamental group is finite (see also [117]). Therefore, the fundamental group is the only obstruction to a compact manifold admitting a positive Ricci curvature metric when there is enough symmetry. It remains open what the obstructions are to positive Ricci curvature besides the restriction on the fundamental group and those coming from positive scalar curvature (such as the $\hat{A}$-genus).

Of course, another interesting class of examples are given by Einstein manifolds. For these, besides the "bible" on Einstein manifolds [16], one can refer to the survey book [77] for the development after [16], and the recent articles [**?**, 18] for Sasakian Einstein metrics and compact homogenous Einstein manifolds.

Contrary to a Ricci curvature lower bound, a Ricci curvature upper bound does not have any topological constraint [85].

Theorem 3.4.1 (Lohkamp, 1994) *If $n \geq 3$, any manifold, M^n, admits a complete metric with* $\mathrm{Ric}_M < 0$.

An upper Ricci curvature bound does have geometric implications, e g. the isometry group of a compact manifold with negative Ricci curvature is finite. In the presence of a lower bound, an upper bound on Ricci curvature forces additional regularity of the metric, see Theorem 8.5.13 in Section **??** by Anderson. It's still unknown whether it will give additional topological control. For example, the following question is still open.

Question 3.4.2 *Does the class of manifolds M^n with* $|\mathrm{Ric}_M| \le H, \mathrm{Vol}_M \ge V$ *and* $\mathrm{diam}_M \le D$ *have finite many homotopy types?*

There are infinitely many homotopy types without the Ricci upper bound [106] . This is the only known topological obstruction to a compact manifold supports a metric with positive Ricci curvature other than topological obstructions shared by manifolds with positive scalar curvature.

What can one say if the dimension n is fixed? For example, is the order of the group modulo an abelian subgroup bounded by the dimension? See [138] for a partial result.

Chapter 4

Gromov-Hausdorff convergence

Gromov-Hausdorff convergence is very useful in studying manfolds with a lower Ricci bound. The starting point is Gromov's precompactness theorem. Let's first recall the Gromov-Hausdorff distance. See [68, Chapter 3,5],[108, Chapter 10], [23, Chapter 7] for more background material on Gromov-Hausdorff convergence.

Given a metric space (X, d) and subsets $A, B \subset X$, the Hausdorff distance is

$$d_H(A, B) = \inf\{\epsilon > 0 : B \subset T_\epsilon(A) \text{ and } A \subset T_\epsilon(B)\},$$

where $T_\epsilon(A) = \{x \in X : d(x, A) < \epsilon\}$.

Definition 4.0.3 (Gromov, 1981) *Given two compact metric spaces X, Y, the Gromov-Hausdorff distance is $d_{GH}(X, Y) = \inf\{d_H(X, Y) :$ all metrics on the disjoint union, $X \coprod Y$, which extend the metrics of X and $Y\}$.*

The Gromov-Hausdorff distance defines a metric on the collection of isometry classes of compact metric spaces. Thus, there is the naturally associated notion of Gromov-Hausdorff convergence of compact metric spaces. While the Gromov-Hausdorff distance make sense for non-compact metric spaces, the following looser definition of convergence is more appropriate. See also [68, Defn 3.14]. These two definitions are equivalent [127, Appendix].

Definition 4.0.4 *We say that non-compact metric spaces (X_i, x_i) converge in the pointed Gromov-Hausdorff sense to (Y, y) if for any $r > 0$, $B(x_i, r)$ converges to $B(y, r)$ in the pointed Gromov-Hausdorff sense.*

Applying the relative volume comparison (1.4.7) to manifolds with lower Ricci bound, we have

Theorem 4.0.5 (Gromov's precompactness theorem) *The class of closed manifolds M^n with $\mathrm{Ric}_M \geq (n-1)H$ and $\mathrm{diam}_M \leq D$ is precompact. The class of pointed complete manifolds M^n with $\mathrm{Ric}_M \geq (n-1)H$ is precompact.*

By the above, for an open manifold M^n with $\mathrm{Ric}_M \geq 0$ any sequence $\{(M^n, x, r_i^{-2}g)\}$, with $r_i \to \infty$, subconverges in the pointed Gromov-Hausdorff topology to a length space M_∞. In general, M_∞ is not unique [105]. Any such limit is called an asymptotic cone of M^n, or a cone of M^n at infinity .

Gromov-Hausdorff convergence defines a very weak topology. In general one only knows that Gromov-Hausdorff limit of length spaces is a length space and diameter is continuous under the Gromov-Hausdorff convergence. When the limit is a smooth manifold with same dimension Colding showed the remarkable result that for manifolds with lower Ricci curvature bound the volume also converges [47] which was conjectured by Anderson-Cheeger. See also [31] for a proof using mod 2 degree.

Theorem 4.0.6 (Volume Convergence, Colding, 1997) *If (M_i^n, x_i) has $\mathrm{Ric}_{M_i} \geq (n-1)H$ and converges in the pointed Gromov-Hausdorff sense to smooth Riemannian manifold (M^n, x), then for all $r > 0$*

$$\lim_{i\to\infty} \mathrm{Vol}(B(x_i, r)) = \mathrm{Vol}(B(x, r)). \tag{4.0.1}$$

The volume convergence can be generalized to the noncollapsed singular limit space (by replacing the Riemannian volume with the n-dimensional Hausdorff measure $\mathcal{H}^n$) [35, Theorem 5.9], and to the collapsing case with smooth limit M^k in terms of the k-dimensional Hausdorff content [36, Theorem 1.39].

As an application of Theorem 4.0.6, Colding [47] derived the rigidity result that if M^n has $\mathrm{Ric}_M \geq 0$ and some M_∞ is isometric to $\mathbb{R}^n$, then M is isometric to $\mathbb{R}^n$.

We also have the following wonderful stability result [35] which sharpens an earlier version in [47].

Theorem 4.0.7 (Cheeger-Colding, 1997) *For a closed Riemannian manifold M^n there exists an $\epsilon(M) > 0$ such that if N^n is a n-manifold with $\mathrm{Ric}_N \geq -(n-1)$ and $d_{GH}(M, N) < \epsilon$ then M and N are diffeomorphic.*

Unlike the sectional curvature case, examples show that the result does not hold if one allows M to have singularities even on the fundamental group level [102, Remark (2)]. Also the ϵ here must depend on M [3].

Cheeger-Colding also showed that the eigenvalues and eigenfunctions of the Laplacian are continuous under measured Gromov-Hausdorff convergence [37]. To state the result we need a definition and some structure result on the limit space (see Section **??** for more structures). Let X_i be a sequence of metric spaces converging to X_∞ and μ_i, μ_∞ are Radon measures on X_i, X_∞.

Definition 4.0.8 *We say (X_i, μ_i) converges in the measured Gromov-Hausdorff sense to (X_∞, μ_∞) if for all sequences of continuous functions $f_i : X_i \to \mathbb{R}$ converging to $f_\infty : X_\infty \to \mathbb{R}$, we have*

$$\int_{X_i} f_i d\mu_i \to \int_{X_\infty} f_\infty d\mu_\infty. \tag{4.0.2}$$

If (M_∞, p) is the pointed Gromov-Hausdorff limit of a sequence of Riemannian manifolds (M_i^n, p_i) with $\mathrm{Ric}_{M_i} \geq -(n-1)$, then there is a natural collection of measures, μ, on M_∞ obtained by taking limits of the normalized Reimannian measures on M_j^n for a suitable subsequence M_j^n [53], [35, Section 1],

$$\mu = \lim_{j\to\infty} \underline{\mathrm{Vol}}_j(\cdot) = \mathrm{Vol}(\cdot)/\mathrm{Vol}(B(p_j, 1)). \tag{4.0.3}$$

In particular, for all $z \in M_\infty$ and $0 < r_1 \leq r_2$, we have the renormalized limit measure μ satisfy the following comparison

$$\frac{\mu(B(z, r_1))}{\mu(B(z, r_2))} \geq \frac{\mathrm{Vol}_{n,-1}\,(B(r_1))}{\mathrm{Vol}_{n,-1}\,(B(r_2))}. \tag{4.0.4}$$

With this, the extension of the segment inequality (1.6.4) to the limit, the gradient estimate (**??**), and Bochner's formula, one can define a canonical self-adjoint Laplacian Δ_∞ on the limit space M_∞ by means of limits of the eigenfunctions and eigenvalues for the sequence of the manifolds. In [29, 37] an intrinsic construction of this operator is also given on a more general metric measure spaces. Let $\{\lambda_{1,i}\cdots,\}, \{\lambda_{1,\infty},\cdots,\}$ denote the eigenvalues for Δ_i, Δ_∞ on M_i, M_∞, and $\phi_{j,i}, \phi_{j,\infty}$ the eigenfunctions of the jth eigenvalues $\lambda_{j,i}, \lambda_{j,\infty}$. In [37] Cheeger-Colding in particular proved the following theorem, establishing Fukaya's conjecture [53].

Theorem 4.0.9 (Spectral Convergence, Cheeger-Colding, 2000) *Let $(M_i^n, p_i, \underline{\mathrm{Vol}}_i)$ with $\mathrm{Ric}_{M_i} \geq -(n-1)$ converges to (M_∞, p, μ) under measured Gromov-Hausdorff sense and M_∞ is compact. Then for each j, $\lambda_{j,i} \to \lambda_{j,\infty}$ and $\phi_{j,i} \to \phi_{j,\infty}$ uniformly as $i \to \infty$.*

As a natural extension, in [52] Ding proved that the heat kernel and Green's function also behave nicely under the measured Gromov-Hausdorff convergence. The natural extension to the p-form Laplacian does not hold, however, there is still very nice work in this direction by John Lott, see [86, 88].

Chapter 5

Comparison for Integral Ricci Curvature

5.1 Integral Curvature: an Overview

What's integral curvature? A natural integral curvature is the L^p-norm of the curvature tensor. For a compact Riemannian manifold M^n, $x \in M$, let $\sigma(x) = \max_{v,w,\in T_xM} |K(v,w)|$, where $K(v,w)$ is the sectional curvature of the plane spanned by v, w. The L^p-norm of the curvature tensor is

$$\|Rm\|_p = \left(\int_M \sigma(x)^p dvol \right)^{1/p}.$$

When the metric g scales by λ^2, the sectional curvature scales by λ^{-2}, volume by λ^n, so $\|Rm\|_p$ scales by $\lambda^{\frac{n}{p}-2}$. Therefore when $p = \frac{n}{2}$, $\|Rm\|_p$ is scale invariant, while for $p < \frac{n}{2}$, one can make $\|Rm\|_p$ small just by choosing λ small, not a very restrictive condition. Sometime the normalized norm,

$$\overline{\|Rm\|}_p = \left(\frac{1}{\text{Vol}M} \int_M \sigma(x)^p dvol \right)^{1/p},$$

which scales like curvature, is more appropriate. When M is noncompact, one can define the integral over a ball (see below).

Bounds on these integral curvatures are extensions of two sided pointwise curvature bounds to integral. What about one sided curvature bound? Or integral curvature lower bound? Here we specify for Ricci curvature. Given $H \in \mathbb{R}$, we can measure the amount of Ricci curvature lying below $(n-1)H$ in L^p norm.

For each $x \in M^n$ let $\rho(x)$ denote the smallest eigenvalue for the Ricci tensor $\text{Ric} : T_xM \to T_xM$, and $\text{Ric}_-^H(x) = ((n-1)H - \rho(x))_+ = \max\{0, (n-1)H - \rho(x)\}$, amount of Ricci curvature below $(n-1)H$. Let

$$\|\mathrm{Ric}_-^H\|_p(R) = \sup_{x\in M}\left(\int_{B(x,R)} (\mathrm{Ric}_-^H)^p \, dvol\right)^{\frac{1}{p}}. \tag{5.1.1}$$

Then $\|\mathrm{Ric}_-^H\|_p$ measures the amount of Ricci curvature lying below $(n-1)H$ in the L^p sense. Clearly $\|\mathrm{Ric}_-^H\|_p(R) = 0$ iff $\mathrm{Ric}_M \geq (n-1)H$.

Similarly we can define Ricci curvature integral upper bound, or for sectional curvature integral lower bound $\|K_-^H\|_p$. When $H = 0$, we will omit the superscript, e.g. denote $\|\mathrm{Ric}_-^0\|_p$ by $\|\mathrm{Ric}_-\|_p$.

Why do we study integral curvature? Many geometric problems lead to integral curvatures, for example, the isospectral problems, geometric variational problems and extremal metrics, and Chern-Weil's formula for characteristic numbers. Integral curvature also makes sense on some singular spaces, e.g. polyhedral surfaces. Since integral curvature bound is much weaker than pointwise curvature bound, one naturally asks what geometric and topological results can be extended to integral curvature.

In general one can not extend results from pointwise curvature bounds to integral curvature bounds. This can be illustrated by an example by D. Yang [144].

Recall a very important result in Riemannian geometry is Cheeger's finiteness theorem [28]. Namely the class of manifolds M^n with

$$|K_M| \leq H, \ \mathrm{Vol}_M \geq v, \ \mathrm{diam}_M \leq D$$

has only finite many diffeomorphism types. A key estimate is Cheeger's estimate on the length of the shortest closed geodesics. This is not true if $|K_M| \leq H$ is replaced by $\|Rm\|_p$ is bounded. In fact we have [144]

Example 5.1.1 (D. Yang 1992) *For all $p \geq \frac{n}{2}$ there are manifolds M_k^n such that*

$$\|Rm\|_p \leq H, \ \mathrm{Vol} \geq v, \ \mathrm{diam} \leq D$$

but $b_2(M_k) \to \infty$ as $k \to \infty$.

Hence some smallness is needed. For $p \leq \frac{n}{2}$, this still does not work as Gromov's Betti number estimate [61] does not extend [56].

Example 5.1.2 (Gallot 1988) *For any $\epsilon > 0, D > 0, n \geq 3$, there are M_k^n such that*

$$\mathrm{diam}(M_k) \leq D, \quad \|K_-\|_{\frac{n}{2}} \leq \epsilon, \quad \overline{\|K_-\|}_{\frac{n}{2}} \leq \epsilon,$$

but $b_2(M_k) \to \infty$ as $k \to \infty$.

This is not the end of story. Most results extend when the normalized L^p norm $\overline{\|Rm\|}_p, \overline{\|K_-^H\|}_p$, or $\overline{\|\mathrm{Ric}_-^H\|}_p$ is small for $p > \frac{n}{2}$. Namely we need the error from pointwise curvature bound to be small in L^p for $p > \frac{n}{2}$. There is a gap phenomenon. Some of the basic tools for these extensions are volume comparison for integral curvature, use Ricci flow to deform the manifolds with integral curvature bounds to pointwise curvature bound (so called smoothing), D. Yang and Gallot's Sobolev estimates [56, 144, 111, 112, 50].

5.2 Mean Curvature Comparison Estimate

Recall the mean curvature comparison theorem (Theorem 1.2.2) states that if $\mathrm{Ric} \geq (n-1)H$, then $m \leq m_H$. In general, without any curvature bound, we can estimate $m_+^H = (m - m_H)_+$ (set it to zero whenever it is not defined), amount of mean curvature comparison failed in L^{2p}, in terms of Ric_-^H, amount of Ricci curvature lying below $(n-1)H$ in L^p when $p > \frac{n}{2}$ [111].

Theorem 5.2.1 (Mean Curvature Estimate, Petersen-Wei 1997) *For any $p > \frac{n}{2}$, $H \in \mathbb{R}$, and when $H > 0$ assume $r \leq \frac{\pi}{2\sqrt{H}}$, we have*

$$\|m_+^H\|_{2p}(r) \leq \left(\frac{(n-1)(2p-1)}{2p-n} \|\mathrm{Ric}_-^H\|_p(r) \right)^{\frac{1}{2}}. \tag{5.2.2}$$

Clearly this generalizes the mean curvature comparison theorem (Theorem 1.2.2). In fact we prove the following which also gives a pointwise estimate.

Proposition 5.2.2 (Mean Curvature Estimate, Petersen-Wei 1997) *For any $p > \frac{n}{2}$, $H \in \mathbb{R}$, and when $H > 0$ assume $r \leq \frac{\pi}{2\sqrt{H}}$, we have*

$$\int_0^r (m_+^H)^{2p} \mathcal{A}(t,\theta)\, dt \quad \leq \quad \left(\frac{(n-1)(2p-1)}{2p-n} \right)^p \int_0^r (\mathrm{Ric}_-^H)^p \mathcal{A}(t,\theta)\, dt, \tag{5.2.3}$$

$$(m_+^H)^{2p-1}(r,\theta)\mathcal{A}(r,\theta) \quad \leq \quad (2p-1)^p \left(\frac{n-1}{2p-n} \right)^{p-1} \int_0^r (\mathrm{Ric}_-^H)^p \mathcal{A}\, dt, \tag{5.2.4}$$

where $\mathcal{A}(t,\theta)$ is the volume element of the volume form $d\,vol = \mathcal{A}(t,\theta) dt \wedge d\theta_{n-1}$ in polar coordinate.

Proof: By (1.2.2) and (1.2.4) we have

$$(m - m_H)' \leq -\frac{(m-m_H)(m+m_H)}{n-1} + (n-1)H - \mathrm{Ric}(\nabla r, \nabla r). \tag{5.2.5}$$

On the interval $m \leq m_H$, we have $m_+^H = 0$, on the interval where $m > m_H$, $m - m_H = m_+^H$, and $(n-1)H - \mathrm{Ric}(\nabla r, \nabla r) \leq \mathrm{Ric}_-^H$. Therefore we have

$$(m_+^H)' + \frac{(m_+^H)^2}{n-1} + 2\frac{m_+^H \cdot m_H}{n-1} \leq \mathrm{Ric}_-^H. \tag{5.2.6}$$

Multiply (5.2.6) by $(m_+^H)^{2p-2}\mathcal{A}$ we get

$$(m_+^H)'(m_+^H)^{2p-2}\mathcal{A} + \frac{(m_+^H)^{2p}}{n-1}\mathcal{A} + \frac{2(m_+^H)^{2p-1}}{n-1}m_H\mathcal{A} \le \mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}.$$

To complete the integral of the first term we compute, using (1.4.3) and $m - m_H \le m_+^H$,

$$\begin{aligned}(2p-1)(m_+^H)'(m_+^H)^{2p-2}\mathcal{A} &= \left((m_+^H)^{2p-1}\mathcal{A}\right)' - (m_+^H)^{2p-1}\mathcal{A}' \\ &= \left((m_+^H)^{2p-1}\mathcal{A}\right)' - (m_+^H)^{2p-1}(m-m_H)\mathcal{A} - (m_+^H)^{2p-1}m_H\mathcal{A} \\ &\ge \left((m_+^H)^{2p-1}\mathcal{A}\right)' - (m_+^H)^{2p}\mathcal{A} - (m_+^H)^{2p-1}m_H\mathcal{A}.\end{aligned}$$

Therefore we have the ODE

$$\begin{aligned}&\left((m_+^H)^{2p-1}\mathcal{A}\right)' + \left(\frac{2p-1}{n-1} - 1\right)(m_+^H)^{2p}\mathcal{A} + \left(\frac{4p-2}{n-1} - 1\right)(m_+^H)^{2p-1}m_H\mathcal{A} \\ &\quad \le (2p-1)\mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}. \end{aligned} \tag{5.2.7}$$

When $p > \frac{n}{2}$, $\frac{2p-1}{n-1} - 1 = \frac{2p-n}{n-1} > 0$. Hence if $m_H \ge 0$ (which is true under our assumption), we can throw away the third term in (5.2.7) and integrate from 0 to r to get

$$\left((m_+^H)^{2p-1}\mathcal{A}\right)(r) + \frac{2p-n}{n-1}\int_0^r (m_+^H)^{2p}\mathcal{A}\,dt \le (2p-1)\int_0^r \mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}\,dt.$$

This gives

$$\left((m_+^H)^{2p-1}\mathcal{A}\right)(r) \le (2p-1)\int_0^r \mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}\,dt, \tag{5.2.8}$$

$$\frac{2p-n}{n-1}\int_0^r (m_+^H)^{2p}\mathcal{A}\,dt \le (2p-1)\int_0^r \mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}\,dt. \tag{5.2.9}$$

By Hölder's inequality

$$\int_0^r \mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}\,dt \le \left(\int_0^r (m_+^H)^{2p}\mathcal{A}\,dt\right)^{1-\frac{1}{p}}\left(\int_0^r (\mathrm{Ric}_-^H)^p\mathcal{A}\,dt\right)^{\frac{1}{p}}.$$

Plug this into (5.2.9) we get (5.2.3). Plug this into (5.2.8) and combine (5.2.3) we get (5.2.4). ■

When $H > 0$ and $r > \frac{\pi}{2\sqrt{H}}$, m_H is negative so we can not throw away the third term in (5.2.7). Following the above estimate with an integrating factor Aubry gets [8]

Proposition 5.2.3 *For $p > \frac{n}{2}, H > 0, \frac{\pi}{2\sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, we have*

$$\begin{aligned}&\sin^{4p-n-1}(\sqrt{H}r)(m_+^H)^{2p-1}(r,\theta)\mathcal{A}(r,\theta) \\ &\quad \le (2p-1)^p\left(\frac{n-1}{2p-n}\right)^{p-1}\int_0^r (\mathrm{Ric}_-^H)^p\mathcal{A}\,dt.\end{aligned} \tag{5.2.10}$$

Proof: Write (5.2.7) as

$$\left((m_+^H)^{2p-1}\mathcal{A}\right)' + \frac{(4p-n-1)m_H}{n-1}(m_+^H)^{2p-1}\mathcal{A} + \left(\frac{2p-n}{n-1}\right)(m_+^H)^{2p}\mathcal{A} \le (2p-1)\mathrm{Ric}_-^H(m_+^H)^{2p-2}\mathcal{A}.$$

The integrating factor of the first two terms is $e^{\int \frac{(4p-n-1)m_H}{n-1}} = \sin^{4p-n-1}(\sqrt{H}r)$. Multiply by the integrating factor and integrate from 0 to r we get

$$\begin{aligned} 0 &\le \sin^{4p-n-1}(\sqrt{H}r)(m_+^H)^{2p-1}(r,\theta)\mathcal{A}(r,\theta) + \frac{2p-n}{n-1}\int_0^r (m_+^H)^{2p}\sin^{4p-n-1}(\sqrt{H}r)\mathcal{A}\,dt \\ &\le (2p-1)\int_0^r \mathrm{Ric}_-^H(m_+^H)^{2p-2}\sin^{4p-n-1}(\sqrt{H}r)\mathcal{A} \end{aligned}$$

Using Hólder's inequality as before we get

$$\int_0^r (m_+^H)^{2p}\sin^{4p-n-1}(\sqrt{H}r)\mathcal{A}(t,\theta)\,dt \le \left(\frac{(n-1)(2p-1)}{2p-n}\right)^p \int_0^r (\mathrm{Ric}_-^H)^p \sin^{4p-n-1}(\sqrt{H}r)\mathcal{A}(t,\theta)\,dt$$

and this gives (5.2.10) as before. ∎

All estimates hold for the mean curvature of hypersurfaces.

5.3 Volume Comparison Estimate

From (1.4.3) one naturally expects that the mean curvature comparison estimates in the last section would give volume comparison estimate for integral Ricci lower bound.

First we give a comparison estimate for the area of geodesics spheres using the pointwise mean curvature estimate (5.2.4). Recall $A(x,r) = \int_{S^{n-1}} \mathcal{A}(r,\theta)d\theta_{n-1}$, the volume of the geodesic sphere $S(x,r) = \{y \in M | d(x,y) = r\}$, and $A_H(r)$ the volume of the geodesic sphere in the model space.

Theorem 5.3.1 *Let $x \in M^n, H \in \mathbb{R}$ and $p > \frac{n}{2}$ be given, and when $H > 0$ assume that $R \le \frac{\pi}{2\sqrt{H}}$. For $r \le R$, we have*

$$\left(\frac{A(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{A(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}} \le C(n,p,H,R)\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}, \quad (5.3.11)$$

where $C(n,p,H,R) = \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R (A_H)^{-\frac{1}{2p-1}}dt$. Furthermore when $r = 0$ we obtain

$$A(x,R) \le \left(1 + C(n,p,H,R)\cdot\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}\right)^{2p-1} A_H(R). \quad (5.3.12)$$

The proof below much simplifies the proof in [51].

Proof: Recall

$$\frac{d}{dt}\left(\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)}\right) = (m-m_H)\cdot\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)} \le m_+^H\cdot\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)}.$$

Hence

$$\begin{aligned}\frac{d}{dt}\left(\frac{A(x,t)}{A_H(t)}\right) &= \frac{1}{\mathrm{Vol}S^{n-1}}\int_{S^{n-1}}\frac{d}{dt}\left(\frac{\mathcal{A}(t,\theta)}{\mathcal{A}_H(t)}\right)d\theta_{n-1}\\ &\le \frac{1}{A_H(t)}\int_{S^{n-1}} m_+^H\mathcal{A}(t,\theta)d\theta_{n-1}.\end{aligned}$$

Using Hölder's inequality and (5.2.4) yields

$$\begin{aligned}&\int_{S^{n-1}} m_+^H\mathcal{A}(t,\theta)d\theta_{n-1}\\ &\le \left(\int_{S^{n-1}}(m_+^H)^{2p-1}\mathcal{A}d\theta_{n-1}\right)^{\frac{1}{2p-1}}(A(x,t))^{1-\frac{1}{2p-1}}\\ &\le C(n,p)\left(\|\mathrm{Ric}_-^H\|_p(t)\right)^{\frac{p}{2p-1}}(A(x,t))^{1-\frac{1}{2p-1}},\end{aligned}$$

where $C(n,p)=\left((2p-1)^p\left(\frac{n-1}{2p-n}\right)^{p-1}\right)^{\frac{1}{2p-1}}$. Hence we have

$$\frac{d}{dt}\left(\frac{A(x,t)}{A_H(t)}\right)\le C(n,p)\left(\|\mathrm{Ric}_-^H\|_p(t)\right)^{\frac{p}{2p-1}}(A_H)^{-\frac{1}{2p-1}}\left(\frac{A(x,t)}{A_H(t)}\right)^{1-\frac{1}{2p-1}}. \quad (5.3.13)$$

Separation of variables and integrate from r to R we get

$$\begin{aligned}&\left(\frac{A(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}}-\left(\frac{A(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}}\\ &\quad\le\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}\int_r^R(A_H)^{-\frac{1}{2p-1}}dt.\end{aligned}$$

The integral $\int_r^R(A_H)^{-\frac{1}{2p-1}}dt\le\int_0^R(A_H)^{-\frac{1}{2p-1}}dt$ converges when $p>\frac{n}{2}$. This gives (5.3.11). ■

Similarly, using (5.2.10) instead of (5.2.4) and that $A_H(t)=\left(\frac{\sin(\sqrt{H}t}{\sqrt{H}}\right)^{n-1}$ for $H>0$, one has for $p>\frac{n}{2}, H>0, \frac{\pi}{2\sqrt{H}}<r\le R<\frac{\pi}{\sqrt{H}}$,

$$\begin{aligned}&\left(\frac{A(x,R)}{A_H(R)}\right)^{\frac{1}{2p-1}}-\left(\frac{A(x,r)}{A_H(r)}\right)^{\frac{1}{2p-1}}\\ &\quad\le\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}\int_r^R\frac{\left(\sqrt{H}\right)^{\frac{n-1}{2p-1}}}{\sin^2(\sqrt{H}t)}dt.\end{aligned} \quad (5.3.14)$$

Using (5.3.13) we have

Theorem 5.3.2 (Volume Comparison Estimate, Petersen-Wei 1997) *Let $x \in M^n, H \in \mathbb{R}$ and $p > \frac{n}{2}$ be given, when $H > 0$ assume that $R \le \frac{\pi}{2\sqrt{H}}$. For $r \le R$ we have*

$$\left(\frac{\operatorname{Vol} B(x,R)}{\operatorname{Vol}_H(B(R))}\right)^{\frac{1}{2p-1}} - \left(\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol}_H(B(r))}\right)^{\frac{1}{2p-1}} \le C(n,p,H,R)\cdot\left(\|\operatorname{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}}, \tag{5.3.15}$$

where $C(n,p,H,R) = \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \int_0^R A_H(t)\left(\frac{t}{\operatorname{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}} dt$, increasing in R.

Note that when $\|\operatorname{Ric}_-^H\|_p(R) = 0$, this gives the Bishop-Gromov relative volume comparison.

Proof of Theorem 5.3.2: Since $\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol}_H(B(r))} = \frac{\int_0^r A(x,t)dt}{\int_0^r A_H(t)dt}$, we have

$$\frac{d}{dr}\left(\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol}_H B(r)}\right) = \frac{A(x,r)\int_0^r A_H(t)dt - A_H(r)\int_0^r A(x,t)dt}{\left(\operatorname{Vol}_H B(r)\right)^2}. \tag{5.3.16}$$

Integrate (5.3.13) from t to r gives

$$\begin{aligned}
\frac{A(x,r)}{A_H(r)} - \frac{A(x,t)}{A_H(t)} &\le C(n,p)\int_t^r \left(\|\operatorname{Ric}_-^H\|_p(s)\right)^{\frac{p}{2p-1}} \frac{A(x,s)^{1-\frac{1}{2p-1}}}{A_H(s)} ds \\
&\le C(n,p)\frac{\left(\|\operatorname{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}}}{A_H(t)} \int_t^r A(x,s)^{1-\frac{1}{2p-1}} ds \\
&\le C(n,p)\frac{\left(\|\operatorname{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}}}{A_H(t)} (r-t)^{\frac{1}{2p-1}} \left(\operatorname{Vol} B(x,r)\right)^{1-\frac{1}{2p-1}}.
\end{aligned} \tag{5.3.17}$$

Hence

$$A(x,r)A_H(t) - A_H(r)A(x,t) \le C(n,p)\left(\|\operatorname{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}} A_H(r) r^{\frac{1}{2p-1}} \left(\operatorname{Vol} B(x,r)\right)^{1-\frac{1}{2p-1}}.$$

Plug this into (5.3.16) gives

$$\frac{d}{dr}\left(\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol}_H B(r)}\right) \le C(n,p)\left(\|\operatorname{Ric}_-^H\|_p(r)\right)^{\frac{p}{2p-1}} A_H(r)\left(\frac{r}{\operatorname{Vol}_H B(r)}\right)^{\frac{2p}{2p-1}} \left(\frac{\operatorname{Vol} B(x,r)}{\operatorname{Vol}_H B(r)}\right)^{1-\frac{1}{2p-1}}.$$

Separation of variables and integrate from r to R we get

$$\left(\frac{\operatorname{Vol}B(x,R)}{\operatorname{Vol}_H(R)}\right)^{\frac{1}{2p-1}} - \left(\frac{\operatorname{Vol}B(x,r)}{\operatorname{Vol}_H(r)}\right)^{\frac{1}{2p-1}} \le \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}} \left(\|\operatorname{Ric}_-^H\|_p(R)\right)^{\frac{p}{2p-1}} \int_r^R A_H(t)\left(\frac{t}{\operatorname{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}} dt.$$

The integral $\int_r^R A_H(t)\left(\frac{t}{\mathrm{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}} dt \le \int_0^R A_H(t)\left(\frac{t}{\mathrm{Vol}_H B(t)}\right)^{\frac{2p}{2p-1}} dt$ converges when $p > \frac{n}{2}$. ■

For applications the volume doubling estimate is often more useful. From (5.3.15)

$$\begin{aligned}&\left(\frac{\mathrm{Vol}\, B\,(x,r)}{\mathrm{Vol}_H\,(B(r))}\right)^{\frac{1}{2p-1}}\\ &\quad\ge \left(\frac{\mathrm{Vol}\, B\,(x,R)}{\mathrm{Vol}_H\,(B(R))}\right)^{\frac{1}{2p-1}}\left(1 - C\,(n,p,H,R)\,(\mathrm{Vol}_H B(R))^{\frac{1}{2p-1}}\cdot\left(\overline{\|\mathrm{Ric}_-^H\|}_p(R)\right)^{\frac{p}{2p-1}}\right),\end{aligned}$$

where

$$C\,(n,p,H,R)\,(\mathrm{Vol}_H B(R))^{\frac{1}{2p-1}} \le R^{\frac{2p}{2p-1}} C(n,p,|H|R^2).$$

Hence

$$\begin{aligned}&\left(\frac{\mathrm{Vol} B(x,r)}{\mathrm{Vol}\,(B(x,R))}\right)^{\frac{1}{2p-1}}\\ &\quad\ge \left(\frac{\mathrm{Vol}_H\, B\,(r)}{\mathrm{Vol}_H\,(B(R))}\right)^{\frac{1}{2p-1}}\left(1 - C\left(n,p,|H|R^2\right)\left(R^2\overline{\|\mathrm{Ric}_-^H\|}_p(R)\right)^{\frac{p}{2p-1}}\right).\end{aligned} \tag{5.3.18}$$

Therefore we have

Corollary 5.3.3 (Volume Doubling Estimate) *Given $\alpha < 1$ and $p > \frac{n}{2}$, there is an $\epsilon = \epsilon(n,p,|H|R^2,\alpha) > 0$ such that if $R^2\overline{\|\mathrm{Ric}_-^H\|}_p(R) < \epsilon$, then for all $x \in M^n$ and $r_1 < r_2 \le R$ (assume $R \le \frac{\pi}{2\sqrt{H}}$ when $H > 0$),*

$$\frac{\mathrm{Vol}B(x,r_1)}{\mathrm{Vol}B(x,r_2)} \ge \alpha\frac{\mathrm{Vol}_H(r_1)}{\mathrm{Vol}_H(r_2)}. \tag{5.3.19}$$

Proof: From above we have

$$\left(\frac{\mathrm{Vol}\, B\,(x,r_1)}{\mathrm{Vol}\,(B(x,r_2))}\right)^{\frac{1}{2p-1}} \ge \left(\frac{\mathrm{Vol}_H\, B\,(r_1)}{\mathrm{Vol}_H\,(B(r_2))}\right)^{\frac{1}{2p-1}}(1-\eta),$$

where

$$\eta = C\,(n,p,H,r_2)\,(\mathrm{Vol}_H B(r_2))^{\frac{1}{2p-1}}\cdot\left(\overline{\|\mathrm{Ric}_-^H\|}_p(r_2)\right)^{\frac{p}{2p-1}}.$$

To control η we note that it is almost increasing in r_2. In fact, since $C(n,p,H,R)$ is increasing in R,

$$\eta(r_2) \le \eta(R)\cdot\left(\frac{\mathrm{Vol}\, B\,(x,R)}{\mathrm{Vol}\,(B(x,r_2))}\right)^{\frac{1}{2p-1}}\cdot\left(\frac{\mathrm{Vol}_H\, B\,(r_2)}{\mathrm{Vol}_H\,(B(R))}\right)^{\frac{1}{2p-1}}.$$

By (5.3.18), when $R^2\overline{\|\mathrm{Ric}_-^H\|}_p(R)$ is small depends only on $n, p, |H|R^2$,

$$\left(\frac{\mathrm{Vol}B(x,R)}{\mathrm{Vol}\,(B(x,r_2))}\right)^{\frac{1}{2p-1}} \le 2\left(\frac{\mathrm{Vol}_H\,(B(R))}{\mathrm{Vol}_H B\,(r_2)}\right)^{\frac{1}{2p-1}}.$$

Hence $\eta(r_2) \le 2\eta(R)$. Now $\eta(R)$ can be made arbitrary small if $R^2\overline{\|\mathrm{Ric}_-^H\|}_p(R)$ is small enough depends on $n, p, |H|R^2$. ■

We note that to apply the volume doubling, one needs a smallness condition on $\overline{\|\mathrm{Ric}_-^H\|}_p(R)$, the normalized one, which is a more natural condition than the smallness of $\|\mathrm{Ric}_-^H\|_p(R)$. As R gets bigger, the smallness needed has to be more stringent.

Definition 5.3.4 *We call a set $T \subset M$ is a star shaped set at x if for any $y \in T$, a minimal geodesic connecting x, y also lies in T.*

Obviously the ball $B(x,r)$ is a star shaped set at x. By integrating only along the direction lies in the start shaped set at x, we get the same volume estimate for any set which is star shaped set at x, with Ric_-^H also only integrate in T. This will be useful in applications.

5.4 Geometric and Topological Results for Integral Curvature

Volume comparison is a powerful tool for studying manifolds with lower Ricci curvature bound and has many applications. As a result of (5.3.15), (5.3.11), (5.3.3) many results with pointwise Ricci lower bound (i.e. $\|\mathrm{Ric}_-^H\|_p(r) = 0$) can be extended to the case when $\overline{\|\mathrm{Ric}_-^H\|}_p(r)(p > n/2)$ is very small, although many times serious extra work is needed. As shown by Examples 5.1.1, 5.1.2 the smallness of $\overline{\|\mathrm{Ric}_-^H\|}_p(r)$ for $p \le n/2$ or general boundedness of $\overline{\|\mathrm{Ric}_-^H\|}_p(r)$ for any $p \ge 1$ does not give any interesting results.

In [56] Gallot obtained lower bound for certain isoperimetric constants when $\overline{\|\mathrm{Ric}_-^H\|}_p$ is small and $p > n/2$. This gives in a standard way lower eigenvalue bounds and Sobolev constant bounds (for $L^{1,1} \subset L^{\frac{2p}{2p-1}}$). Using this and Bochner technique Gallot proved various interesting topological results for integral curvature [56]. For example,

Theorem 5.4.1 (Gallot 1988) *Given $p > n/2, D > 0, H \le 0$ there exist constants $\epsilon(p,H,D), C(p,H,D)$ such that if M^n is a compact Riemannian manifold with $diam_M \le D$, $\overline{\|\mathrm{Ric}_-^H\|}_p \le \epsilon(p, H.D)$, then $b_1(M) \le n \cdot C(p,H,D)$.*

This extends Theorem 3.1.3 (for $H \le 0$) to integral Ricci curvature lower bound, see also [72]. Gallot also obtained bound for all higher Betti numbers. In this case the bound depends in addition on $\overline{\|Rm\|}_p$. Replacing that by the smallness of $\overline{\|K_-^H\|}_p$ would give a true generalization of Gromov's Betti number estimate.

In [110] using the volume estimate (5.3.15) for tube of hypersurface instead of balls P. Levy-Gromov's isoperimetric inequality (Theorem **??**) is generalized to integral Ricci curvature, improving Gallot's estimate. In particular one gets a bound for the classical Sobolev constant coming from the embedding $L^{1,1} \subset L^{\frac{n}{n-1}}$.

With (5.3.3) several pinching and compactness results can be extended quickly [111]. For example Gromov's precompactness theorem (Theorem 4.0.5) extends immediately when $\overline{\|\mathrm{Ric}_-^H\|}_p$ is small. In fact ???? [9]. (5.3.3) also gives volume growth when the volume is bounded from below. Therefore combining with D. Yang's compactness [144] one has

Theorem 5.4.2 (Petersen-Wei 1997) *Given an integer $n \geq 2$ and numbers $p > n/2, H \leq 0, v > 0, D < \infty, \Lambda < \infty$, one can find $\epsilon(n,p,H,D) > 0$ such that the class of closed Riemannian n-manifolds with $\mathrm{Vol}_M \geq v, \mathrm{diam}_M \leq D, \|Rm\|_p \leq \Lambda, \overline{\|\mathrm{Ric}_-^H\|}_p \leq \epsilon(n,p,H,D)$ is precompact in the $C^\alpha, \alpha < 2 - \frac{n}{p}$, topology.*

Example 5.1.1 shows the smallness of $\overline{\|\mathrm{Ric}_-^H\|}_p$ is necessary.

In [112] using (5.3.3) and D. Yang's estimate on Sobolev constants [144] the basic tools — maximal principle, gradient estimate, excess estimate are extended to integral Ricci curvature. With these and (5.2.3) some of Colding's [46, 47] and Cheeger-Colding's [35] work (e.g. Theorems 4.0.6, 4.0.7) are also generalized.

Using (5.3.11) Cheeger-Yau's lower bound of the heat kernel (Theorem **??**) is extended in [51].

On the other hand, unlike pointwise Ricci curvature lower bound, Ricci curvature bounded from below in L^p does not automatically lift to the covering spaces. Therefore certain topological implications, such as those on the fundamental group, for Ricci curvature bounded from below in L^p does not follow immediately since we need to apply volume comparison on the covering space. Aubry [8, 9] showed the mean of the integral Ricci curvature on the geodesic balls of the covering space can be controlled by the mean of the manifold, allowing several topological extensions of pointwise Ricci curvature to integral Ricci curvature.

Proposition 5.4.3 (Aubry 2009) *Given an integer $n \geq 2$ and numbers $p > n/2, H \leq 0, D < \infty$, one can find $\epsilon(n,p,|H|D^2) > 0$ such that if M^n satisfies $\mathrm{diam}_M \leq D, D^2\overline{\|\mathrm{Ric}_-^H\|}_p \leq \epsilon(n,p,|H|D^2)$ then for any non-negative function φ on M, any normal covering $\pi : \bar{M} \to M$, $\bar{x} \in \bar{M}$ and $R \geq 2D$, we have*

$$\frac{1}{3^{n+1}e^{2(n-1)|H|D^2}} \fint_M \varphi \leq \fint_{B(\bar{x},R)} \bar{\varphi} \leq 3^{n+1}e^{2(n-1)|H|D^2} \fint_M \varphi, \tag{5.4.20}$$

where $\bar{\varphi} = \pi^*\varphi$ and $\fint_M \varphi = \frac{1}{\mathrm{Vol} M}\int_M \varphi$.

Proof: To relate the integral on $B(\bar{x}, R)$ to the one in the base, it is natural to cover $B(\bar{x}, R)$ with a subset $\bar{T} \subset \bar{M}$ which is union of fundamental domains. Let $N = \max_{y \in M} \#\{\pi^{-1}(y) \cap B(\bar{x}, R)\}$. Since $R \geq D$, we have $N \geq 1$. Now for each $y \in M$, choose N distinct points $\bar{y}_i \in \pi^{-1}(y), i = 1, \cdots, N$ such that $d(\bar{x}, \bar{y}_i) \leq d(\bar{x}, \bar{y})$ for any $\bar{y} \in \pi^{-1}(y) \setminus \{\bar{y}_i, i = 1, \cdots, N\}$. Let $\bar{T}$ be the union of these $\bar{y}_i, i = 1, \cdots, N$ for all $y \in M$, i.e. $\bar{T}$ is the smallest Dirichlet domains contains $B(\bar{x}, R)$. Hence $B(\bar{x}, R) \subset \bar{T} \subset B(\bar{x}, R + D)$ and $⨍_{\bar{T}} \bar{\varphi} = ⨍_M \varphi$.

Now we show $\bar{T}$ is also star shaped at $\bar{x}$. Given $\bar{y} \in \bar{T}$, connect $\bar{x}, \bar{y}$ with a minimal geodesic γ. Assume there exists $\bar{z} \in \gamma \setminus T$. Then there are distinct nontrivial decktranformations $\sigma_1, \cdots, \sigma_N$ such that each $\sigma_i \bar{z} \in T$. Since $\bar{y} \in T$, there exists $1 \leq i_0 \leq N$ such that $\sigma_{i_0} \bar{y} \notin T$. I.e.

$$d(\bar{x}, \bar{y}) \leq d(\bar{x}, \sigma_{i_0} \bar{y}), \quad d(\bar{x}, \bar{z}) \geq d(\bar{x}, \sigma_{i_0} \bar{z}).$$

Now

$$\begin{aligned} d(\bar{x}, \bar{y}) &= d(\bar{x}, \bar{z}) + d(\bar{z}, \bar{y}) \\ &\geq = d(\bar{x}, \sigma_{i_0} \bar{z}) + d(\sigma_{i_0} \bar{z}, \sigma_{i_0} \bar{y} \\ &\geq d(\bar{x}, \sigma_{i_0} \bar{y}). \end{aligned}$$

Combining above we have equalities everywhere. We have a minimal geodesics connecting $\bar{x}, \sigma_{i_0} \bar{y}$ which contains $\sigma_{i_0} \bar{z}$. Hence the geodesic $\sigma_{i_0} \gamma$ contain $\bar{x}$ and $\sigma_{i_0} \bar{x} = \bar{x}$, a contradiction.

Hence

$$⨍_{B(\bar{x}, R)} \varphi \leq \frac{\mathrm{Vol} T}{\mathrm{Vol} B(\bar{x}, R)} ⨍_M \varphi$$

and we only need to control $\frac{\mathrm{Vol} T}{\mathrm{Vol} B(\bar{x}, R)} = \frac{\mathrm{Vol}(T \cap B(\bar{x}, R+D))}{\mathrm{Vol}(T \cap B(\bar{x}, R))}$. Apply Corollary 5.3.3 to T, we immediately get (5.4.20) with smallness depends on R. To get smallness only depends on D, we need to integrate a little more carefully.

Denote $A_T(x, r) = A(x, r) \cap T, B_T(x, r) = B(x, r) \cap T$. By (5.3.17), for $R - D \leq t \leq r \leq R + D$, applying to the start shaped set T, we have

$$\begin{aligned} &A_T(x, r) A_H(t) - A_H(r) A_T(x, t) \\ &\quad \leq C(n, p) \left(\overline{\|\mathrm{Ric}_-^H\|_p}(T) \right)^{\frac{p}{2p-1}} A_H(r) (2D)^{\frac{1}{2p-1}} \mathrm{Vol}\, T. \end{aligned}$$

Integrate this with respect to t from $R - D$ to R and r from R to $R + D$ gives

$$\begin{aligned} &(\mathrm{Vol} B_T(x, R + D) - \mathrm{Vol} B_T(x, R)) \int_{R-D}^{R} A_H(t)\, dt \\ &\quad \leq \left(\mathrm{Vol} B_T(x, R) + 2C(n, p) \mathrm{Vol}\, T \left(D^2 \overline{\|\mathrm{Ric}_-^H\|_p}(T) \right)^{\frac{p}{2p-1}} \right) \int_R^{R+D} A_H(r)\, dr. \end{aligned}$$

Namely

$$\begin{aligned} &\frac{\mathrm{Vol} B_T(x, R + D)}{\mathrm{Vol} B_T(x, R)} \left(\int_{R-D}^{R} A_H - 2C(n, p) \left(D^2 \overline{\|\mathrm{Ric}_-^H\|_p}(T) \right)^{\frac{p}{2p-1}} \int_R^{R+D} A_H \right) \\ &\quad \leq \int_{R-D}^{R+D} A_H. \end{aligned}$$

Since $A_H(R)$ is increasing and $\frac{sn_H(R+D)}{sn_H(R-D)}$ is decreasing in R, for any $R \in [2D, \infty)$, we have

$$\frac{\int_R^{R+D} A_H}{\int_{R-D}^{R} A_H} \leq \frac{2D}{D}\left(\frac{sn_H(R+D)}{sn_H(R-D)}\right)^{n-1} \leq 2\left(\frac{sn_H(3D)}{sn_H(D)}\right)^{n-1} \leq 3^n e^{2(n-1)|H|D^2}.$$

Therefore if $D^2\overline{\|\mathrm{Ric}_-^H\|}_p \leq \left(\frac{1}{3^{n+2}C(n,p)e^{2(n-1)|H|D^2}}\right)^{\frac{2p-1}{p}}$ then $\frac{\mathrm{Vol}B_T(x,R+D)}{\mathrm{Vol}B_T(x,R)} \leq 3^{n+1}e^{2(n-1)|H|D^2}$, proving the right part of the inequality in (5.4.20). The left part of the inequality follows similarly with by constructing T for $B(\bar{x}, R-D)$ and controlling $\frac{\mathrm{Vol}B_T(x,R-D)}{\mathrm{Vol}B_T(x,R)}$. ■

With Proposition 5.4.3, Theorems 3.1.3, 3.2.10, **??**, **??**, **??** can be easily extended to the integral curvature, since in these cases one only needs to use volume comparison on balls of radius comparable to the diameter [8, 72, 9]. One does not have extension of Milnor's result (Theorem 3.2.11) directly to integral curvature, even when one assumes the manifold is compact since one needs to use volume comparison for balls of arbitrary large radius in the cover. We conjecture it is still true. Namely

Conjecture 5.4.4 *For $n \in \mathbb{N}$, $p > \frac{n}{2}$, there exists a constant $\epsilon(n,p) > 0$ such that if a compact Riemannian manifold M^n has $Diam_M \leq 1$ and $\overline{\|\mathrm{Ric}_-^0\|}_p(1) \leq \epsilon(n,p)$ then the fundamental group of M is almost nilpotent.*

This would recover the recent result of Kapovitch-Wilking [74] (Theorem **??**).

When $H > 0$ the mean curvature estimate (5.2.4) only hold for $0 < r \leq \frac{\pi}{2\sqrt{H}}$. Therefore the extension of Myers' theorem (Theorem 1.2.3) to integral curvature is not immediate at all [110, 8]. Using (5.2.10) and volume comparison on a star shaped set, Aubry obtained a complete extension [8].

Theorem 5.4.5 (Aubry, 2007) *Given $p > n/2$, there exists an $\epsilon = \epsilon(n,p) > 0$ such that if M^n is a complete Riemannian manifold with $\overline{\|\mathrm{Ric}_-^{n-1}\|}_p \leq \epsilon$, then M is compact with finite fundamental group and*

$$\mathrm{diam}_M \leq \pi\left(1 + C(n,p)\epsilon^{\frac{1}{10}}\right).$$

Here one implicitly assumes the volume of M is finite. This is easily obtained with the following estimate also by Aubry. For $p > n/2$,

$$\mathrm{Vol}M \leq \mathrm{Vol}\mathbb{S}^n\left(1 + C(n,p)\|\mathrm{Ric}_-^{n-1}\|_p\right). \tag{5.4.21}$$

Namely the volume is finite whenever $\|\mathrm{Ric}_-^{n-1}\|_p$ is finite. The volume and diameter estimates are proved together.

Proof: Let $\{B(x_i, 2\pi)\}_{i\in I}$ be a maximal family of disjoint balls in M. Consider the Dirichlet domains $T_i = \{y \in M | d(y,x_i) < d(y,x_j), \forall j \neq i\}$. Then

$B(x_i, 2\pi) \subset T_i \subset B(x_i, 4\pi)$, T_i is star shaped at x_i and $M = \cup_i T_i$ up to a set of measure zero.Therefore

$$\begin{aligned}
\int_M (\mathrm{Ric}_-^{n-1})^p &= \sum_{i\in I}\int_{T_i}(\mathrm{Ric}_-^{n-1})^p \\
&\geq \alpha^p \sum_{i\in I}\mathrm{Vol}T_i = \alpha^p \mathrm{Vol}M,
\end{aligned}$$

where $\alpha = \inf_{i\in I}\overline{\|\mathrm{Ric}_-^{n-1}\|}_p(T_i)$. Now the diameter and volume estimate follow from the following key local diameter estimate.

Lemma 5.4.6 *If M^n contains a subset T such that $B(x, R_0) \subset T \subset B(x, R_T)$ with $R_T \geq R_0 > \pi$, T is star shaped at x, and*

$$\epsilon = R_T^2\overline{\|\mathrm{Ric}_-^{n-1}\|}_p(T) \leq B(n,p)\left(1 - \frac{\pi}{R_0}\right)^4,$$

then $\mathrm{diam}_M \leq \pi\left(1 + C(n,p)\epsilon^{\frac{1}{20}}\right)$ *(and $M \subset T$).*

Proof: First we show for $\pi \leq r \leq R_T$,

$$A_T(x,r) \leq \frac{C(n,p)}{r}\epsilon^{\frac{p(n-1)}{2p-1}}\mathrm{Vol}T. \tag{5.4.22}$$

(For this estimate we only need $\epsilon^{\frac{p}{2p-1}} \leq \frac{\pi}{6}$.) In order to do comparison for $r \geq \pi$, we take the model space with constant sectional curvature $H_r = \left(\frac{\pi-\epsilon'}{r}\right)^2 < 1$, where $\epsilon' = \epsilon^{\frac{p}{2p-1}}$. From (5.3.14) for star shaped set, we have for $t \in [\frac{\pi}{2(\pi-\epsilon')}r, r]$,

$$\begin{aligned}
&\left(\frac{A_T(x,r)}{\sin^{n-1}(\sqrt{H_r}r)}\right)^{\frac{1}{2p-1}} - \left(\frac{A_T(x,t)}{\sin^{n-1}(\sqrt{H_r}t)}\right)^{\frac{1}{2p-1}} \\
&\quad\leq \left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\|\mathrm{Ric}_-^{n-1}\|_p(T)\right)^{\frac{p}{2p-1}}\int_t^r \frac{1}{\sin^2(\sqrt{H_r}s)}ds.
\end{aligned}$$

Since $\sqrt{H_r}s \in (\frac{\pi}{2}, \pi)$,

$$\begin{aligned}
\int_t^r \frac{1}{\sin^2(\sqrt{H_r}s)}ds &\leq \left(\frac{\pi}{2}\right)^2\int_t^r \frac{1}{(\pi - \sqrt{H_r}s)^2}ds = \frac{\pi^2(r-t)}{4(\pi-\sqrt{H_r}t)(\pi-\sqrt{H_r}r)} \\
&= \frac{\pi^2(r-t)}{4(\pi-\sqrt{H_r}t)\epsilon'} \leq \frac{\pi r}{4\epsilon'}.
\end{aligned}$$

Here we use the fact that $t \to \frac{r-t}{\pi-\sqrt{H_r}t}$ is decreasing. Since $\sin(\sqrt{H_r}r) = \sin(\pi - \epsilon') = \sin(\epsilon') \leq \epsilon'$, we have

$$\begin{aligned}
A_T(x,r)^{\frac{1}{2p-1}} &\leq A_T(x,t)^{\frac{1}{2p-1}}\left(\frac{\epsilon'}{\sin((\pi-\epsilon')\frac{t}{r})}\right)^{\frac{n-1}{2p-1}} \\
&\quad + \frac{\pi}{4}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{\frac{p-1}{2p-1}}\left(\frac{\mathrm{Vol}T}{R_T}\right)^{\frac{1}{2p-1}}\epsilon'^{\frac{n-1}{2p-1}}.
\end{aligned}$$

For $t \in [\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$, $\sin((\pi-\epsilon')\frac{t}{r}) \geq \sin\frac{\pi}{6} = \frac{1}{2}$. When $\epsilon' \leq \frac{\pi}{6}$, one has $\frac{5\pi}{6(\pi-\epsilon')}r \leq r$. Now using the inequality $(a+b)^{2p-1} \leq 2^{2p-2}(a^{2p-1}+b^{2p-1})$ we have

$$A_T(x,r) \leq 2^{2p+n-3}(\epsilon')^{n-1}A_T(x,t) + \frac{\pi^{2p-1}}{4^p}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1}\frac{\mathrm{Vol}T}{R_T}(\epsilon')^{n-1}$$

for all $t \in [\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$.

By mean value theorem, there exists $t_0 \in [\frac{\pi}{2(\pi-\epsilon')}r, \frac{5\pi}{6(\pi-\epsilon')}r]$ such that

$$A_T(x,t_0) = \frac{3(\pi-\epsilon')}{\pi r}\int_{\frac{\pi}{2(\pi-\epsilon')}r}^{\frac{5\pi}{6(\pi-\epsilon')}r} A_T(x,t)dt \leq \frac{3}{r}\int_0^{R_T} A_T(x,t)dt = \frac{3}{r}\mathrm{Vol}T.$$

Hence

$$A_T(x,r) \leq \left[3\cdot 2^{2p+n-3} + \frac{\pi^{2p-1}}{4^p}\left(\frac{n-1}{(2p-1)(2p-n)}\right)^{p-1}\right]\frac{\mathrm{Vol}T}{r}\epsilon^{\frac{p(n-1)}{2p-1}},$$

which is (5.4.22).

Therefore, for $\pi < R_0 \leq 2\pi$, $T = B(x,R_0)$, $\delta \in (0, \frac{R_0-\pi}{2})$, if $y \in M$ with $d(x,y) \geq \pi+\delta$, then $B(y,\delta) \subset B(x,\pi+2\delta) \setminus B(x,\pi)$, and

$$\mathrm{Vol}B(y,\delta) \leq \int_{\pi}^{\pi+2\delta} A(x,r)dr \leq 2\delta C(p,n)\mathrm{Vol}B(x,R_0)\epsilon^{\frac{p(n-1)}{2p-1}}. \qquad (5.4.23)$$

Next we give a lower bound for $\mathrm{Vol}B(y,\delta)$.

From (5.3.18), for $0 < r \leq R \leq R_0$, we have

$$\left(\frac{\mathrm{Vol}B(x,r)}{\mathrm{Vol}B(x,R)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{r}{R}\right)^{\frac{n}{2p-1}}\left(1 - C(n,p)\bar{\epsilon}^{\frac{p}{2p-1}}\right),$$

where $\bar{\epsilon} = R_0^2\overline{\|\mathrm{Ric}_-^0\|}_p(R_0) \leq R_0^2\overline{\|\mathrm{Ric}_-^{n-1}\|}_p(R_0) = \epsilon$, and for $z \in B(x,R_0)$ with $0 < r \leq R \leq R_0 - d(x,z)$,

$$\left(\frac{\mathrm{Vol}B(z,r)}{\mathrm{Vol}B(z,R)}\right)^{\frac{1}{2p-1}} \geq \left(\frac{r}{R}\right)^{\frac{n}{2p-1}}\left(1 - C(n,p)\left(\frac{R^2\epsilon}{R_0^2}\right)^{\frac{p}{2p-1}}\right).$$

Iterate this estimate with a sequence of balls of increasing size as in Proposition 1.4.12, but with $\frac{1}{2} \leq \alpha = \alpha(p,n) < 1$ close to 1 such that $\alpha^{\frac{n}{2p-1}} \geq \frac{2}{3}$ and for $\frac{n}{2} < p \leq n$, $(2-\alpha)^{2p-n}\alpha^n < 1$, for $\epsilon \leq \epsilon(n,p)$ small, we have

$$\frac{\mathrm{Vol}B(y,r)}{\mathrm{Vol}B(x,R_0)} \geq \frac{r^n}{R_0^n}\left[\left(\frac{2}{3} - C(n,p)\epsilon^{\frac{p'}{2p'-1}}\right)\left(\frac{r}{R_0}\right)^{\frac{2n}{2p'-1}} - C(n,p)\epsilon^{\frac{p'}{2p'-1}}\right]^{2p'-1},$$

when $p' = \max(n,p)$. Hence for ϵ small,

$$\mathrm{Vol}B(y,\delta) \geq \frac{\delta^n}{R_0^n}\left[\frac{1}{2}\left(\frac{\delta}{R_0}\right)^{\frac{2n}{2p'-1}} - C(n,p)\epsilon^{\frac{p'}{2p'-1}}\right]^{2p'-1}\mathrm{Vol}B(x,R_0).(5.4.24)$$

To finish the proof, note that when $\epsilon \leq \epsilon(n,p)$, $\frac{\mathrm{Vol}B_T(x,R)}{\mathrm{Vol}T} \geq \frac{1}{2}\frac{R^n}{R_T^n}$ for $R \leq R_T$. Hence $\epsilon \geq \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{R_T}{R}\right)^{2-\frac{n}{p}} R^2 \overline{\|\mathrm{Ric}_-^{n-1}\|}_p(B_T(x,R)) \geq \frac{1}{2}R^2\overline{\|\mathrm{Ric}_-^{n-1}\|}_p(B_T(x,R))$ and we can assume $R_T = 2\pi$ so $\pi < R_0 \leq 2\pi$. Now from (5.4.23) and (5.4.24), either $\left(\frac{\delta}{2\pi}\right)^{\frac{2n}{2p'-1}} \leq 4C(n,p)\epsilon^\beta$, where $\beta = \frac{2np(n-1)}{(2p-1)(2p'-1)(3n-1)} \leq \frac{p'}{2p'-1}$, or

$$\left(\frac{\delta}{2\pi}\right)^n \epsilon^{\beta(2p'-1)} \leq C(n,p)\delta\epsilon^{\frac{p(n-1)}{2p-1}}.$$

In either case we have $\delta \leq C(n,p)\epsilon^{\frac{p(n-1)}{(2p-1)(3n-1)}} \leq C(n,p)\epsilon^{\frac{1}{10}}$, hence $M \subset B(x,R_0)$. Let z be any point of M. By above $B(x, R_0 - \pi - C(n,p)\epsilon^{\frac{1}{10}}) \subset B(z,R_0)$. Therefore

$$\begin{aligned}\frac{\mathrm{Vol}B(z,R_0)}{\mathrm{Vol}B(x,R_0)} &\geq \frac{\mathrm{Vol}B(x,R_0-\pi-C(n,p)\epsilon^{\frac{1}{10}})}{\mathrm{Vol}B(x,R_0)}\\ &\geq \frac{1}{2}\left(\frac{R_0-\pi-C(n,p)\epsilon^{\frac{1}{10}}}{2\pi}\right)^n \geq \frac{1}{4}\left(\frac{R_0-\pi}{2\pi}\right)^n\end{aligned}$$

and $\epsilon \geq \frac{1}{4}\left(\frac{R_0-\pi}{2\pi}\right)^{\frac{n}{p}} \overline{\|\mathrm{Ric}_-^{n-1}\|}_p(B(z,R_0))$. Now the same estimate for x applies to z by replacing ϵ with $4\left(\frac{2\pi}{R_0-\pi}\right)^{\frac{n}{p}}\epsilon$, this gives the diameter estimate. ■

5.5 Smoothing

Another method to [50].

Chapter 6

Comparison Geometry for Bakry-Emery Ricci Tensor

6.1 N-Bakry-Emery Ricci Tensor

The Bakry-Emery Ricci tensor is a Ricci tensor for smooth metric measure spaces, which are Riemannian manifold with a measure conformal to the Riemannian measure. Formally a smooth metric measure space is a triple $(M^n, g, e^{-f}dvol_g)$, where M is a complete n-dimensional Riemannian manifold with metric g, f is a smooth real valued function on M, and $dvol_g$ is the Riemannian volume density on M. This is also sometimes called a manifold with density. physics dilaton, analytical reasons. These spaces occur naturally as smooth collapsed limits of manifolds with lower Ricci curvature bound under the measured Gromov-Hausdorff convergence [53].

Definition 6.1.1 *We say (X_i, μ_i) converges in the measured Gromov-Hausdorff sense to (X_∞, μ_∞) if for all sequences of continuous functions $f_i : X_i \to \mathbb{R}$ converging to $f_\infty : X_\infty \to \mathbb{R}$, we have*

$$\int_{X_i} f_i d\mu_i \to \int_{X_\infty} f_\infty d\mu_\infty. \tag{6.1.1}$$

Example 6.1.2 *Let $(M^n \times F^N, g_\epsilon)$ be a product manifold with warped product metric $g_\epsilon = g_M + (\epsilon e^{-f})^2 g_F$, where f is a function on M, F is compact. Then, as $\epsilon \to 0$, the Riemannian measure $dvol_{g_\epsilon}$ goes to zero, but with respect to the renormalized Riemannian measure $\widetilde{dvol_{g_\epsilon}} = dvol_{g_\epsilon}/$volume of a unit ball of$g_\epsilon$, $(M^n \times F^N, \widetilde{dvol_{g_\epsilon}})$ converges to $(M^n, e^{-Nf}dvol_{g_M})$ under the measured Gromov-Hausdorff convergence.*

The N-Bakry-Emery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \mathrm{Hess} f - \frac{1}{N} df \otimes df \quad \text{for } N > 0. \tag{6.1.2}$$

As we will discuss below, N is related to the dimension of the model space. We allow N to be infinite, in this case we denote $\mathrm{Ric}_f = \mathrm{Ric}_f^\infty = \mathrm{Ric} + \mathrm{Hess} f$. Note that when f is a constant function $\mathrm{Ric}_f^N = \mathrm{Ric}$ for all N and we can take $N = 0$ in this case. Moreover, if $N_1 \geq N_2$ then $\mathrm{Ric}_f^{N_1} \geq \mathrm{Ric}_f^{N_2}$ so that $\mathrm{Ric}_f^N \geq \lambda g$ implies $\mathrm{Ric}_f \geq \lambda g$.

The Bakry Emery Ricci tensor (for N finite and and infinite) has a natural extension to non-smooth metric measure spaces [**?**, **?**, 130] and diffusion operators [10]. Moreover, the equation $\mathrm{Ric}_f = \lambda g$ for some constant λ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow; the equation $\mathrm{Ric}_f^N = \lambda g$, for N positive integer, corresponds to warped product Einstein metric on $M \times_{e^{-\frac{f}{N}}} F^N$. See [**?**] for a modification of the Ricci tensor which is conformally invariant.

We are interested in investigating what geometric and topological results for the Ricci tensor extend to the Bakry-Emery Ricci tensor. This was studied by Lichnerowicz [**?**, **?**] almost forty years ago, though this work does not seem to be widely known. Recently this has been actively investigated and there are many interesting results in this direction which we will discuss below, see for example [**?**, 114, 87, 11, 107, 12, **?**, **?**, **?**, **?**, **?**, **?**]. In this note we first recall the Bochner formulas for Bakry-Emery Ricci tensors (stated a little differently from how they have appeared in the literature). The derivation of these from the classical Bochner formula is elementary, so we present the proof. Then we quickly derive the first eigenvalue comparison from the Bochner formulas as in the classical case. In the rest of the paper we focus on mean curvature and volume comparison theorems and their applications. When N is finite, this work is mainly from [114, 12], and when N is infinite, it's mainly from our recent work [**?**].

The most well known example is the following soliton, often referred to as the Gaussian soliton.

Example 6.1.3 *Let $M = \mathbb{R}^n$ with Euclidean metric g_0, $f(x) = \frac{\lambda}{2}|x|^2$. Then* $\mathrm{Hess} f = \lambda g_0$ *and* $\mathrm{Ric}_f = \lambda g_0$.

This example shows that, unlike the case of Ricci curvature uniformly bounded from below by a positive constant, the manifold could be noncompact when $\mathrm{Ric}_f \geq \lambda g$ and $\lambda > 0$.

Example 6.1.4 *Let $M = \mathbb{H}^n$ be the hyperbolic space. Fixed any $p \in M$, let $f(x) = (n-1)r^2 = (n-1)d^2(p,x)$. Now Hess $r^2 = 2|\nabla r|^2 + 2r\mathrm{Hess} r \geq 2I$, therefore* $\mathrm{Ric}_f \geq (n-1)$.

This example shows that the Cheeger-Gromoll splitting theorem and Abresch-Gromoll's excess estimate do not hold for $\mathrm{Ric}_f \geq 0$, in fact they don't even hold for $\mathrm{Ric}_f \geq \lambda > 0$. Note that the only properties of hyperbolic space used are that $\mathrm{Ric} \geq -(n-1)$ and that Hess $r^2 \geq 2I$. But $\mathrm{Hess}\, r^2 \geq 2I$ for any Cartan-Hadamard manifold, therefore any Cartan-Hadamard manifold with Ricci curvature bounded below has a metric with $\mathrm{Ric}_f \geq 0$ on it. On the other hand, in these examples $\mathrm{Ric} < 0$. When $\mathrm{Ric} < 0$ ($\mathrm{Ric} \leq 0$) and $\mathrm{Ric}_f \geq 0$($\mathrm{Ric}_f > 0$), then $\mathrm{Hess} f > 0$ and f is strictly convex. Therefore M has to homeomorphic to $\mathbb{R}^n$.

Example 6.1.5 *Let $M = \mathbb{R}^n$ with Euclidean metric, $f(x_1, \cdots, x_n) = x_1$. Since Hess $f = 0$, $\mathrm{Ric}_f = \mathrm{Ric} = 0$. On the other hand $\mathrm{Vol}_f(B(0,r))$ is of exponential growth. Along the x_1 direction, $m_f - m_H = -1$ which does not goes to zero.*

6.2 Bochner formulas for the N-Bakry-Emery Ricci tensor

With respect to the measure $e^{-f} dvol$ the natural self-adjoint f-Laplacian is $\Delta_f = \Delta - \nabla f \cdot \nabla$. In this case we have

$$\begin{aligned}\Delta_f |\nabla u|^2 &= \Delta |\nabla u|^2 - 2\mathrm{Hess}\, u(\nabla u, \nabla f), \\ \langle \nabla u, \nabla(\Delta_f u)\rangle &= \langle \nabla u, \nabla(\Delta u)\rangle - \mathrm{Hess}\, u(\nabla u, \nabla f) - \mathrm{Hess} f(\nabla u, \nabla u).\end{aligned}$$

Plugging these into (1.1.1) we immediately get the following Bochner formula for the N-Bakry-Emery Ricci tensor.

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u) + \frac{1}{N}|\langle \nabla f, \nabla u\rangle|^2. \quad (6.2.3)$$

When $N = \infty$, we have

$$\frac{1}{2}\Delta_f |\nabla u|^2 = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f(\nabla u, \nabla u). \quad (6.2.4)$$

This formula is virtually the same as (1.1.1) except for the important fact that $tr(\mathrm{Hess}\, u) = \Delta u$ not $\Delta_f(u)$. In the case where N is finite, however, we can get around this difficulty by using the inequality

$$\frac{(\Delta u)^2}{n} + \frac{1}{N}|\langle \nabla f, \nabla u\rangle|^2 \geq \frac{(\Delta_f(u))^2}{N+n} \quad (6.2.5)$$

which implies

$$\frac{1}{2}\Delta_f |\nabla u|^2 \geq \frac{(\Delta_f(u))^2}{N+n} + \langle \nabla u, \nabla(\Delta_f u)\rangle + \mathrm{Ric}_f^N(\nabla u, \nabla u). \quad (6.2.6)$$

In other words, a Bochner formula holds for Ric_f^N that looks like the Bochner formula for the Ricci tensor of an $n+N$ dimensional manifold. Note that (6.2.5) is an equality if and only if $\Delta u = \frac{n}{N}\langle \nabla f, \nabla u\rangle$, so equality in (6.2.6) is seldom achieved when f is nontrivial. When f is constant, we can take $N = 0$ so (6.2.6) recovers (1.1.5).

6.3 Eigenvalue and Mean Curvature Comparison

From the Bochner formulas we can now prove eigenvalue and mean curvature comparisons which generalize the classical ones. First we consider the eigenvalue comparison.

Let M^n be a complete Riemannian manifold with $\mathrm{Ric}_f^N \geq (n-1)H > 0$. Applying (6.2.6) to the first eigenfunction u of Δ_f, $\Delta_f u = -\lambda_1 u$, and integrating with respect to the measure $e^{-f}dvol$, we have

$$0 \geq \int_M \left(\frac{(\lambda_1 u)^2}{N+n} - \lambda_1 |\nabla u|^2 + (n-1)H|\nabla u|^2 \right) e^{-f} dvol.$$

Since $\int_M |\nabla u|^2 e^{-f} dvol = \lambda_1 \int_M u^2 e^{-f} dvol$, we deduce the eigenvalue estimate [?]

$$\lambda_1 \geq (n-1)H \left(1 + \frac{1}{N+n-1} \right). \tag{6.3.1}$$

When f is constant, taking $N = 0$ gives the classical Lichnerowicz's first eigenvalue estimate $\lambda_1 \geq nH$ [82]. When $N = \infty$, we have [10]

$$\lambda_1 \geq (n-1)H. \tag{6.3.2}$$

This also can be derived from (6.2.4) directly. One may expect that the estimate (6.3.2) is weaker than the classical one. In fact (6.3.2) is optimal as the following example shows.

Example 6.3.1 *Let $M = \mathbb{R}^1 \times S^2$ with standard product metric g_0, $f(x,y) = \frac{1}{2}x^2$. Then $\mathrm{Hess} f(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = 1$ and zero on all other directions. We have $\mathrm{Ric}_f = 1 g_0$. Now for the linear function $u(x,y) = x$, $\Delta_f u = -x$. So $\lambda_1 = 1$.*

On the other hand (6.3.2) is never optimal for compact manifolds since equality in (6.3.2) implies $\mathrm{Hess}\, u = 0$. Note that $\mathrm{Ric}_f \geq (n-1)H > 0$ on a compact manifold implies $\mathrm{Ric}_f^N \geq (n-1)H' > 0$ for some N big, hence one can use estimate (6.3.1).

Now we turn to the mean curvature (or Laplacian) comparison. Recall that the mean curvature measures the relative rate of change of the volume element. Therefore, for the measure $e^{-f}dvol$, the associated mean curvature is $m_f = m - \partial_r f$, where m is the mean curvature of the geodesic sphere with inward pointing normal vector. Also $m_f = \Delta_f(r)$, where r is the distance function.

Let m_H^k be the mean curvature of the geodesic sphere in the model space M_H^k, the complete simply connected k-manifold of constant curvature H. When we drop the superscript k and write m_H we mean the mean curvature from the model space whose dimension matches the dimension of the manifold. Since $\mathrm{Hess}\, r$ is zero along the radial direction, applying the Bochner formula (6.2.3) to the distance function r, the Schwarz inequality $|\mathrm{Hess} r|^2 \geq \frac{(\Delta r)^2}{n-1}$ and (6.2.5) gives

$$m_f' \leq -\frac{(m_f)^2}{n+N-1} - \mathrm{Ric}_f^N(\partial_r, \partial_r). \tag{6.3.3}$$

Thus, using the standard Sturm-Liouville comparison argument, one has the mean curvature comparison [12].

Theorem 6.3.2 (Mean curvature comparison for N-Bakry-Emery) *If $\mathrm{Ric}_f^N \geq (n+N-1)H$, then*

$$m_f(r) \leq m_H^{n+N}(r). \tag{6.3.4}$$

Namely the mean curvature is less or equal to the one of the model with dimension $n+N$. This does not give any information when N is infinite.

In fact, such a strong, uniform estimate is not possible when N is infinite. To see this note that, when $H > 0$, the model space M_H^{n+N} is a round sphere so that $m_H^{n+N}(r)$ goes to $-\infty$ as r goes to $\frac{\pi}{\sqrt{H}}$. Thus (6.3.4) implies that if N is finite and $\mathrm{Ric}_f^N \geq \lambda > 0$ then M is compact (See Theorem 6.4.5 in the next section for the diameter bound). On the other hand, this is not true when $N = \infty$ as the following example shows.

Thus, when N is infinite, one can not expect such a strong mean curvature comparison to be true. However, we can show a weaker, nonuniform estimate and also give some uniform estimates if we make additional assumptions on f such as f being bounded or $\partial_r f$ bounded from below. In these cases we have the following mean curvature comparisons [**?**] which generalizes the classical one.

Theorem 6.3.3 (Mean Curvature Comparison for ∞-Bakry-Emery) *Let $p \in M^n$. Assume $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$,*
a) given any minimal geodesic segment and $r_0 > 0$,

$$m_f(r) \leq m_f(r_0) - (n-1)H(r - r_0) \quad \text{for } r \geq r_0. \tag{6.3.5}$$

b) if $\partial_r f \geq -a$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/2\sqrt{H}$) then

$$m_f(r) - m_H(r) \leq a \tag{6.3.6}$$

along that minimal geodesic segment from p. Equality holds if and only if the radial sectional curvatures are equal to H and $f(t) = f(p) - at$ for all $t < r$.
c) if $|f| \leq k$ along a minimal geodesic segment from p (when $H > 0$ assume $r \leq \pi/4\sqrt{H}$) then

$$m_f(r) \leq m_H^{n+4k}(r) \tag{6.3.7}$$

along that minimal geodesic segment from p. In particular when $H = 0$ we have

$$m_f(r) \leq \frac{n+4k-1}{r} \tag{6.3.8}$$

Proof:

$$m_f' \le -\frac{m^2}{n-1} - \mathrm{Ric}_f(\partial r, \partial r). \tag{6.3.9}$$

If $\mathrm{Ric}_f \ge (n-1)H$, we have

$$m_f' \le -(n-1)H.$$

This immediately gives the inequality (6.3.5).

From the Riccati inequality (1.2.2), equality (1.2.4), and assumption on Ric_f, we have

$$(m - m_H)' \le -\frac{m^2 - m_H^2}{n-1} + \mathrm{Hess}\, f(\partial_r, \partial_r). \tag{6.3.10}$$

As in the third proof of the mean curvature comparison theorem (Theorem **??**), we compute

$$\begin{aligned}
\left(\mathrm{sn}_H^2(m - m_H)\right)' &= 2\mathrm{sn}_H'\mathrm{sn}_H(m - m_H) + sn_H^2(m - m_H)' \\
&\le \mathrm{sn}_H^2\left(\frac{2m_H}{n-1}(m - m_H) - \frac{m^2 - m_H^2}{n-1} + \mathrm{Hess}\, f(\partial_r, \partial_r)\right) \\
&= \mathrm{sn}_H^2\left(-\frac{(m - m_H)^2}{n-1} + \mathrm{Hess}\, f(\partial_r, \partial_r)\right) \\
&\le sn_H^2 \mathrm{Hess}\, f(\partial_r, \partial_r).
\end{aligned} \tag{6.3.11}$$

Here in the 2nd line we have used (6.3.10) and (1.2.6).

Integrating (6.3.11) from 0 to r yields

$$\mathrm{sn}_H^2(r)m(r) \le \mathrm{sn}_H^2(r)m_H(r) + \int_0^r \mathrm{sn}_H^2(t)\partial_t\partial_t f(t)dt. \tag{6.3.12}$$

When f is constant (the classical case) this gives the usual mean curvature comparison.

Proof or Part b. Using integration by parts on the last term we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) - \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt. \tag{6.3.13}$$

Under our assumptions $(\mathrm{sn}_H^2(t))' = 2\mathrm{sn}_H'(t)\mathrm{sn}_H(t) \ge 0$ so if $\partial_t f(t) \ge -a$ we have

$$\mathrm{sn}_H^2(r)m_f(r) \le \mathrm{sn}_H^2(r)m_H(r) + a\int_0^r (\mathrm{sn}_H^2(t))'dt = \mathrm{sn}_H^2(r)\left(m_H(r) + a\right).$$

This proves the inequality (6.3.6).

To see the rigidity statement suppose that $\partial_t f \ge -a$ and $m_f(r) = m_H(r) + a$ for some r. Then from (6.3.13) we see

$$a\mathrm{sn}_H^2 \le \int_0^r (\mathrm{sn}_H^2(t))'\partial_t f(t)dt \le a\mathrm{sn}_H^2. \tag{6.3.14}$$

So that $\partial_t f \equiv -a$. But then $m(r) = m_f - a = m_H(r)$ so that the rigidity follows from the rigidity for the usual mean curvature comparison.

Proof of Part c. Integrate (6.3.13) by parts again

$$\mathrm{sn}_H^2(r) m_f(r) \leq \mathrm{sn}_H^2(r) m_H(r) - f(r)(\mathrm{sn}_H^2(r))' + \int_0^r f(t)(sn_H^2)''(t)dt. \tag{6.3.15}$$

Now if $|f| \leq k$ and $r \in (0, \frac{\pi}{4\sqrt{H}}]$ when $H > 0$, then $(sn_H^2)''(t) \geq 0$ and we have

$$\mathrm{sn}_H^2(r) m_f(r) \leq \mathrm{sn}_H^2(r) m_H(r) + 2k(\mathrm{sn}_H^2(r))'. \tag{6.3.16}$$

From (1.2.6) we can see that

$$(\mathrm{sn}_H^2(r))' = 2\mathrm{sn}_H' \mathrm{sn}_H = \frac{2}{n-1} m_H \mathrm{sn}_H^2.$$

so we have

$$m_f(r) \leq \left(1 + \frac{4k}{n-1}\right) m_H(r) = m_H^{n+4k}(r). \tag{6.3.17}$$

■

Now when $H > 0$ and $r \in [\frac{\pi}{4\sqrt{H}}, \frac{\pi}{2\sqrt{H}}]$,

$$\begin{aligned} \int_0^r f(t)(sn_H^2)''(t)dt &\leq k\left(\int_0^{\frac{\pi}{4\sqrt{H}}} (sn_H^2)''(t)dt - \int_{\frac{\pi}{4\sqrt{H}}}^r (sn_H^2)''(t)dt\right) \\ &= k\left(\frac{2}{\sqrt{H}} - sn_H(2r)\right). \end{aligned}$$

Hence

$$m_f(r) \leq \left(1 + \frac{4k}{n-1} \cdot \frac{1}{\sin(2\sqrt{H}r)}\right) m_H(r). \tag{6.3.18}$$

This estimate will be used later to prove the Myers' theorem in Section **??**. ■

In the case $H = 0$, we have $sn_H(r) = r$ so (6.3.15) gives the estimate in [**?**] that

$$m_f(r) \leq \frac{n-1}{r} - \frac{2}{r} f(r) + \frac{2}{r^2} \int_0^r f(t)dt. \tag{6.3.19}$$

These mean curvature comparisons can be used to prove some Myers' type theorems for Ric_f, and is related to volume comparison theorems, both of which we discuss in the next section.

6.4 Volume Comparison and Myers' Theorems

For $p \in M^n$, we use exponential polar coordinates around p and write the volume element $d\,vol = \mathcal{A}(r,\theta)dr \wedge d\theta_{n-1}$, where $d\theta_{n-1}$ is the standard volume element on the unit sphere $S^{n-1}(1)$. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$. Using (1.4.3) we have

$$\frac{\mathcal{A}_f'}{\mathcal{A}_f}(r,\theta) = (\ln(\mathcal{A}_f(r,\theta)))' = m_f(r,\theta). \tag{6.4.1}$$

And for $r \geq r_0 > 0$

$$\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}_f(r_0,\theta)} = e^{\int_{r_0}^r m_f(r,\theta)}. \tag{6.4.2}$$

Combining this equation with the mean curvature comparisons we obtain volume comparisons. Let $\mathrm{Vol}_f(B(p,r)) = \int_{B(p,r)} e^{-f} dvol_g$, the weighted (or f-)volume, $\mathrm{Vol}_H^k(r)$ be the volume of the radius r-ball in the model space M_H^k.

Theorem 6.4.1 (Volume comparison for N-Bakry-Emery) *[114] If $\mathrm{Ric}_f^N \geq (n+N-1)H$, then $\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_H^{n+N}(R)}$ is nonincreasing in R.*

In [87] Lott shows that if M is compact (or just $|\nabla f|$ is bounded) with $\mathrm{Ric}_f^N \geq \lambda$ for some positive integer $2 \leq N < \infty$, then, in fact, there is a family of warped product metrics on $M \times S^N$ with Ricci curvature bounded below by λ, recovering the comparison theorems for Ric_f^N.

When $N = \infty$ we have the following volume comparison results which generalize the classical one. Part a) is originally due to Morgan [**?**] where it follows from a hypersurface volume estimate(also see [**?**]). For the proofs of parts b) and c) see [**?**].

Theorem 6.4.2 (Volume Comparison for ∞-Bakry-Emery) *Let $(M^n, g, e^{-f}dvol_g)$ be complete smooth metric measure space with $\mathrm{Ric}_f \geq (n-1)H$. Fix $p \in M^n$.*
a) If $H > 0$, then $\mathrm{Vol}_f(M)$ is finite.
b) If $\partial_r f \geq -a$ along all minimal geodesic segments from p then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$) ,

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq e^{aR}\frac{\mathrm{Vol}_H^n(R)}{\mathrm{Vol}_H^n(r)}. \tag{6.4.3}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H and $\partial_r f \equiv -a$. In particular if $\partial_r f \geq 0$ and $\mathrm{Ric}_f \geq 0$ then M has f-volume growth of degree at most n.
c) If $|f(x)| \leq k$ then for $R \geq r > 0$ (assume $R \leq \pi/4\sqrt{H}$ if $H > 0$),

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{\mathrm{Vol}_H^{n+4k}(R)}{\mathrm{Vol}_H^{n+4k}(r)}. \tag{6.4.4}$$

In particular, if f is bounded and $\mathrm{Ric}_f \geq 0$ then M has polynomial f-volume growth.

For Part a) we compare with a model space, however, we modify the measure according to a. Namely, the model space will be the pointed metric measure space $M^n_{H,a} = (M^n_H, g_H, e^{-h}dvol, O)$ where (M^n_H, g_H) is the n-dimensional simply connected space with constant sectional curvature H, $O \in M^n_H$, and $h(x) = -a \cdot d(x, O)$. We make the model a pointed space because the space only has $\mathrm{Ric}_f(\partial_r, \partial_r) \geq (n-1)H$ in the radial directions from O and we only compare volumes of balls centered at O.

Let $\mathcal{A}^a_H$ be the h-volume element in $M^n_{H,a}$. Then $\mathcal{A}^a_H(r) = e^{ar}\mathcal{A}_H(r)$ where $\mathcal{A}_H$ is the Riemannian volume element in M^n_H. By the mean curvature comparison we have $(\ln(\mathcal{A}_f(r,\theta))' \leq a + m_H = (\ln(\mathcal{A}^a_H))'$ so for $r < R$,

$$\frac{\mathcal{A}_f(R,\theta)}{\mathcal{A}_f(r,\theta)} \leq \frac{\mathcal{A}^a_H(R,\theta)}{\mathcal{A}^a_H(r,\theta)}. \tag{6.4.5}$$

Namely $\frac{\mathcal{A}_f(r,\theta)}{\mathcal{A}^a_H(r,\theta)}$ is nonincreasing in r. Using Lemma 3.2 in [150], we get for $0 < r_1 < r$, $0 < R_1 < R$, $r_1 \leq R_1$, $r \leq R$,

$$\frac{\int_{R_1}^R \mathcal{A}_f(t,\theta)dt}{\int_{r_1}^r \mathcal{A}_f(t,\theta)dt} \leq \frac{\int_{R_1}^R \mathcal{A}^a_H(t,\theta)dt}{\int_{r_1}^r \mathcal{A}^a_H(t,\theta)dt}. \tag{6.4.6}$$

Integrating along the sphere direction gives

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \leq \frac{\mathrm{Vol}^a_H(R_1,R)}{\mathrm{Vol}^a_H(r_1,r)}. \tag{6.4.7}$$

Where $\mathrm{Vol}^a_H(r_1,r)$ is the h-volume of the annulus $B(O,r) \setminus B(O,r_1) \subset M^n_H$. Since $\mathrm{Vol}_H(r_1,r) \leq \mathrm{Vol}^a_H(r_1,r) \leq e^{ar}\mathrm{Vol}_H(r_1,r)$ this gives (6.4.3) when $r_1 = R_1 = 0$ and proves Part b).

In the model space the radial function h is not smooth at the origin. However, clearly one can smooth the function to a function with $\partial_r h \geq -a$ and $\partial^2_r h \geq 0$ such that the h-volume taken with the smoothed h is arbitrary close to that of the model. Therefore, the inequality (6.4.7) is optimal. Moreover, one can see from the equality case of the mean curvature comparison that if the annular volume is equal to the volume in the model then all the radial sectional curvatures are H and f is exactly a linear function.

Proof of Part b): In this case let $\mathcal{A}^{n+4k}_H$ be the volume element in the simply connected model space with constant curvature H and dimension $n+4k$.

Then from the mean curvature comparison we have $\ln(\mathcal{A}_f(r,\theta))' \leq \ln(\mathcal{A}^{n+4k}_H(r))'$. So again applying Lemma 3.2 in [150] we obtain

$$\frac{\mathrm{Vol}_f(A(p,R_1,R))}{\mathrm{Vol}_f(A(p,r_1,r))} \leq \frac{Vol^{n+4k}_H(R_1,R)}{Vol^{n+4k}_H(r_1,r)}. \tag{6.4.8}$$

With $r_1 = R_1 = 0$ this implies the relative volume comparison for balls

$$\frac{\mathrm{Vol}_f(B(p,R))}{\mathrm{Vol}_f(B(p,r))} \leq \frac{Vol^{n+4k}_H(R)}{Vol^{n+4k}_H(r)}. \tag{6.4.9}$$

Equivalently

$$\frac{\mathrm{Vol}_f(B(p,R))}{V_H^{n+4k}(R)} \le \frac{\mathrm{Vol}_f(B(p,r))}{V_H^{n+4k}(r)}. \tag{6.4.10}$$

Since $n+4k > n$ we note that the right hand side blows up as $r \to 0$ so one does not obtain a uniform upper bound on $\mathrm{Vol}_f(B(p,R))$. Indeed, it is not possible to do so since one can always add a constant to f and not effect the Bakry-Emery tensor.

By taking $r = 1$ we do obtain a volume growth estimate for $R > 1$

$$\mathrm{Vol}_f(B(p,R)) \le \mathrm{Vol}_f(B(p,1)) Vol_H^{n+4k}(R). \tag{6.4.11}$$

Note that, from Part a) $\mathrm{Vol}_f(B(p,1)) \le e^{-f(p)} e^a \omega_n$ if $\partial_r f \ge -a$ on $B(p,1)$. ■

Part a) should be viewed as a weak Myers' theorem for Ric_f. Namely if $\mathrm{Ric}_f > \lambda > 0$ then the manifold may not be compact but the measure must be finite. In particular the lifted measure on the universal cover is finite. Since this measure is invariant under the deck transformations, this weaker Myers' theorem is enough to recover the main topological corollary of the classical Myers' theorem.

Corollary 6.4.3 *If M is complete and $Ric_f \ge \lambda > 0$ then M has finite fundamental group.*

Using a different approach the second author has proven that the fundamental group is, in fact, finite for spaces satisfying $\mathrm{Ric} + \mathcal{L}_X g \ge \lambda > 0$ for some vector field X [**?**]. This had earlier been shown under the additional assumption that the Ricci curvature is bounded by Zhang [**?**]. See also [**?**]. When M is compact the finiteness of fundamental group was first shown by X. Li [**?**, Corollary 3] using a probabilistic method.

On the other hand, the volume comparison Theorem 6.4.1 and Theorem 6.4.2 Part c) also give the following generalization of Calabi-Yau's theorem [147].

Theorem 6.4.4 *If M is a noncompact, complete manifold with $\mathrm{Ric}_f^N \ge 0$, assume f is bounded when N is infinite, then M has at least linear f-volume growth.*

Theorem 6.4.2 Part a) and Theorem 6.4.4 then together show that any manifold with $\mathrm{Ric}_f^N \ge \lambda > 0$ and f bounded if N is infinite must be compact. In fact, from the mean curvature estimates one can prove this directly and obtain an upper bound on the diameter. For finite N this is due to Qian [114], for Part b) see [**?**].

Theorem 6.4.5 (Myers' Theorem) *Let M be a complete Riemannian manifold with $\mathrm{Ric}_f^N \ge (n-1)H > 0$,*

a) when N is finite, then M is compact and $\mathrm{diam}_M \le \sqrt{\frac{n+N-1}{n-1}}\frac{\pi}{\sqrt{H}}$.

b) when N is infinite and $|f| \le k$ then M is compact and $\mathrm{diam}_M \le \frac{\pi}{\sqrt{H}} + \frac{4k}{(n-1)\sqrt{H}}$.

For some other Myers' Theorems for manifolds with measure see [**?**] and [**?**]. The relative volume comparison Theorem 6.4.2 also implies the following extensions of theorems of Gromov [68] and Anderson [5].

Theorem 6.4.6 *For the class of manifolds M^n with* $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{diam}_M \leq D$ *and* $|f| \leq k$ ($|\nabla f| \leq a$)*, the first Betti number* $b_1 \leq C(n+4k, HD^2)$ ($C(n, HD^2, aD)$).

Theorem 6.4.7 *For the class of manifolds M^n with* $\mathrm{Ric}_f \geq (n-1)H$, $\mathrm{Vol}_f \geq V$, $\mathrm{diam}_M \leq D$ *and* $|f| \leq k$ ($|\nabla f| \leq a$) *there are only finitely many isomorphism types of* $\pi_1(M)$.

Question 6.4.8 *If M^n has a complete metric and measure such that* $\mathrm{Ric}_f \geq 0$ *and f is bounded, does M^n has a metric with* $\mathrm{Ric} \geq 0$*?*

Chapter 7

Comparison Geometry in Ricci Flow

Perelman's reduced volume monotonicity [107], a basic and powerful tool in his work on Thurston's geometrization conjecture, is a generalization of Bishop-Gromov's volume comparison to Ricci flow.

7.1 Reduced Volume Monotonicity

7.2 Heuristic Argument

This little note presents explicit curvature formulas in §6.1 of Perelman's paper [**?**]. In particular it verifies (mod N^{-1}) the geometric interpretation of Hamilton's matrix (trace) Harnarck quadratic and that the Ricci tensor of the warped metric are equal to zero. The mod N^{-1} computation of the curvatures is also done in [**?**] using Christoffel symbols. Here we do the computation using Gauss equation and Koszul's formula.

Recall that $\tilde{M} = M \times \mathbb{S}^N \times \mathbb{R}^+$ with the metric:

$$\tilde{g}_{ij} = g_{ij},\ \tilde{g}_{\alpha\beta} = \tau g_{\alpha\beta},\ \tilde{g}_{00} = \frac{N}{2\tau} + R,\ \tilde{g}_{i\alpha} = \tilde{g}_{i0} = \tilde{g}_{\alpha 0} = 0, \tag{7.2.1}$$

where i, j denote coordinate indices on the M factor, α, β denote those on the $\mathbb{S}^N$ factor, and the coordinate τ on $\mathbb{R}^+$ had index 0; g_{ij} evolves with τ by the backward Ricci flow $(g_{ij})_\tau = 2R_{ij}$, $g_{\alpha\beta}$ is the metric on $\mathbb{S}^N$ of constant curvature $\frac{1}{2N}$.

We first compute the curvatures of $\tilde{M}$ without τ-direction using Gauss equation by viewing $M \times \mathbb{S}^N$ as isometrically embedded submanifold of $\tilde{M}$. Let $n = (\frac{N}{2\tau} + R)^{-1/2}\frac{\partial}{\partial\tau}$ be its unit normal vector. Denote $\{X_i\}$ local coordinate fields of M and $\{U_\alpha\}$ local coordinate fields of $\mathbb{S}^N$. Then the second fundamental form

$$\begin{aligned}\langle B(X_i,X_j),n\rangle &= \langle\tilde{\nabla}_{X_i}X_j,n\rangle = -\frac{1}{2}n\langle X_i,X_j\rangle\\ &= -\frac{1}{2}(\frac{N}{2\tau}+R)^{-1/2}(g_{ij})_\tau = -(\frac{N}{2\tau}+R)^{-1/2}R_{ij}.\end{aligned}$$

So

$$B(X_i,X_j) = -(\frac{N}{2\tau}+R)^{-1/2}R_{ij}n. \tag{7.2.2}$$

Similarly

$$B(U_\alpha,U_\beta) = -\frac{1}{2}(\frac{N}{2\tau}+R)^{-1/2}g_{\alpha\beta}n,\ \ B(U_\alpha,X_i)=0. \tag{7.2.3}$$

By Gauss equation

$$\begin{aligned}\langle\tilde{R}(X_i,X_j)X_k,X_l\rangle &= \langle R(X_i,X_j)X_k,X_l\rangle - \langle B(X_i,X_l),B(X_j,X_k)\rangle + \langle B(X_j,X_l),B(X_i,X_k)\rangle\\ &= \langle R(X_i,X_j)X_k,X_l\rangle - (\frac{N}{2\tau}+R)^{-1}\left[R_{il}R_{jk}-R_{jl}R_{ik}\right], \qquad (7.2.4)\\ \langle\tilde{R}(X_i,U_\alpha)U_\beta,X_j\rangle &= 0-\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}R_{ij}g_{\alpha\beta}, \qquad (7.2.5)\\ \langle\tilde{R}(U_\alpha,U_\beta)U_\gamma,U_\theta\rangle &= \langle R(U_\alpha,U_\beta)U_\gamma,U_\theta\rangle - \frac{1}{4}(\frac{N}{2\tau}+R)^{-1}\left[g_{\alpha\theta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\theta}\right] \qquad (7.2.6)\\ &= \frac{1}{2N\tau}\left[\tilde{g}_{\alpha\theta}\tilde{g}_{\beta\gamma}-\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\theta}\right] - \frac{1}{4}(\frac{N}{2\tau}+R)^{-1}\left[g_{\alpha\theta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\theta}\right]\\ &= (\frac{N}{2\tau}+R)^{-1}\frac{\tau R}{2N}\left[g_{\alpha\theta}g_{\beta\gamma}-g_{\alpha\gamma}g_{\beta\theta}\right] \qquad (7.2.7)\\ \langle\tilde{R}(X_i,X_j)X_k,U_\alpha\rangle &= 0 \qquad (7.2.8)\\ \langle\tilde{R}(X_i,U_\beta)U_\gamma,U_\theta\rangle &= 0. \qquad (7.2.9)\end{aligned}$$

For curvatures involve normal direction, note that

$$[U_\alpha,n]=0,\quad [n,X_i]=\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}(X_iR)n.$$

By Koszul's formula we have

$$\begin{aligned}\tilde{\nabla}_{U_\alpha}X_i &= 0,\\ \tilde{\nabla}_n n &= -\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}\sum_{l,k}(X_lR)g^{lk}X_k,\\ \tilde{\nabla}_{U_\alpha}n &= \frac{1}{2\tau}(\frac{N}{2\tau}+R)^{-1/2}U_\alpha,\\ \tilde{\nabla}_{X_i}n &= (\frac{N}{2\tau}+R)^{-1/2}\sum_{l,k}R_{il}g^{lk}X_k.\end{aligned}$$

Therefore

$$\begin{aligned}\langle \tilde{R}(U_\alpha, n)n, X_i\rangle &= 0, &(7.2.10)\\ \langle \tilde{R}(U_\alpha, n)n, U_\beta\rangle &= -\langle \tilde{\nabla}_n \tilde{\nabla}_{U_\alpha} n, U_\beta\rangle \\ &= -(\frac{N}{2\tau}+R)^{-1/2}\frac{d}{d\tau}\left[(\frac{N}{2\tau}+R)^{-1/2}\frac{1}{2\tau}\right]\tau g_{\alpha\beta} - \frac{1}{4\tau^2}(\frac{N}{2\tau}+R)^{-1}\tau g_{\alpha\beta}\\ &= \frac{1}{4}(\frac{N}{2\tau}+R)^{-2}(R_\tau + \frac{R}{\tau})g_{\alpha\beta}. &(7.2.11)\end{aligned}$$

By (7.2.2)

$$\tilde{\nabla}_{X_i} X_j = -(\frac{N}{2\tau}+R)^{-1/2}R_{ij}n + \nabla_{X_i}X_j. \tag{7.2.12}$$

So

$$\begin{aligned}\langle \tilde{\nabla}_{X_i}\tilde{\nabla}_n n, X_j\rangle &= -\frac{1}{2}X_i[(\frac{N}{2\tau}+R)^{-1}\sum_{l,k}(X_lR)g^{lk}]g_{kj} - \frac{1}{2}(\frac{N}{2\tau}+R)^{-1}\sum_{l,k}(X_lR)g^{lk}\langle\nabla_{X_i}X_k, X_j\rangle\\ &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}(X_iR)(X_jR) - \frac{1}{2}(\frac{N}{2\tau}+R)^{-1}Hess(R)(X_i,X_j),\\ -\langle \tilde{\nabla}_n\tilde{\nabla}_{X_i} n, X_j\rangle &= -(\frac{N}{2\tau}+R)^{-1/2}\sum_{l,k}\frac{d}{d\tau}\left[(\frac{N}{2\tau}+R)^{-1/2}R_{il}g^{lk}\right]g_{kj} - (\frac{N}{2\tau}+R)^{-1}\sum_{l,k}R_{il}g^{lk}R_{kj}\\ &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\left(-\frac{N}{2\tau^2}+R_\tau\right)R_{ij} + (\frac{N}{2\tau}+R)^{-1}\left[\sum_{l,k}R_{il}g^{lk}R_{kj} - (R_{ij})_\tau\right],\\ -\langle \tilde{\nabla}_{[X_i,n]} n, X_j\rangle &= -\frac{1}{4}(\frac{N}{2\tau}+R)^{-2}(X_iR)(X_jR).\end{aligned}$$

Thus

$$\begin{aligned}\langle \tilde{R}(X_i, n)n, X_j\rangle &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\left[\left(-\frac{N}{2\tau^2}+R_\tau\right)R_{ij} + \frac{1}{2}(X_iR)(X_jR)\right]\\ &\quad +(\frac{N}{2\tau}+R)^{-1}\left[\sum_{l,k}R_{il}g^{lk}R_{kj} - \frac{1}{2}Hess(R)(X_i,X_j) - (R_{ij})_\tau\right] &(7.2.13)\end{aligned}$$

Last we need to look at the normal component of the curvature tensor. By (7.2.3) we have

$$\tilde{\nabla}_{U_\alpha} U_\beta = -\frac{1}{2}(\frac{N}{2\tau}+R)^{-1/2}g_{\alpha\beta}n + \nabla_{U_\alpha}U_\beta. \tag{7.2.14}$$

Therefore

$$\begin{aligned}\langle \tilde{R}(X_i, X_j)X_k, n\rangle &= \langle \tilde{\nabla}_{X_i}\tilde{\nabla}_{X_j}X_k - \tilde{\nabla}_{X_j}\tilde{\nabla}_{X_i}X_k, n\rangle\\ &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-\frac{3}{2}}[(X_iR)R_{jk} - (X_jR)R_{ik}]\\ &\quad -(\frac{N}{2\tau}+R)^{-\frac{1}{2}}[(\nabla_{X_i}Ric)(X_j,X_k) - (\nabla_{X_j}Ric)(X_i,X_k)]\end{aligned}$$

$$\begin{aligned}
\langle \tilde{R}(U_\alpha, U_\beta)U_\gamma, n\rangle &= 0 \\
\langle \tilde{R}(X_i, U_\alpha)U_\beta, n\rangle &= \frac{1}{4}(\frac{N}{2\tau}+R)^{-\frac{3}{2}}(X_i R)g_{\alpha\beta} \\
\langle \tilde{R}(U_\alpha, X_i)X_j, n\rangle &= 0.
\end{aligned}$$

From above, mod N^{-1}, all curvature tensors of $\tilde{M}$ are zero except

$$\begin{aligned}
\langle \tilde{R}(X_i, X_j)X_k, X_l\rangle &= \langle R(X_i, X_j)X_k, X_l\rangle \\
\langle \tilde{R}(X_i, \frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}, X_j\rangle &= -\frac{1}{2\tau}R_{ij} + \sum_{l,k} R_{il}g^{lk}R_{kj} - \frac{1}{2}Hess(R)(X_i, X_j) - (R_{ij})_\tau \\
&= -\frac{1}{2\tau}R_{ij} + \Delta R_{ij} + 2R_{ikjl}R_{kl} - \sum_{l,k} R_{il}g^{lk}R_{kj} - \frac{1}{2}Hess(R)(X_i, X_j) \\
\langle \tilde{R}(X_i, X_j)X_k, \frac{\partial}{\partial\tau}\rangle &= -(\nabla_{X_i}Ric)(X_j, X_k) + (\nabla_{X_j}Ric)(X_i, X_k).
\end{aligned}$$

These are exactly the coefficients $R_{ijkl}, M_{ij}, P_{ijk}$ of Hamilton's Harnack quadratic.

Any two form ω on $\tilde{M}$ can be written as

$$\omega = U_{ij}X_i^* \wedge X_j^* + W_i d\tau \wedge X_i^* + \text{two form with sphere components.}$$

Then mod N^{-1}, the curvature operator $\tilde{\mathcal{R}}$ acts on ω is

$$\langle \tilde{\mathcal{R}}(\omega), \omega\rangle = \langle \tilde{R}(X_i, X_j)X_k, X_l\rangle U_{ij}U_{kl} + 2\langle \tilde{R}(X_i, X_j)X_k, \frac{\partial}{\partial\tau}\rangle U_{ij}W_k + \langle \tilde{R}(X_i, \frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}, X_j\rangle W_i W_j,$$

which is exactly Hamilton's matrix Harnack quadratic. Therefore Hamilton's matrix Harnack inequality can be interpreted as the curvature operator $\tilde{\mathcal{R}}$ is nonnegative (mod N^{-1}). This is suggested to me by John Lott.

By taking trace in the manifold directions we get the trace Harnack quadratic. Namely let

$$\omega_k = e_k^* \wedge (d\tau + X^*),$$

where $\{e_k\}$ is an orthonormal basis of TM and X is a vector field on M. Then

$$\begin{aligned}
\sum_k \langle \tilde{\mathcal{R}}(\omega_k), \omega_k\rangle &= \sum_k \langle \tilde{R}(X, e_k)e_k, X\rangle + 2\sum_k \langle \tilde{R}(X, e_k)e_k, \frac{\partial}{\partial\tau}\rangle + \sum_k \langle \tilde{R}(e_k, \frac{\partial}{\partial\tau})\frac{\partial}{\partial\tau}, e_k\rangle \\
&= Ric(X, X) - \langle \nabla R, X\rangle - \frac{1}{2\tau}R - \frac{1}{2}R_\tau \quad \text{mod } N^{-1},
\end{aligned}$$

which is exactly Hamilton's trace Harnack quadratic. Note that this is not $\tilde{Ric}(\frac{\partial}{\partial\tau} + X, \frac{\partial}{\partial\tau} + X)$, since we need to take trace in all directions for $\tilde{Ric}$.

For Ricci curvatures we take trace of the curvature tensors and get

$$\begin{aligned}
\tilde{Ric}(X_i, X_j) &= \langle \tilde{R}(X_i, n)n, X_j\rangle + R_{ij} - (\frac{N}{2\tau}+R)^{-1}\sum_{k,l}\left[R_{ij}R_{kl} - R_{kj}R_{il}\right]g^{kl} - \frac{N}{2\tau}(\frac{N}{2\tau}+R)^{-1}R \\
&= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\left[\left(-\frac{N}{2\tau^2}+R_\tau\right)R_{ij} + \frac{1}{2}(X_i R)(X_j R)\right]
\end{aligned}$$

$$+(\frac{N}{2\tau}+R)^{-1}\left[2\sum_{l,k}R_{il}g^{lk}R_{kj}-(R_{ij})_\tau-\frac{1}{2}Hess(R)(X_i,X_j)\right] \tag{7.2.15}$$

$$\tilde{Ric}(X_i,U_\alpha) = 0 \tag{7.2.16}$$

$$\begin{aligned}\tilde{Ric}(U_\alpha,U_\beta) &= \langle\tilde{R}(U_\alpha,n)n,U_\beta\rangle-\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}\left(\sum_{ij}R_{ij}g^{ij}-\frac{N-1}{N}R\right)g_{\alpha\beta}\\ &= \frac{1}{4}(\frac{N}{2\tau}+R)^{-2}(R_\tau-\frac{2R^2}{N})g_{\alpha\beta}\end{aligned} \tag{7.2.17}$$

$$\tilde{Ric}(n,n) = \frac{1}{4}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2 \tag{7.2.18}$$

$$\tilde{Ric}(X_i,n) = -\frac{1}{2}(\frac{N}{2\tau}+R)^{-3/2}[\sum_{j,k}(X_jR)R_{ik}g^{jk}] \tag{7.2.19}$$

$$\tilde{Ric}(U_\alpha,n) = 0. \tag{7.2.20}$$

So $\tilde{Ric}=0 \bmod N^{-1}$.

The scalar curvature of $\tilde{M}$ is

$$\begin{aligned}\tilde{R} &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2-\frac{1}{2}(\frac{N}{2\tau}+R)^{-1}(\Delta R+\frac{R}{\tau}+R_\tau)\\ &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2+(\frac{N}{2\tau}+R)^{-1}(\|\mathrm{Ric}\|^2-\frac{R}{2\tau})\\ &= \frac{1}{2}(\frac{N}{2\tau}+R)^{-2}\|\nabla R\|^2+(\frac{N}{2\tau}+R)^{-1}[\|\mathrm{Ric}_o\|^2+\frac{R}{n}(R-\frac{n}{2\tau})].\end{aligned}$$

So the scalar curvature is positive when $R<0$ or $R>\frac{n}{2\tau}$.

$\tilde{\Delta}=\frac{\partial}{\partial\tau}+\Delta \pmod{N^{-1}}$.

Consider a metric ball in $(\tilde{M},\tilde{g})$ centered at some point p where $\tau=0$. The shortest geodesic between p and an arbitrary point q is always orthogonal to the $\mathbb{S}^N$ fiber. The length of such curve $\gamma(\tau)$ is

$$\begin{aligned}l(\gamma(\tau)) &= \int_0^{\tau(q)}\sqrt{\frac{N}{2\tau}+R+|\dot\gamma_M(\tau)|^2}d\tau\\ &= \int_0^{\tau(q)}\sqrt{\frac{N}{2\tau}}\sqrt{1+\frac{2\tau(R+|\dot\gamma_M(\tau)|^2)}{N}}d\tau\\ &= \sqrt{2N\tau(q)}+\frac{1}{\sqrt{2N}}\int_0^{\tau(q)}\sqrt{\tau}(R+|\dot\gamma_M(\tau)|^2)d\tau+O(N^{-\frac{3}{2}}).\end{aligned}$$

The shortest geodesic should minimize

$$\mathcal{L}(\gamma)=\int_0^{\tau(q)}\sqrt{\tau}(R+|\dot\gamma_M(\tau)|^2)d\tau.$$

The metric sphere $S(p,\sqrt{2N\tau(q)})\subset\tilde{M}$ is $O(N^{-1})$-close to the hypersurface $\tau=\tau(q)$. $\bmod N^{-1}$,

$$\frac{A((0,p),\sqrt{2N\tau(q)})}{A(N+n+1,0,\sqrt{2N\tau})} = \frac{\int_{(M,g(\tau))}(2N\tau(x))^{N/2}\mathrm{Vol}(\mathbb{S}^N)dvol_M}{(\sqrt{2N\tau})^{N+n}\mathrm{Vol}(\mathbb{S}^{N+n})}$$

$$
\begin{aligned}
&= \frac{(2N)^{N/2}\mathrm{Vol}(\mathbb{S}^N)\int_{(M,g(\tau))}\left(\sqrt{\tau(q)}-\frac{1}{2N}L(x)+O(N^{-2})\right)^N dx}{(\sqrt{2N\tau})^{N+n}\mathrm{Vol}(\mathbb{S}^{N+n})} \\
&= \frac{\mathrm{Vol}(\mathbb{S}^N)}{\mathrm{Vol}(\mathbb{S}^{N+n})(2N)^{n/2}}\int_M \tau^{-\frac{n}{2}}\left(1-\frac{L}{2\sqrt{\tau}}\frac{1}{N}\right)^N dx + O(N^{-1}).
\end{aligned}
$$

As $N\to\infty$, $\left(1-\frac{L}{2\sqrt{\tau}}\frac{1}{N}\right)^N \to e^{-\frac{L}{2\sqrt{\tau}}}$. In fact Perelman gave a heuristic argument that volume comparison on an infinite dimensional space (incorporating the Ricci flow) gives the reduced volume monotonicity.

7.3 Laplacian Comparison for Ricci Flow

Chapter 8

Ricci Curvature for Metric Measure Spaces

8.1 Metric Space and Optimal Transportation

Ricci curvature lower bound of a metric measure space is closely related to the convexity of an entropy functional on the space of probability measures. In this section we state the basic definition and properties of metric spaces, length spaces, and optimal transport problem and the space of probability measures. These materials can be found at [23, 68, 131, 132].

8.1.1 Metric and Length Spaces

Some good references of this subsection are [23, 68], we refer to these for proofs and more details.

Given a metric space (X, d), $\gamma : [a, b] \to X$ a continuous curve, the length of γ is defined by

$$L(\gamma) = \sup_{k\in\mathbb{N}} \sup_{a=y_0\le y_1\le\cdots\le y_k=b} \sum_{i=1}^{k} d(\gamma(y_{i-1}), \gamma(y_i)).$$

γ is called rectifiable if it has finite length. A curve is called a geodesic if it is locally a distance minimizer and has a constant speed.

Definition 8.1.1 *A metric space (X, d) is called a length space if $d(x, y) = \inf_\gamma(\mathrm{length}(\gamma))$ for all $x, y \in X$, where the infimum runs over all continuous curve γ connecting x, y. (X, d) is called a geodesic space if for all $x, y \in X$, there exists continuous curve γ connecting x, y such that $d(x, y) = L(\gamma)$.*

For a geodesic space (length space), there is a (an ϵ-) midpoint point between every two points. For complete metric space, the converse is also true.

Proposition 8.1.2 *Let (X,d) be a complete metric space. (X,d) is a geodesic space (length space) if and only if for every $x,y \in X$ there exits a (an ϵ-) midpoint.*

Proof: We will show the equivalence in the case of geodesic space. The other case is similar.

$\Rightarrow$: If (X,d) is a geodesic space, for every two points x,y, there is curve $\gamma : [a,b] \to X$ with $\gamma(a) = x, \gamma(b) = y, L(\gamma) = d(x,y)$. Let $L(t) = L(\gamma_{[a,t]}$. Since $L(t)$ is continuous in t, there exists $c \in [a,b]$ such that $L(c) = \frac{1}{2}L(\gamma)$. Choose $z = \gamma(c)$, we have $d(x,z) = d(y,z) = \frac{1}{2}d(x,y)$. So z is a midpoint of x,y. Here we did not use the completeness assumption.

$\Leftarrow$: Given $x,y \in X$, we need to construct $\gamma : [0,1] \to X$ such that $\gamma(0) = x, \gamma(1) = y$ and $L(\gamma) = d(x,y)$. Assign $\gamma(\frac{1}{2})$ to be a midpoint of x,y, $\gamma(\frac{1}{4})$ to be a midpoint of $x = \gamma(0)$ and $\gamma(\frac{1}{2})$, and $\gamma(\frac{3}{4})$ to be a midpoint of $\gamma(\frac{1}{2})$ and $y = \gamma(1)$. Proceeding this way, we define γ for all dyadic rationals between 0 and 1. From the construction, for every two dyadic rational t_i, t_j

$$d(\gamma(t_i), \gamma(t_j)) = |t_i - t_j| \cdot d(x,y).$$

So the map γ defined on the set of dyadic rationals is Lipschitz. Since X is complete and the set of dyadic rational is dense in $[0,1]$, this map can be extended to a continuous map on the entire interval $[0,1]$. Thus we obtained $\gamma : [0,1] \to X$ connecting x and y with $L(\gamma) = d(x,y)$. ■

X is locally compact if every point has a compact neighborhood. These spaces enjoy the following nice property [23, Theorem 2.5.23].

Proposition 8.1.3 *If (X,d) is a complete, locally compact length space, then every closed metric ball in X is compact and (X,d) is a geodesic space.*

8.1.2 Optimal Transportation

If $f : X \to Y$ is measurable, μ a measure on X, the push forward of μ under f is $(f_*\mu)(B) = \mu(f^{-1}(B))$ for all measurable subsets B of Y.

Let (X,μ) and (Y,ν) be probability spaces. A probability measure π on $X \times Y$ is a transference plan between μ and ν if

$$(\mathrm{Proj}_X)_*\pi = \mu, \quad (\mathrm{Proj}_Y)_*\pi = \nu, \tag{8.1.1}$$

where Proj_X and Proj_Y are projections of $X \times Y$ onto X and Y respectively. If π satisfies (8.1.1) we say π has marginal μ on X and marginal ν on Y. In this case, π is also called a coupling of μ and ν. Intuitively, $\pi(x,y)$ represents the amount of mass transported from x to y. The equation (8.1.1) means

$$\pi(A \times Y) = \mu(A), \quad \pi(X \times B) = \nu(B)$$

for all measurable subsets A of X and B of Y, which is equivalent to

$$\int_{X\times Y} (\varphi(x) + \psi(y)) d\pi(x,y) = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu$$

for all measurable functions $(\varphi, \psi) \in L^1(d\mu) \times L^1(d\nu)$.

A Polish space X is a separable (i.e. it contains a countable dense subset), complete metric space. Denote by $P(X)$ the space of Borel probability measures on X. We equip $P(X)$ with the weak topology, namely $\mu_k \in P(X)$ converges weakly to μ if for all $\varphi \in C_b(X)$ (i.e. φ is bounded and continuous), $\int \varphi d\mu_k$ converges to $\int \varphi d\mu$ as $k \to \infty$.

Define $\Pi(\mu, \nu)$ to be the set of all Borel probability measures π on $X \times Y$ with marginals μ on X and ν on Y. It is a convex set, which is also nonempty since the product measure $\mu \times \nu \in \Pi(\mu, \nu)$. Given two Polish spaces X, Y, $\mu \in P(X), \nu \in P(Y)$, and c a nonnegative measurable function (the cost function) on $X \times Y$, the Kantorovich's (an Nobel Laureate in Economics, 1975) mass transportation problem is to minimize the linear functional (the total cost)

$$\pi \to \int_{X \times Y} c(x, y) d\pi(x, y)$$

on $\Pi(\mu, \nu)$.

Originally arising from economics, the mass transportation problem has turns out to be actually very useful in physics, PDE, geometry etc..

Example 8.1.4 (The discrete case) *Suppose* $X = \{x_1, \cdots, x_k\}, Y = \{y_1, \cdots, y_k\}$ *are discrete spaces where all points have the same mass:*

$$\mu = \frac{1}{k} \sum_{i=1}^{k} \delta_{x_i}, \quad \nu = \frac{1}{k} \sum_{j=1}^{k} \delta_{y_j}.$$

Any measure in $\Pi(\mu, \nu)$ *is represented by a* $k \times k$ *matrix* $\pi = (\pi_{ij})$, *where* $\pi_{ij} \geq 0$, $\sum_i \pi_{ij} = \frac{1}{k}$ *for all* j, *and* $\sum_j \pi_{ij} = \frac{1}{k}$ *for all* i. *The solution of Kantorovich's minimizing problem is given by a permutation matrix divided by* k *[**?**, Page 5]. (distance and cost function?)*

A fundamental property of Polish space is the following characterization of precompactness of probability measures which underlies the proofs of several basic facts in optimal transportation problem.

Theorem 8.1.5 (Prokhorov's theorem) *If* X *is a Polish space, then a subset* $\mathcal{P}$ *of the probability measures of* X *is precompact for the weal topology iff it is tight, i.e. for any* $\epsilon > 0$, *there is a compact set* K_ϵ *such that* $\mu(X \setminus K_\epsilon) \leq \epsilon$ *for all* $\mu \in \mathcal{P}$. *(reference?)*

By Ulam's lemma, a probability measure on a Polish space is automatically tight (i.e., the set of single probability measure is tight). Also note that a Borel probability measure on a Polish space is automatically regular. See [**?**] for the proofs of these results.

Using these one can readily prove the existence of a minimizer of Kantorovich's mass transportation problem.

Theorem 8.1.6 (Existence of a minimizer) *Let X, Y be two Polish spaces, $\mu \in P(X), \nu \in P(Y)$, and let $c : X \times Y \to [a, +\infty]$ be a lower semi-continuous cost function with $a \in \mathbb{R}$. Then there is a coupling of (μ, ν) which minimizes the total cost $C(\pi) = \int_{X\times Y} c(x, y)d\pi(x, y)$ on $\Pi(\mu, \nu)$.*

Proof: Since X, Y are Polish spaces, we have μ, ν are tight. Namely for any $\epsilon > 0$, there are compact sets $K_\epsilon \subset X$ and $L_\epsilon \subset Y$ such that $\mu(X \setminus K_\epsilon) \le \epsilon/2$ and $\nu(Y \setminus L_\epsilon) \le \epsilon/2$. Since $X \times Y \setminus K_\epsilon \times L_\epsilon = X \times (Y \setminus L_\epsilon) \cup (X \setminus K_\epsilon) \times Y$, for any coupling $\pi \in \Pi(\mu, \nu)$, we have

$$\pi(X \times Y \setminus K_\epsilon \times L_\epsilon) \le \mu(X \setminus K_\epsilon) + \nu(Y \setminus L_\epsilon) \le \epsilon.$$

Hence $\Pi(\mu, \nu)$ is tight. By Prokhorov's theorem, $\Pi(\mu, \nu)$ is precompact. The equations (8.1.1) for the marginals pass to the limit, so $\Pi(\mu, \nu)$ is also closed. Therefore $\Pi(\mu, \nu)$ is compact. Let $\{\pi_k\} \in \Pi(\mu, \nu)$ be a sequence such that the total cost $C(\pi_k)$ converges to the infimum. We have $\{\pi_k\}$ (a subsequence if necessary) converges to some $\pi \in \Pi(\mu, \nu)$. Since $c \ge a$ and is lower semi-continuous, we can write $c = \lim_{l\to\infty} c_l$, where c_l is a nondecreasing sequence of continuous functions with $a \le c_l \le c$. By monotone convergence,

$$\int c\, d\pi = \lim_{l\to\infty} \int c_l\, d\pi = \lim_{l\to\infty} \lim_{k\to\infty} \int c_l\, d\pi_k \le \lim_{k\to\infty} \int c\, d\pi_k.$$

Thus π is an minimizer. ■

We refer the reader to [132, Theorem 4.1] for a more general case. Any minimizer π is called an optimal transference plan.

When $X = Y$, the optimal transport cost $C(\mu, \nu) = \inf_{\pi\in\Pi(\mu,\nu)} \int_{X\times Y} c(x, y)d\pi(x, y)$ defines a distance on $P(X)$ when the cost function c is a distance of X. So we can use $P(X)$ to study X.

8.1.3 The Monge transport

As we discussed above, given two Polish spaces X, Y, two probability measures $\mu \in P(X), \nu \in P(Y)$, and a cost function $c : X \times Y \to \mathbb{R}$, the mass transportation problem seeks a Borel probability measure π on $X \times Y$ that couples μ and ν, and minimizes the functional

$$\int_{X\times Y} c(x, y)d\pi(x, y)$$

on $\Pi(\mu, \nu)$. Such minimizers always exist when the cost function $c(x, y)$ is lower semicontinuous and bounded from below. A minimizer π will be called an optimal transport from μ to ν.

In applications, a special type of optimal transport, formulated by Monge (hence the name), plays an important role. A Monge transport is a transference plan that comes from a map $F : X \to Y$ such that

$$F_*(\mu) = \nu. \tag{8.1.2}$$

In this case, the transference plan $\pi = (\mathrm{Id} \times F)_*(\mu)$. It is called optimal if the transference plan π is optimal.

When $X = Y$ and we consider measures that are absolutely continuous with respect to a fixed measure, a Monge transport satisfies the so-called Monge-Amperé equation. Fix $\mu \in P(X)$. Consider $\mu_0 = \rho_0(x)\mu \in P(X)$ and $\mu_1 = \rho_1(x)\mu \in P(X)$. When $F : X \to X$ is differentiable almost everywhere (such as Lipschitz) and gives rise to a Monge transport from μ_0 to μ_1, (8.1.2) (with μ replaced by μ_0 and ν by μ_1) is equivalent to the Monge-Amperé equation

$$\rho_1(F(x)) \det DF(x) = \rho_0(x), \quad \mu_0\text{-a.e.}. \tag{8.1.3}$$

Here the determinant of Jacobian factor $\det DF(x)$ is associated with the measure μ. That is

$$\det DF(x) = \lim_{\epsilon \to 0} \frac{\mu(F(B_\epsilon(x)))}{\mu(B_\epsilon(x))}. \tag{8.1.4}$$

When the cost function is given by the square of the distance, there is an important case when an optimal transport is given by a Monge transport. Moreover, there are more structures and regularities there. It was first proved in the Euclidean space with its usual distance by Brenier [**?**] and later generalized to general Riemannian manifolds by McCann [**?**].

Theorem 8.1.7 (Brenier-McCann Theorem) *Let $(X, d) = (M, g)$ be a complete connected Riemannian manifold and $c(x, y) = d^2(x, y)$ where d is the Riemannian distance. If probability measures $\mu_0 = \rho_0(x)d\mathrm{Vol}_g, \mu_1 = \rho_1(x)d\mathrm{Vol}_g$ are absolutely continuous with respect to the Riemannian measure and both ρ_0, ρ_1 are compactly supported, then there exists a convex function $f : M \to \mathbb{R}$ such that*

$$F(x) = \exp_x(\nabla f(x)) : M \to M$$

defines an optimal Monge transport between μ_0 and μ_1. Moreover, it is the unique optimal transport between μ_0 and μ_1.

8.1.4 Topology and Geometry of $P(X)$

Given a Polish space X, the space of Borel probability measures $P(X)$ inherits several topological and geometric structures of X.

Definition 8.1.8 (Wasserstein distances) *Let (X, d) be a Polish metric space. For $p \geq 1$, $\mu, \nu \in P(X)$, the (p-)Wassenstein distance is*

$$W_p(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\int_{X \times X} d(x, y)^p d\pi(x, y) \right)^{1/p}. \tag{8.1.5}$$

If π is an optimal transport from μ to ν with respect to the cost function $c(x, y) = d^p(x, y)$, then

$$W_p(\mu, \nu) = \left(\int_{X \times X} d(x, y)^p d\pi(x, y) \right)^{1/p}.$$

In particular, when $\pi = (\mathrm{Id} \times F)_*(\mu)$ is given by a Monge transport $F: X \to X$,

$$W_p(\mu, \nu) = \left(\int_X d(x, F(x))^p d\mu(x) \right)^{1/p}. \tag{8.1.6}$$

Proposition 8.1.9 *W_p defines a metric on*

$$\mathcal{P}_p(X) = \left\{ \mu \in P(X) | \int d(x_0, x)^p d\mu(x) < +\infty \ \ \textit{for some } x_0 \in X \right\}$$

.

Proof: It is clear that W_p is finite, symmetric, nonnegative and $W_p(\mu, \mu) = 0$. If $W_p(\mu, \nu) = 0$, let π be an optimal transportation plan, then $d\pi(x, y)$ is supported on the diagonal $(y = x)$. Thus for all $\varphi \in C_b(X)$, $\int \varphi d\mu = \int \varphi(x)\, d\pi(x, y) = \int \varphi(y)\, d\pi(x, y) = \int \varphi d\nu$, which implies $\mu = \nu$.

To show the triangle inequality, we need the following gluing lemma [131, Lemma 7.6].

Lemma 8.1.10 *Given Polish spaces X_1, X_2, X_3, $\mu_i \in P(X_i)$, and $\pi_{12} \in \Pi(\mu_1, \mu_2)$, $\pi_{23} \in \Pi(\mu_2, \mu_3)$, then there exists $\pi \in P(x_1 \times X_2 \times X_3)$ with marginals π_{12} on $X_1 \times X_2$ and π_{23} on $X_2 \times X_3$.*

Now consider $\mu_1, \mu_2, \mu_3 \in \mathcal{P}_p(X)$, and optimal transference plans π_{12} between μ_1, μ_2, and π_{23} between μ_2, μ_3. Denote X_i the support of μ_i. Let π be as in the gluing lemma and π_{13} the marginal of π on $X_1 \times X_3$. Then $\pi_{13} \in \Pi(\mu_1, \mu_3)$. We have, using the triangle inequality, the Minkowski inequality and the definition of coupling,

$$\begin{aligned}
W_p(\mu_1, \mu_3) &\leq \left(\int_{X_1 \times X_3} d(x_1, x_3)^p d\pi_{13}(x_1, x_3) \right)^{1/p} \\
&= \left(\int_{X_1 \times X_2 \times X_3} d(x_1, x_3)^p d\pi(x_1, x_2, x_3) \right)^{1/p} \\
&\leq \left(\int_{X_1 \times X_2 \times X_3} [d(x_1, x_2) + d(x_2, x_3)]^p \, d\pi(x_1, x_2, x_3) \right)^{1/p} \\
&\leq \left(\int_{X_1 \times X_2 \times X_3} d(x_1, x_2)^p d\pi(x_1, x_2, x_3) \right)^{1/p} \\
&\quad + \left(\int_{X_1 \times X_2 \times X_3} d(x_2, x_3)^p d\pi(x_1, x_2, x_3) \right)^{1/p} \\
&= \left(\int_{X_1 \times X_2} d(x_1, x_2)^p d\pi(x_1, x_2) \right)^{1/p} \\
&\quad + \left(\int_{X_2 \times X_3} d(x_2, x_3)^p d\pi(x_2, x_3) \right)^{1/p} \\
&= W_p(\mu_1, \mu_2) + W_p(\mu_2, \mu_3).
\end{aligned}$$

■

Since $W_p(\delta_x, \delta_y) = d(x, y)$, the map $x \to \delta_x$ gives an isometric imbedding of $X \to \mathcal{P}_p(X)$. Note also, by the Hölder inequality, $W_p \le W_q$ when $1 \le p \le q$, so W_1 is the weakest of all.

Proposition 8.1.11 (W_p metrizes $\mathcal{P}_p$) *Let (X, d) be a Polish space. Then the Wasserstein distance W_p metrizes the weak convergence in $\mathcal{P}_p(X)$. Namely $W_P(\mu_k, \mu) \to 0$ iff μ_k converges weakly to μ.*

For a proof, see [132, Theorem 6.8]. A quick corollary of this is

Corollary 8.1.12 (Metrizability of the weak topology) *Let (X, d) be a Polish space. If $\tilde{d} = \frac{d}{1+d}$ (or any bounded distance inducing the same topology as d), then W_p of $\tilde{d}$ metrizes the weak topology of $P(X)$.*

Proposition 8.1.13 (Topological properties of $\mathcal{P}_p(X)$) *Let (X, d) be a Polish space. Then*

a) The set of all normalized configurations $\frac{1}{k}\sum_{i=1}^{k} \delta_{x_i}$ with $k \in \mathbb{N}$ and $x_1, \cdots, x_k \in X$ is dense in $(\mathcal{P}_p(X), W_p)$.

b) $\mathcal{P}_p(X)$ is a Polish space.

c) $(\mathcal{P}_p(X), W_p)$ is a compact space or a length space if and only if (X, d) is so.

d) $(\mathcal{P}_2(X), W_2)$ has nonnegative Alexandrov curvature if and only if (X, d) does.

Proof: a) Let $\mathcal{D}$ be a dense sequence in X, $\mathcal{P}$ be the space of probability measures that can be written as $\sum a_i \delta_{x_i}$, where the $a_i \ge 0$ are rational coefficients, and the x_i are finitely many elements in $\mathcal{D}$. By choosing k to be the common denominators of a_i and repeating some x_i, $\sum a_i \delta_{x_i} = \frac{1}{k}\sum_{i=1}^{k} \delta_{x_i}$. From the proof of Theorem 6.16 in [**?**], $\mathcal{P}$ is dense in $\mathcal{P}_p(X)$. Therefore $\frac{1}{k}\sum_{i=1}^{k} \delta_{x_i}$ is dense in $\mathcal{P}_p(X)$.

b) See the proof of Theorem 6.16 in [132].

c) Since X isometrically imbeds in $\mathcal{P}_p(X)$, one direction ($\Rightarrow$) is clear. Also $\mathrm{diam}(\mathcal{P}_p(X)) = \mathrm{diam}(X)$.

$\Leftarrow$: When X is compact, by Prokhorov's theorem (Theorem 8.1.5) $\mathcal{P}_p(X)$ is compact in weak topology, by Proposition 8.1.11, it also compact in W_p.

Assume that (X, d) is a length space and let $\epsilon > 0$ and $\mu, \nu \in \mathcal{P}_p(X)$ be given. By Proposition 8.1.2 it is enough to construct an ϵ-midpoint η of μ, ν. From b) and Example 8.1.4 there are $\bar{\mu} = \frac{1}{k}\sum_{i=1}^{k} \delta_{x_i}, \bar{\nu} = \frac{1}{k}\sum_{i=1}^{k} \delta_{y_i}$ such that $W_p(\mu, \bar{\mu}) \le \epsilon/3, W_p(\nu, \bar{\nu}) \le \epsilon/3$ and

$$W_p(\bar{\mu}, \bar{\nu}) = \left(\frac{1}{k}\sum_{i=1}^{k} d(x_i, y_i)^p\right)^{1/p}.$$

For each $i = 1, \cdots, k$ let z_i be an $\epsilon/3$ midpoints of x_i and y_i and put $\eta = \frac{1}{k}\sum_{i=1}^k \delta_{z_i}$. Then

$$\begin{aligned} W_p(\bar{\mu}, \eta) &\leq \left(\frac{1}{k}\sum_{i=1}^k d(x_i, z_i)^p\right)^{1/p} \leq \left(\frac{1}{k}\sum_{i=1}^k \left[\frac{1}{2}d(x_i, y_i) + \frac{\epsilon}{3}\right]^p\right)^{1/p} \\ &\leq \frac{1}{2}\left(\frac{1}{k}\sum_{i=1}^k d(x_i, y_i)^p\right)^{1/p} + \frac{\epsilon}{3} = \frac{1}{2}W_p(\bar{\mu}, \bar{\nu}) + \frac{\epsilon}{3}. \end{aligned}$$

Hence $W_p(\mu, \eta) \leq \frac{1}{2}W_p(\mu, \nu) + \epsilon$. Similarly $W_p(\nu, \eta) \leq \frac{1}{2}W_p(\mu, \nu) + \epsilon$. We have that η is an ϵ-midpoint of μ, ν.

d) See [129, Prop. 2.10 (iv)] for a proof . ■

Minimizing geodesics in $\mathcal{P}_p(X)$ are also related to minimal geodesics in X. Given a length space (X, d), let $\Gamma(X)$ be the set of minimal geodesics equipped with the topology of uniform convergence:

$$\Gamma(X) = \{\gamma \,|\, \gamma : [0, 1] \to X \text{ with } d(\gamma(0), \gamma(1)) = L(\gamma)\}.$$

For each $t \in [0, 1]$, $e_t : \Gamma(X) \to X$ with $e_t(\gamma) = \gamma(t)$ is the evaluation map. $E : \Gamma(X) \to X \times X$ with $E(\gamma) = (\gamma(0), \gamma(1))$ is the "endpoints" map.

A dynamical transference plan consists of a transference plan π and a Borel measure Π on $\Gamma(X)$ such that $E_*\Pi = \pi$. It is optimal if π is optimal. If Π is an optimal dynamical transference plan, then for $t \in [0, 1]$, put $\mu_t = (e_t)_*\Pi$. The family $\{\mu_t\}_{t\in[0,1]}$ is called a displacement interpolation.

Proposition 8.1.14 (Geodesics of $\mathcal{P}_p(X)$) *Let (X, d) be a complete, separable, locally compact length space. Assume $p > 1$. Given any two $\mu_0, \mu_1 \in \mathcal{P}_p(X)$, and a continuous curve $\{\mu_t\}_{0\leq t\leq 1}$ in $\mathcal{P}_p(X)$, the followings are equivalent:*
a) $\{\mu_t\}_{0\leq t\leq 1}$ is a minimizing geodesic in $\mathcal{P}_p(X)$;
b) $\{\mu_t\}_{0\leq t\leq 1}$ is a displacement interpolation.
Moreover, $\mathcal{P}_p(X)$ is a geodesic space.

Proof: b) $\Rightarrow$ a): Let Π be a dynamical optimal transference plan such that $(e_t)_*\Pi = \mu_t$. Given $0 \leq t \leq t' \leq 1$, $(e_t, e_{t'})_*\Pi$ is a particular coupling of $(\mu_t, \mu_{t'})$, so

$$\begin{aligned} W_p(\mu_t, \mu_{t'}) &\leq \left(\int_{X\times X} d(x_0, x_1)^p \, d((e_t, e_{t'})_*\Pi)(x_0, x_1)\right)^{1/p} \\ &= \left(\int_{\Gamma(X)} d(\gamma(t), \gamma(t'))^p \, d\Pi(\gamma)\right)^{1/p} \\ &= (t' - t)\left(\int_\Gamma L(\gamma)^p \, d\Pi(\gamma)\right)^{1/p} = (t' - t)\, W_p(\mu_0, \mu_1). \end{aligned}$$

Let $c : [0, 1] \to P(X)$ with $c(t) = \mu_t$. Then $L(c) \leq W_p(\mu_0, \mu_1)$, so $L(c) = W_p(\mu_0, \mu_1)$ and $\{\mu_t\}_{0\leq t\leq 1}$ is a minimal geodesic.

Now we will show the existence of minimal geodesics (and hence $\mathcal{P}_p(X)$ is a geodesic space).

By Proposition 8.1.3 X is a geodesic space, so the endpoints map $E : \Gamma(X) \to X \times X$ is Borel and surjective. Given $(x_0, x_1) \in X \times X$, $E^{-1}(x_0, x_1)$ is compact. It follows that there is a Borel map $S : X \times X \to \Gamma(X)$ so that $E \circ S = \mathrm{Id}_{X\times X}$ []. Given $\mu_0, \mu_1 \in \mathcal{P}_p(X)$, let π be an optimal transference plan between μ_0 and μ_1 and put $\Pi = S_*(\pi)$. The corresponding displacement interpolation joins μ_0 and μ_1. By above this is a minimal geodesic connecting μ_0 and μ_1.

a) $\Rightarrow$ b): We can assume that for all $t, t' \in [0, 1]$,

$$W_p(\mu_t, \mu_{t'}) = |t - t'| W_p(\mu_0, \mu_1). \tag{8.1.7}$$

Let $\pi^{(0)}_{(0,1)}$ be an optimal coupling of (μ_0, μ_1), $\pi^{(1)}_{(0,\frac{1}{2})}$ an optimal coupling of $(\mu_0, \mu_{\frac{1}{2}})$, and $\pi^{(1)}_{(\frac{1}{2},1)}$ an optimal coupling of $(\mu_{\frac{1}{2}}, \mu_1)$. By the gluing lemma (Lemma 8.1.10), we have a probability measure $\pi^{(1)}_{(0,\frac{1}{2},1)}$ on $X \times X \times X$ with marginals $\pi^{(1)}_{(0,\frac{1}{2})}$ on the product of the first two Xs and $\pi^{(1)}_{(\frac{1}{2},1)}$ on the product of the last two Xs. Let $\pi^{(1)}_{(0,1)}$ be the marginal of $\pi^{(1)}_{(0,\frac{1}{2},1)}$ on the product of the first and last Xs. Since $\pi^{(1)}_{(0,\frac{1}{2})}$ and $\pi^{(1)}_{(\frac{1}{2},1)}$ are optimal couplings, we have

$$\begin{aligned}
&\left(\int_{X\times X} d(x_0, x_1)^p \, d\pi^{(1)}_{(0,1)}\right)^{1/p} \\
&\quad \le \left(\int_{X\times X\times X} \left[d(x_0, x_{\frac{1}{2}}) + d(x_{\frac{1}{2}}, x_1)\right]^p d\pi^{(1)}_{(0,\frac{1}{2},1)}\right)^{1/p} \\
&\quad \le \left(\int_{X\times X} d(x_0, x_{\frac{1}{2}})^p \, d\pi^{(1)}_{(0,\frac{1}{2})}\right)^{1/p} + \left(\int_{X\times X} d(x_{\frac{1}{2}}, x_1)^p \, d\pi^{(1)}_{(\frac{1}{2},1)}\right)^{1/p} \\
&\quad = W_p(\mu_1, \mu_{\frac{1}{2}}) + W_p(\mu_{\frac{1}{2}}, \mu_1) = W_p(\mu_0, \mu_1),
\end{aligned}$$

using (8.1.7). Thus $\pi^{(1)}_{(0,1)}$ is an optimal coupling of (μ_0, μ_1) and equality holds everywhere in the above. It follows that the measure $\pi^{(1)}_{(0,\frac{1}{2},1)}$ is supported on $B^{(1)}$ where

$$B^{(1)} = \left\{(x_0, x_{\frac{1}{2}}, x_1) \in X \times X \times X : d(x_0, x_{\frac{1}{2}}) = d(x_{\frac{1}{2}}, x_1) = \frac{1}{2} d(x_0, x_1)\right\}.$$

For $t \in \{0, \frac{1}{2}, 1\}$, define $e_t : B^{(1)} \to X$ by $e_t(x_0, x_{\frac{1}{2}}, x_1) = x_t$. Then $(e_t)_* \pi^{(1)}_{(0,\frac{1}{2},1)} = \mu_t$.

Proceeding in the same manner, for each $k \ge 1$, we obtain a probability measure $\Pi^{(k)} = \pi^{(k)}_{(0,\frac{1}{2^k},\frac{2}{2^k},\frac{3}{2^k},\cdots,1)}$ on X^{2^k+1} such that its marginals $\pi^{(k)}_{(i2^{-k},j2^{-k})}$ are optimal couplings for all $0 \le i, j \le 2^k$. $\Pi^{(k)}$ is supported on

$$B^{(k)} = \Big\{(x_0, x_{\frac{1}{2^k}}, x_{\frac{2}{2^k}}, \cdots, x_1) \in X^{2^k+1} :$$

$$\left. d(x_0, x_{\frac{1}{2^k}}) = d(x_{\frac{1}{2^k}}, x_{\frac{2}{2^k}}) = \cdots = d(x_{\frac{2^k-1}{2^k}}, x_1) = 2^{-k} d(x_0, x_1) \right\}.$$

For $t = \frac{i}{2^k}, 0 \le i \le 2^k$, define $e_t : B^{(k)} \to X$ by $e_t(x_0, \cdots, x_1) = x_t$, then $(e_t)_* \Pi^{(k)} = \mu_t$.

Given $(x_0, \cdots, x_1) \in B^{(k)}$, one extends it to a continuous curve $\gamma^{(k)} : [0,1] \to X$ by using the map S above (in the proof of the existence of geodesics). Namely for $t \in (\frac{i}{2^k}, \frac{i+1}{2^k})$, define $\gamma^{(k)}(t) = e_t(S(x_{\frac{i}{2^k}}, x_{\frac{i+1}{2^k}}))$. $L(\gamma^{(k)}) = d(x_0, x_1)$ so it is a minimal geodesic. Now extend $\Pi^{(k)}$ for $t \in (\frac{i}{2^k}, \frac{i+1}{2^k})$ by $\Pi^{(k)} = S_*(\pi^{(k)}_{(\frac{i}{2^k}, \frac{i+1}{2^k})})$. So we have a family of probability measure $\Pi^{(k)} \in P(\Gamma(X))$ which satisfy $(e_t)_* \Pi^{(k)} = \mu_t$ for all $t = i/2^k$. To be able to pass to limit, we shall check the tightness of the sequence. For any $\epsilon > 0$, since μ_0, μ_1 are tight, there are compact sets K_0, K_1 such that $\mu_0(X \setminus K_0) \le \epsilon, \mu_1(X \setminus K_1) \le \epsilon$. The set $\Gamma^{0,1}_{K_0 \to K_1}$ of minimal geodesics joining K_0 to K_1 is compact. Now

$$\Pi^{(k)}(\Gamma \setminus \Gamma^{0,1}_{K_0 \to K_1}) \le \mu_0(X \setminus K_0) + \mu_1(X \setminus K_1) \le 2\epsilon.$$

Hence the family $\{\Pi^{(k)}\}$ is tight. There is a subsequence converges weakly to some probability measure Π. $\Gamma(X)$ is closed, so Π is still supported in Γ. Moreover for all dyadic time $t = i/2^l$ in $[0,1]$, we have, if k is larger than l, $(e_t)_* \Pi^{(k)} = \mu_t$, and by passing to the limit, we have $(e_t)_* \Pi = \mu_t$ also. $(e_t)_* \Pi$ is weak-$*$ continuous in t. It follows that $(e_t)_* \Pi = \mu_t$ for all $t \in [0,1]$. ■

8.2 N-Ricci Lower Bound for Measured Length Spaces

In this section we discuss various notions of Ricci curvature lower bound for measured length spaces.

8.2.1 Via Localized Bishop-Gromov

In this subsection a metric measure space is a triple (X, d, μ), where (X, d) is a complete separable metric space and μ is a Borel measure on X which is locally finite, positive and has full support, i.e. $0 < \mu(B(x,r)) < \infty$ for all $x \in X$ and $r > 0$. If μ do not have full support, one can just work on supp$[\mu]$.

From Proposition 1.6.1, one can define Ricci curvature lower bound for metric measure spaces using (1.4.8). However, there is serious difficulty defining the set B_t in (1.4.8), especially for general metric spaces where one has to deal with the issue of branching. Following [101] we will define B_t using optimal transportation. Denote (see (1.2.5) for the definition of sn_H)

$$\zeta^{(t)}_{H,N}(r) = \left(\frac{\mathrm{sn}_H(tr)}{\mathrm{sn}_H(r)} \right)^{N-1}.$$

Definition 8.2.1 *Given $H \in \mathbb{R}, N \in [1, \infty)$, we say a metric measure space (X, d, μ) satisfies the (H, N)-Localized Bishop-Gromov property, $LBG(H, N)$, if for μ-a.e. $x \in X$, every measurable set $A \subset X$ ($A \subset B(x, \pi/\sqrt{H})$ if $H > 0$), there exists a displacement interpolation $\{\mu_t\}_{0 \le t \le 1} \subset \mathcal{P}_2(X)$ such that $\mu_0 = \delta_x$, $\mu_1 = \mu(A)^{-1} \cdot \mu|_A$, and*

$$d\mu \ge (e_t)_* \left(t\, \zeta^{(t)}_{H,N}(d(x, \gamma(1)))\, \mu(A)\, d\Pi(\gamma) \right) \tag{8.2.1}$$

holds as measures on X, where Π is an optimal dynamical transference plan associated to $\{\mu_t\}_{0 \le t \le 1}$.

This notion is from [101], where it is called the measure contraction property; for a very similar version see [**?**], also [] for other versions.

To see the relation with (1.4.8), assume that there exists a measurable map $\Phi : A \to \Gamma$ satisfying $e_0 \circ \Phi \equiv x$, $e_1 \circ \Phi = id_A$, and $\Pi = \Phi_* \mu_1$. Then the inequality (8.2.1) can be rewritten as

$$d\mu \ge (e_t \circ \Phi)_* \left(t\, \zeta^{(t)}_{H,N}(d(x, y))\, \chi_A(y)\, d\mu(y) \right), \tag{8.2.2}$$

where χ_A is the characteristic function of A.

The geometric meaning of the map Φ is that for each point of A it assigns, in a measurable way, a minimal geodesic from x to the given point. This is the case, when for almost all $y \in A$, there exists exactly one geodesic connecting x to y, e.g. complete Riemannian manifolds or more generally, non-branching metric spaces.

Now set $\Psi_t = e_t \circ \Phi$ and for any measurable subset $B \subset A$, $B_t = \Psi_t(B)$. Then integrating (8.2.2) yields

$$\mu(B_t) \ge \int_B t\, \zeta^{(t)}_{H,N}(d(x, y))\, d\mu(y), \tag{8.2.3}$$

which is exactly (1.4.8)

Thus, (8.2.1) is slightly stronger than (1.4.8). In certain sense, (8.2.1) is an infinitesimal version of Bishop-Gromov rather than just localized, but the difference is minimal and we will not differentiate. Instead, we note that a complete Riemannian manifold M^n with $\mathrm{Ric}_M \ge (n-1)H$ satisfies $LBG(H, n)$. Another class of examples is Alexandrov spaces with lower curvature bound. Namely if (X, d) is an n-dimensional complete locally compact length space with curvature $\ge K$ in the sense of Alexandrov, and $\mathcal{H}^n$ is the n-dimensional Hausdorff measure on X, then $(X, d, \mathcal{H}^n)$ satisfies $LBG(K, n)$ [**?**, 101].

As a function of H, r, $\zeta^{(t)}_{H,N}(r)$ depends only on Hr^2, so if (X, d, μ) satisfies $LBG(H, N)$, then the scaled metric space $(X, \alpha d, \beta \mu)$ with $\alpha, \beta > 0$ satisfies $LBG(H/\alpha^2, N)$, as in the case of usual curvature bounds. Since $\zeta^{(t)}_{H,N}(r)$ is increasing in H when $H \le 0$ or $H > 0$ and $r \le \frac{\pi}{2\sqrt{H}}$, for $0 \le t \le 1$, $0 \le \frac{\mathrm{sn}_H(tr)}{\mathrm{sn}_H(r)} \le 1$, (X, d, μ) satisfies $LBG(H, N)$ implies (X, d, μ) satisfies $LBG(H', N')$ for all $H' \le H, N' \ge N$.

For metric measure spaces satisfying $LBG(H,N)$ the Bishop-Gromov relative volume comparison theorem also holds. Let $A(x,r,R) = B(x,R) \setminus B(x,r)$ be an annulus and $A(x,r) = \limsup_{\delta\to 0} \frac{1}{\delta}\,\mu(A(x,r,r+\delta))$.

Theorem 8.2.2 (Bishop-Gromov Volume Comparison) *Let (X,d,μ) be a metric measure space satisfying $LBG(H,N)$ for real numbers $H \in \mathbb{R}$, $N > 1$. Then for any $x \in X$, the functions*

$$\frac{A(x,r)}{\mathrm{sn}_H^{N-1}(r)} \text{ and } \frac{\mu(B(x,r))}{\displaystyle\int_0^r \mathrm{sn}_H^{N-1}(t)dt} \text{ are non-increasing in } r, \tag{8.2.4}$$

where $r < \frac{\pi}{\sqrt{H}}$ if $H > 0$. In particular, if $H = 0$ then

$$\frac{A(x,r)}{A(x,R)} \geq \left(\frac{r}{R}\right)^{N-1}, \qquad \frac{\mu(B(x,r))}{\mu(B(x,R))} \geq \left(\frac{r}{R}\right)^{N}.$$

The latter also holds if $N = 1$ and $H \leq 0$.

Proof: Given $0 < r < R, \delta > 0$, apply $LBG(H,N)$ for $x = x, A = A(x,R,(1+\delta)R), t = r/R$ and integrate both sides of (8.2.1) on $A(x,r,(1+\delta)r)$ yields

$$\mu(A(x,r,(1+\delta)r)) \geq \frac{r}{R}\,\mu(A(x,R,(1+\delta)R)) \inf_{1\leq\lambda\leq 1+\delta} \zeta_{H,N}^{(r/R)}(\lambda R)\cdot$$
$$((e_{r/R})_*\Pi)(A(x,r,(1+\delta)r)).$$

Now

$$\begin{aligned}((e_{r/R})_*\Pi)(A(x,r,(1+\delta)r)) &= \Pi((e_{r/R})^{-1}[A(x,r,(1+\delta)r)]) \\ &\geq \Pi((e_1)^{-1}[A(x,R,(1+\delta)R)]) = 1\end{aligned}$$

for we have $(e_1)^{-1}[A(x,R,(1+\delta)R)] \cap \mathrm{supp}\Pi \subset (e_{r/R})^{-1}[A(x,r,(1+\delta)r)]$ by the condition that $(e_0)_*\Pi = \delta_x$. Hence

$$\mu(A(x,r,(1+\delta)r)) \geq \mu(A(x,R,(1+\delta)R))\frac{r}{R}\inf_{1\leq\lambda\leq 1+\delta} \zeta_{H,N}^{(r/R)}(\lambda R). \tag{8.2.5}$$

Similarly we have

$$\mu(A(x,(1-\delta)r),r) \geq \mu(A(x,(1-\delta)R,R))\frac{r}{R}\inf_{1-\delta\leq\lambda\leq 1} \zeta_{H,N}^{(r/R)}(\lambda R). \tag{8.2.6}$$

By construction $\mu(B(x,r))$ is nondecreasing, so it has at most countably many discontinuities. Hence given any $r > 0$, there is $0 < r_0 < r$ such that $\mu(B(x,r))$ is continuous at r_0. But $\mu(B(x,(1+\delta)r)) - \mu(B(x,r)) = \mu(A(x,r,(1+\delta)r))$, so from (8.2.5) and (8.2.6), the fact that $\mu(B(x,r))$ is continues at $r_0 > 0$ implies that it is continuous for all $r > r_0$ ($r < \frac{\pi}{\sqrt{H}}$ if $H > 0$). Therefore $\mu(B(x,r))$ is continuous on $\mathbb{R}_+$. In particular we have $\mu(\partial B(x,r)) = 0$ for all $r > 0$.

Inequality (8.2.5) can be rewritten as

$$\frac{1}{\delta r}\mu(A(x,r,(1+\delta)r)) \geq \frac{1}{\delta R}\,\mu(A(x,R,(1+\delta)R)) \inf_{1\leq\lambda\leq 1+\delta} \zeta_{H,N}^{(r/R)}(\lambda R).$$

Similarly for (8.2.6). Since $\mu(B(x,r))$ is monotonic, it is differentiable almost everywhere. Therefore the above inequality shows that $\mu(B(x,r))$ is in fact differentiable everywhere on $\mathbb{R}_+$. Thus $A(x,r)$ is well-defined and finite and equals the derivative of $\mu(B(x,r))$. Letting $\delta \to 0$ gives

$$A(x,r) \geq A(x,R)\left(\frac{\mathrm{sn}_H(r)}{\mathrm{sn}_H(R)}\right)^{N-1},$$

which is the first part of (8.2.4).

By Lemma 1.4.10 the ratio of the integrals $\frac{\int_0^r A(x,t)dt}{\int_0^r \mathrm{sn}_H^{N-1}(t)dt}$ is also non-increasing. Since $\mu(B(x,r)) = \int_0^r A(x,t)dt$, this completes the proof. ■

Corollary 8.2.3 (Doubling) *If (X,d,μ) satisfies $LBG(H,N)$ for real numbers $H \in \mathbb{R}$, $N \geq 1$, the doubling property holds on every bounded subset $X' \subset X$. In particular, every bounded closed subset $X' \subset X$ is compact.*

Proof: If $H \geq 0$ or $N = 1$, the doubling constant is $\leq 2^N$. If $H < 0$ and $N > 1$, by (8.2.4),

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq \frac{2\int_0^r \sinh(2\sqrt{-H}t)^{N-1}dt}{\int_0^r \sinh(\sqrt{-H}t)^{N-1}dt} \leq 2^N \cosh(\sqrt{-H}r)^{N-1}.$$

The doubling condition implies that every bounded closed ball in X is totally bounded, therefore compact. ■

Corollary 8.2.4 (Hausdorff dimension) *If (X,d,μ) satisfies $LBG(H,N)$ for real numbers $H \in \mathbb{R}$, $N \geq 1$, then X has Hausdorff dimension $\leq N$.*

Proof: By (8.2.4), the function $f(x) = \limsup_{r\to 0} r^N \mu(B(x,r))^{-1}$ on X is locally bounded. Thus the N-dimensional Hausdorff measure $\mathcal{H}^N$ on X is also locally bounded and the Hausdorff dimension of X is $\leq N$. ■

Corollary 8.2.5 (Bonnet-Myers theorem) *If a complete metric measure space (X,d,μ) satisfies $LBG(H,N)$ for real numbers $H > 0$ and $N > 1$, then X is compact and has diameter $\leq \pi/\sqrt{H}$.*

Proof: By (8.2.4) and Lemma 1.4.10, we have for any $x \in X, s,t \in [0, \frac{\pi}{2\sqrt{H}}]$ with $s < t$,

$$\frac{\mu(A(x,s,t))}{\int_s^t (\mathrm{sn}_H(r))^{N-1}dr} \geq \frac{\mu(A(x,\frac{\pi}{\sqrt{H}}-t,\frac{\pi}{\sqrt{H}}-s))}{\int_{\frac{\pi}{\sqrt{H}}-t}^{\frac{\pi}{\sqrt{H}}-s}(\mathrm{sn}_H(r))^{N-1}dr}.$$

I.e.,

$$\mu(A(x,s,t)) \geq \mu(A(x,\frac{\pi}{\sqrt{H}}-t,\frac{\pi}{\sqrt{H}}-s)). \tag{8.2.7}$$

If there are $x_0, x_1 \in X$ with $d(x_0, x_1) \geq \frac{\pi}{\sqrt{H}} + \epsilon$ for some $\epsilon > 0$ (WLOG we choose $\epsilon < \frac{\pi}{4\sqrt{H}}$) , connect x_0, x_1 with a minimal geodesic $\gamma : [0, \frac{\pi}{\sqrt{H}} + \epsilon] \to X$ with $\gamma(0) = x_0, \gamma(\frac{\pi}{\sqrt{H}} + \epsilon) = x_1$. With $\delta \in (0, \epsilon)$, apply $LBG(H, N)$ for $x = \gamma(\epsilon + 2\delta), A = B(x_1, \delta)$ and $t = \frac{\pi/\sqrt{H} - \epsilon - 2\delta}{\pi/\sqrt{H} - \delta}$ and integrate both sides of (8.2.1) on $A_t = e_t(\operatorname{supp}\Pi)$ to obtain

$$\begin{aligned}
\mu(A_t) &\geq t\,\mu(B(x_1, \delta)) \inf_{y \in A} \left(\frac{\operatorname{sn}_H(td(x,y))}{\operatorname{sn}_H(d(x,y))} \right)^{N-1} ((e_t)_*\Pi)(A_t) \\
&\geq \left(1 - \frac{\epsilon + \delta}{\pi/\sqrt{H} - \delta}\right) \mu(B(x_1, \delta)) \left(\frac{\operatorname{sn}_H(\frac{\pi}{\sqrt{H}} - \epsilon - 2\delta)}{\operatorname{sn}_H(\frac{\pi}{\sqrt{H}} - 3\delta)} \right)^{N-1} \\
&= \left(1 - \frac{\epsilon + \delta}{\pi/\sqrt{H} - \delta}\right) \mu(B(x_1, \delta)) \left(\frac{\operatorname{sn}_H(\epsilon + 2\delta)}{\operatorname{sn}_H(3\delta)} \right)^{N-1}.
\end{aligned}$$

On one hand,

$$A_t \subset B(x, t(d(x, x_1) + \delta)) \subset B(x_0, \pi/\sqrt{H}).$$

On the other hand,

$$A_t \subset X \setminus B(x_0, d(x_0, x) + t(d(x, x_1) - \delta)) \subset X \setminus B(x_0, \pi/\sqrt{H} - 2\delta).$$

Thus we have, by (8.2.7),

$$\mu(A_t) \leq \mu(A(x_0, \pi/\sqrt{H} - 2\delta, \pi/\sqrt{H}) \leq \mu(B(x_0, 2\delta)) \leq 2^N \mu(B(x_0, \delta)).$$

Therefore we obtain, since $N > 1$,

$$\frac{\mu(B(x_0, \delta))}{\mu(B(x_1, \delta))} \geq 2^{-N} \left(1 - \frac{\epsilon + \delta}{\pi/\sqrt{H} - \delta}\right) \left(\frac{\operatorname{sn}_H(\epsilon + 2\delta)}{\operatorname{sn}_H(3\delta)} \right)^{N-1} \to \infty$$

as $\delta \to 0$. This is a contradiction since we can reverse the roles of x_0, x_1. ■

While the definition of Ricci lower bound using the localized Bishop-Gromov comparison is very geometric and enjoys very nice geometric consequence, the following example shows it not ideal.

Example 8.2.6 *If M is a compact Riemannian manifold with $\operatorname{Ric}_M \geq 0, \dim_M \leq N - 1 (N > 1)$ and $\operatorname{diam}_M \leq L$, then M satisfies $LBG(H, N)$ for some $H > 0$.*

Proof: For r small, $\frac{\operatorname{sn}_H(tr)}{\operatorname{sn}_H(r)}$ has the Taylor expansion $t + \frac{t}{6}(1 - t^2)|H|r^2 + \cdots$. Hence for $t \in [0, 1]$ and $r \in [0, L]$, there exists $0 < c_N \leq 1$ such that $\zeta_{H,N}^{(t)}(r) \leq t^{N-1}\left(1 + (N-1)(1-t^2)\frac{Hr^2}{2}\right)$ for all $0 \leq Hr^2 \leq c_N$. Choose $c_N \leq \frac{1}{N-1}$, then we also have

$$t^{N-2} \geq t^{N-1}\left(1 + (N-1)(1-t^2)\frac{Hr^2}{2}\right)$$

Since M satisfies $LBG(0, N-1)$, we have M satisfies $LBG(H, N)$ for $0 < H \leq c_N/L^2$. ■

Hence an n-torus will satisfy $LBG(H, N)$ for some $H > 0$ and $N = n+1$, which cannot happen in the classical sense. Also a small convex subset of $\mathbb{R}^{N-1}$ satisfies $LBG(H, N)$ with $H > 0$ and thus the property $LBG(H, N)$ is not local.

8.2.2 Entropy And Ricci Curvature

Given a metric measure space (X, d, μ), let $\mathcal{P}_2(X, \mu)$ be the subspace of all $\nu \in \mathcal{P}_2(X)$ which are absolutely continuous with respect to μ, i.e.

$$\mathcal{P}_2(X, \mu) = \{\nu \in \mathcal{P}_2(X) \,|\, \nu = \rho\mu\},$$

where $\rho : X \to [0, \infty)$ is a Borel measurable function satisfying $\int_X \rho(x) d\mu(x) = 1$ and $\int_X d^2(x_0, x)\rho(x) d\mu(x) < \infty$ for some $x_0 \in X$.

The relative (Shannon) entropy of $\nu \in \mathcal{P}_2(X)$ with respect to μ is defined by

$$H_\mu(\nu) = \begin{cases} \lim_{\epsilon \to 0} \int_{\rho > \epsilon} \rho \log \rho \, d\mu & \text{if } \nu \in \mathcal{P}_2(X, \mu) \\ +\infty & \text{otherwsie} \end{cases}. \tag{8.2.8}$$

Given a real number $N \geq 1$, the Rényi entropy functional $H_{N,\mu} : \mathcal{P}_2(X) \to [0, \infty]$ with respect to μ is

$$H_{N,\mu}(\nu) = -\int_X \rho^{-1/N} d\nu, \tag{8.2.9}$$

where ρ denotes the density of absolutely continuous part ν^c in the Lebesgue decomposition $\nu = v^c + v^s = \rho\mu + v^s$ of $\nu \in \mathcal{P}_2(X)$.

Lemma 8.2.7 *H_μ and $H_{N,\mu}(N > 1)$ are lower semicontinuous on $\mathcal{P}_2(X)$ and satisfy*

$$\lim_{N \to \infty} N[1 + H_{N,\mu}(\nu)] = H_\mu(\nu).$$

Furthermore, $-\mu(X)^{1/N} \leq H_{N,\mu} \leq 0$ and $-\log(\mu(X)) \leq H_\mu$.

Proof: The lower semicontinuity follows essentially from definition. To see the relationship between the two entropy functions, note that

$$N[1 + H_{N,\mu}(\nu)] = \int_X U_N(\rho) d\nu,$$

where $U_N(r) = Nr(1 - r^{-1/N})$. Since $\lim_{N \to \infty} U_N(r) = r \log r$ (by L'Hopstal), the desired equation follows from the Lebesgue monotone convergence theorem (strictly speaking, one applies it to the region $\{0 < \rho < 1\}$ and $\{1 \geq \rho\}$).

Clearly $H_{N,\mu} \leq 0$. On the other hand

$$\begin{aligned} H_{N,\mu}(\nu) &= -\int_{\operatorname{supp} \nu} (\rho^{-1})^{1/N} d\nu \geq -\left(\int_{\operatorname{supp} \nu} \rho^{-1} d\nu \right)^{1/N} \\ &= -\mu(\operatorname{supp} \nu)^{1/N} \geq -\mu(X)^{1/N}, \end{aligned} \tag{8.2.10}$$

where we have used Jensen's inequality.

Similarly,

$$\begin{aligned} H_\mu(\nu) &= -\int_{\operatorname{supp}\nu} \log(\rho^{-1})\,d\nu \ge -\log\left(\int_{\operatorname{supp}\nu} \rho^{-1} d\nu\right) \\ &= -\log(\mu(\operatorname{supp}\nu)) \ge -\log(\mu(X)). \end{aligned} \tag{8.2.11}$$

■

Some remark is in order. For any subset $A \subset X$, the uniform distribution measure (with respect to μ) on A is

$$\nu_A = \frac{\chi_A}{\mu(A)}\mu,$$

where χ_A is the characteristic function of A. Note that $H_{N,\mu}(\nu_A) = -\mu(A)^{1/N}$ (and $H_\mu(\nu_A) = -\log(\mu(A))$). Hence (8.2.10) ((8.2.11) resp.) expresses the intuitive idea that the uniform distribution measure on $\operatorname{supp}\nu$ achieves the minimal entropy (among probability measures with support contained in $\operatorname{supp}\nu$).

Originally arising from thermodynamics and statistical mechanics, the notions of entropy have played fundamental roles in information theory. The relevance to the Ricci curvature lower bound comes from the following observation which relates Ricci curvature lower bound to certain convexity of the entropies.

Recall that $\mathcal{P}_2(X,\mu)$ is a length (geodesic) space. Given a length space (X,d) and a real number $k \in \mathbb{R}$, a function $F : X \to \mathbb{R}$ is called k-convex if for each geodesic $\gamma : [0,1] \to X$,

$$F(\gamma(t)) \le tF(\gamma(1)) + (1-t)F(\gamma(0)) - \frac{k}{2}t(1-t)d^2(\gamma(0),\gamma(1)), \tag{8.2.12}$$

for all $t \in [0,1]$. In case that $(X,d) = (M,g)$ is a Riemannian manifold and $F \in C^2(M)$, (8.2.12) is equivalent to $\operatorname{Hess} F \ge kg$. The k-convex functions on $\mathcal{P}_2(X,\mu)$ are also called displacement k-convex.

Theorem 8.2.8 *For a Riemannian manifold (M,g), the followings are equivalent:*
1). The Ricci curvature of (M,g) is bounded from below by k, $\operatorname{Ric}_M \ge k$;
2). The Shannon entropy $H_\mu(\nu)$ (or the Rényi entropy $H_{N,\mu}(\nu)$) is displacement k-convex on $\mathcal{P}_2(M)$ (for $N \ge n$, resp.).

Proof: 1) ⇒ 2): Given $\operatorname{Ric}_M \ge k$, we need to derive the inequality

$$H(\mu_t) \le tH(\mu_1) + (1-t)H(\mu_0) - \frac{k}{2}t(1-t)(W_2(\mu_0,\mu_1))^2, \tag{8.2.13}$$

for each geodesic μ_t in $(P(M), W_2)$. By Theorem 8.1.7, $\mu_t = (F_t)_*(\mu_0)$, where $F_t(x) = \exp_x(-t\nabla\varphi(x))$ for a convex function φ and F_t gives rise to Monge transports. Write $\mu_t = \rho_t d\mathrm{vol}$. Then the Monge-Amperé equation (8.1.3) yields

$$\rho_t(F_t(x))J_t(x) = \rho_0(x), \quad a.e.$$

where $J_t = \det(dF_t(x))$ is the determinant of Jacobian.

Thus,

$$\begin{aligned} H(\mu_t) &= \int_M \rho_t(x) \log \rho_t(x) d\mathrm{vol}_x \\ &= \int_M \rho_t(F_t(y)) \log \rho_t(F_t(y)) J_t(y) d\mathrm{vol}_y \\ &= \int_M \rho_0(y) \log[\rho_0(y) J_t^{-1}(y) d\mathrm{vol}_y \\ &= H(\mu_0) - \int_M \log J_t(y) d\mu_0(y), \end{aligned}$$

using the Monge-Amperé equation. Consequently,

$$\begin{aligned} & -H(\mu_t) + tH(\mu_1) + (1-t)H(\mu_0) \\ &= -H(\mu_0) + \int_M \log J_t d\mu_0 + t[H(\mu_0) - \int_M \log J_1 d\mu_0] + (1-t)H(\mu_0) \\ &= \int_M \log J_t d\mu_0 - t \int_M \log J_1 d\mu_0. \end{aligned} \tag{8.2.14}$$

On the other hand, $y_t = \log J_t$ satisfies the differential inequality

$$y_t'' + \frac{1}{n}(y_t')^2 + \mathrm{Ric}(F_t', F_t') \leq 0.$$

Hence,

$$y_t'' + k|F_t'|^2 \leq 0$$

using $\mathrm{Ric}_M \geq k$. Note that, since $F_t(x) = \exp_x(-t\nabla\varphi(x))$ is a geodesic (for fixed x), for $t \in [0,1]$,

$$|F_t'(x)| = d(x, F_1(x)) \tag{8.2.15}$$

is independent of t. Thus,

$$y_t(x) \geq t y_1(x) + \frac{t(1-t)}{2} k |F_t'|^2.$$

Plus this back into (8.2.14), we obtain

$$-H(\mu_t)+tH(\mu_1)+(1-t)H(\mu_0) \geq \frac{t(1-t)}{2} k \int_M |F_t'|^2 d\mu_0 = \frac{t(1-t)}{2} k (W_2(\mu_0, \mu_1))^2.$$

using (8.1.6) and (8.2.15).

2) $\Rightarrow$ 1):

■

8.2.3 The Case of Smooth Metric Measure Spaces

Recall that a smooth metric measure space is a metric measure space (M^n, g, μ) where M is an n dimensional smooth manifold, g is the Riemannian metric which gives rise to the distance function, and the measure $\mu = e^{-f} d\mathrm{vol}_g$, where f is a smooth real valued function on M and $d\mathrm{vol}_g$ is the Riemannian measure. There is well-defined notion of Ricci curvature for smooth metric measure spaces, via the N-Bakry-Emery Ricci curvature Ric_f^N. Besides interests of its own in diffusion process, Ricci flow etc., the class of smooth metric measure spaces gives us a good collection of examples which interpolates between the smooth manifolds and the general metric measure spaces.

For $t \in [0, 1]$, $N > n$, and $K \in \mathbb{R}$, set $H = \frac{K}{N-1}$ and define

$$\beta_{K,N}^{(t)}(r) = \left(\frac{\mathrm{sn}_H(rt)}{t\,\mathrm{sn}_H(r)}\right)^{N-1} = \frac{\zeta_{H,N}^{(t)}(r)}{t^{N-1}}. \tag{8.2.16}$$

Theorem 8.2.9 *For a smooth metric measure space (M^n, g, μ), $\mu = e^{-f} d\mathrm{vol}_g$, the followings are equivalent:*
1). The Bakry-Emery Ricci curvature of (M, g, μ) is bounded from below by K, $\mathrm{Ric}_f \geq K$;
2). The Shannon entropy $H_\mu(\nu)$ is displacement K-convex on $\mathcal{P}_2(M)$.
Furthermore, the followings are equivalent:
1)$_N$. The N-Bakry-Emery Ricci curvature of (M, g, μ) is bounded from below by K, $\mathrm{Ric}_f^N \geq K$;
2)$_N$. The Rényi entropy $H_{N,\mu}(\nu)$ satisfies the following (convexity) inequality: for any $\mu_0 = \rho_0\mu$, $\mu_1 = \rho_1\mu \in P^{ac}(M)$, if $F: M \to M$ is the optimal Monge transport and μ_t is the geodesic from μ_0 to μ_1 in $P^{ac}(M)$, then

$$\begin{aligned} H_{N,\mu}(\mu_t) \quad \leq \quad & -(1-t)\int_M \beta_{K,N}^{(1-t)}(d(x, F(x)))^{\frac{1}{N}} \rho_0(x)^{-\frac{1}{N}} d\mu_0(x) \\ & -t \int_M \beta_{K,N}^{(t)}(d(x, F(x)))^{\frac{1}{N}} \rho_1(F(x))^{-\frac{1}{N}} d\mu_0(x) \end{aligned} \tag{8.2.17}$$

for all $t \in [0, 1]$.

Remark For $K = 0$, $\beta_{K,N}^{(t)} \equiv 1$ and the entropy inequality simplifies to the usual convexity inequality

$$H_{N,\mu}(\mu_t) \leq (1-t)H_{N,\mu}(\mu_0) + tH_{N,\mu}(\mu_1).$$

Remark The entropy inequality can be reformulated using the optimal transport $\pi = (\mathrm{Id} \times F)_*(\mu_0)$ as follows.

$$\begin{aligned} H_{N,\mu}(\mu_t) \quad \leq \quad & -(1-t)\int_{M\times M} \beta_{K,N}^{(1-t)}(d(x, y))^{\frac{1}{N}} \rho_0(x)^{-\frac{1}{N}} d\pi(x, y) \\ & -t \int_{M\times M} \beta_{K,N}^{(t)}(d(x, y))^{\frac{1}{N}} \rho_1(y)^{-\frac{1}{N}} d\pi(x, y) \end{aligned} \tag{8.2.18}$$

for all $t \in [0,1]$.

The reason that (8.2.17) and (8.2.18) are convexity type inequalities can be found in the following lemma (using (8.2.16)), which also plays a basic role in our proof of Theorem 8.2.9.

Lemma 8.2.10 *A C^2 function $\Phi(x)$ on $[0,1]$ satisfies the differential inequality*

$$\Phi'' \leq -k\Phi \tag{8.2.19}$$

for $k < \pi^2$ iff for all $x_0, x_1 \in [0,1]$ and $t \in [0,1]$,

$$\Phi((1-t)x_0 + tx_1) \geq \delta_k^{(1-t)}(|x_1 - x_0|)\Phi(x_0) + \delta_k^{(t)}(|x_1 - x_0|)\Phi(x_1). \tag{8.2.20}$$

Here

$$\delta_k^{(t)}(r) = (\zeta_{k,N}^{(t)}(r))^{\frac{1}{N-1}} = \frac{\mathrm{sn}_k(rt)}{\mathrm{sn}_k(r)}. \tag{8.2.21}$$

If $k = \pi^2$ and Φ is nonnegative, then (8.2.19) implies (and hence is equivalent to) $\Phi(x) = c\sin \pi x,\ c \geq 0$, while for $k > \pi^2$ and Φ is nonnegative, (8.2.19) implies (and hence is equivalent to) $\Phi(x) = 0$.

Proof: $\Longrightarrow$: Without loss of generality, we assume that $x_0 = 0, x_1 = 1$. Let

$$\Phi_0(t) = \delta_k^{(1-t)}(1)\Phi(0) + \delta_k^{(t)}(1)\Phi(1) = \frac{\mathrm{sn}_k(1-t)}{\mathrm{sn}_k(1)}\Phi(0) + \frac{\mathrm{sn}_k(t)}{\mathrm{sn}_k(1)}\Phi(1).$$

Then

$$\Phi_0'' = -k\Phi_0, \qquad \Phi_0(0) = \Phi(0), \quad \Phi_0(1) = \Phi(1).$$

Hence by using the maximal principle, one obtain $\Phi(t) \geq \Phi_0(t)$. (For $k < 0$, the maximal principle applied to $\Psi = \Phi - \Phi_0$ yields right away that $\Psi \geq 0$. For $k = 0$, the statement is the usual convexity. For $k > 0$, we refer to [**?**].)

$\Longleftarrow$: For any $x \in (0,1)$, apply (8.2.20) to $x_0 = x - \epsilon,\ x_1 = x + \epsilon$ and $t = \frac{1}{2}$ for $\epsilon > 0$ sufficiently small. From the Taylor expansions,

$$\begin{aligned}
\Phi(x_0) &= \Phi(x) - \Phi'(x)\epsilon + \frac{1}{2}\Phi''(x)\epsilon^2 + O(\epsilon^3), \\
\Phi(x_1) &= \Phi(x) + \Phi'(x)\epsilon + \frac{1}{2}\Phi''(x)\epsilon^2 + O(\epsilon^3), \\
\delta_k^{(\frac{1}{2})}(2\epsilon) &= \frac{1}{2}(1 + \frac{k}{2}\epsilon^2) + O(\epsilon^3),
\end{aligned}$$

we derive (8.2.19). ■

We now give the proof of Theorem 8.2.9.

Proof: We show $1)_N \Longleftrightarrow 2)_N$; the other case being similar (and also to the proof of Theorem 8.2.8).

$1)_N \Longrightarrow 2)_N$: For $\mu_0 = \rho_0\mu,\ \mu_1 = \rho_1\mu \in P^{ac}(M)$, by the Brennier-McCann Theorem, there is a convex function $\varphi:\ M \to \mathbb{R}$ such that

$$F_t(x) = \exp_x t\nabla\varphi(x)$$

provides the unique minimal geodesic $\mu_t = (F_t)_*(\mu_0) = \rho_t \mu$ from μ_0 to μ_1.

Taking into account of the weight e^{-f} in our measure μ, we introduce the Jacobian determinant

$$J_t^f(x) = e^{f(x)-f(F_t(x))} \det DF_t(x) = e^{f(x)-f(F_t(x))}\mathcal{J}. \tag{8.2.22}$$

Indeed, $J_t^f(x)$ is simply the ratio of change of the infinitesimal measure of μ under F_t, and the Monge-Amperé equation (8.1.3) becomes

$$\rho_0(x) = \rho_t(F_t(x))J_t^f(x).$$

Claim:

$$(J_t^f(x))^{\frac{1}{N}} \geq (1-t)\beta_{K,N}^{(1-t)}(d(x,F(x)))^{\frac{1}{N}} + t\beta_{K,N}^{(t)}(d(x,F(x)))^{\frac{1}{N}}(J_1^f(x))^{\frac{1}{N}} \tag{8.2.23}$$

for all $t \in [0,1]$ (note that $(J_0^f(x))^{\frac{1}{N}} \equiv 1$).

Granted, we have

$$\begin{aligned} H_{N,\mu}(\mu_t) &= -\int_M \rho_t^{1-\frac{1}{N}} d\mu = -\int_M \rho_t(F_t)^{1-\frac{1}{N}} J_t^f(x) d\mu \\ &= -\int_M \rho_0 \rho_t(F_t)^{-\frac{1}{N}} d\mu = -\int_M \left(\frac{J_t^f}{\rho_0}\right)^{\frac{1}{N}} d\mu_0 \end{aligned}$$

using the change of variable formula and the Monge-Ampère equation. The desired inequality (8.2.17) then follows by plugging in the claim.

To show the claim, set

$$\Phi_\mu(t) = (J_t^f(x))^{\frac{1}{N}} = e^{\frac{f(x)-f(F_t(x))}{N}}\mathcal{J}^{\frac{1}{N}}. \tag{8.2.24}$$

Using that $t \to F_t(x)$ is a geodesic, we compute

$$\begin{aligned} \Phi_\mu'' &= \frac{\Phi_\mu}{N}\left[-\mathrm{Hess} f(\dot{F}_t, \dot{F}_t) + \frac{df(\dot{F}_t)^2}{N} - \frac{2df(\dot{F}_t)}{N}\frac{\dot{\mathcal{J}}}{\mathcal{J}} \right. \\ &\quad \left. + \frac{1-N}{N}\left(\frac{\dot{\mathcal{J}}}{\mathcal{J}}\right)^2 + \frac{\ddot{\mathcal{J}}}{\mathcal{J}}\right] \\ &= \frac{\Phi_\mu}{N}\left[-\mathrm{Hess} f(\dot{F}_t, \dot{F}_t) + \frac{df(\dot{F}_t)^2}{N} + \frac{\ddot{\mathcal{J}}}{\mathcal{J}} - (1-\frac{1}{n})\left(\frac{\dot{\mathcal{J}}}{\mathcal{J}}\right)^2 \right. \\ &\quad \left. - \frac{N-n}{nN}\left(\frac{\dot{\mathcal{J}}}{\mathcal{J}} + \frac{n}{N-n}df(\dot{F}_t)\right)^2\right] \\ &\leq -\frac{\mathrm{Ric}_f^N(\dot{F}_t)}{N}\Phi_\mu \leq -\frac{K|\dot{F}_t|^2}{N}\Phi_\mu. \end{aligned}$$

Here we have made use of (1.5.5).

Since $t \to F_t(x)$ is a geodesic, $|\dot{F}_t(x)| \equiv d(x, F(x)$ is independent of t. Hence

$$\Phi_\mu'' \le -\frac{Kd(x,F(x))^2}{N}\Phi_\mu.$$

At this point one is tempted to appeal to Lemma 8.2.10 to derive the Claim. However, one runs into a discrepancy of power $\frac{N}{N-1}$. This is remedied by separating out the direction of motion as detailed in §1.5. Indeed, using the notations there, we set

$$\Phi_{\mu,11}(t) = \mathcal{J}_{11}(t), \qquad \Phi_{\mu,\perp}(t) = \left(e^{f(x)-f(F_t(x))}\mathcal{J}_\perp\right)^{\frac{1}{N-1}}.$$

Then $\Phi_\mu = \Phi_{\mu,11}^{\frac{1}{N}}\Phi_{\mu,\perp}^{\frac{N-1}{N}}$. By (1.5.8) $\Phi_{\mu,11}(t)$ is concave:

$$\Phi_{\mu,11}(t) \ge (1-t) + t\Phi_{\mu,11}(1).$$

On the other hand, the same computation as above, but using (1.5.11) instead, yields

$$\Phi_{\mu,\perp}'' \le -\frac{Kd(x,F(x))^2}{N-1}\Phi_{\mu,\perp}.$$

Hence Lemma 8.2.10 gives, for $k = \frac{Kd(x,F(x))^2}{N-1}$,

$$\Phi_{\mu,\perp}(t) \ge \delta_k^{(1-t)}(1) + \delta_k^{(t)}(1)\Phi_{\mu,\perp}(1).$$

Combining these using the Hölder inequality, we arrive at

$$\begin{aligned}
\Phi_\mu(t) &\ge ((1-t) + t\Phi_{\mu,11}(1))^{\frac{1}{N}}\left(\delta_k^{(1-t)}(1) + \delta_k^{(t)}(1)\Phi_{\mu,\perp}(1)\right)^{\frac{N-1}{N}} \\
&\ge (1-t)^{\frac{1}{N}}(\delta_k^{(1-t)}(1))^{\frac{N-1}{N}} + t^{\frac{1}{N}}(\delta_k^{(t)}(1))^{\frac{N-1}{N}}\Phi_\mu(1),
\end{aligned}$$

which is precisely the Claim. ■

8.2.4 Via Entropy Convexity

The discussion on the Ricci curvature lower bound in the case of smooth metric measure spaces leads directly to the following generalization of the Ricci curvature lower bound for general metric measure spaces, introduced first by Lott-Villani [] and Sturm [] independently. The entropy convexity condition characterized a metric measure space satisfying a Ricci curvature lower bound and a dimensional upper bound, hence named the curvature dimension condition.

Definition: Let (X, d, μ) be a metric measure space satisfying

$$0 < \mu(B(x,r)) < \infty, \qquad \forall x \in X,\ r \in (0,\infty).$$

For $K \in \mathbb{R}$, $N \in (1, \infty)$, we say that (X, d, μ) satisfies the curvature dimension condition $\mathrm{CD}(K, N)$ if for any $\mu_0 = \rho_0 \mu$, $\mu_1 = \rho_1 \mu \in P^{ac}(X)$, there exists a minimal geodesic μ_t from μ_0 to μ_1 such that

$$\begin{aligned} H_{N,\mu}(\mu_t) \quad \leq \quad & -(1-t) \int_{X\times X} \beta^{(1-t)}_{K,N}(d(x,y))^{\frac{1}{N}} \rho_0(x)^{-\frac{1}{N}} d\pi(x,y) \\ & -t \int_{X\times X} \beta^{(t)}_{K,N}(d(x,y))^{\frac{1}{N}} \rho_1(y)^{-\frac{1}{N}} d\pi(x,y) \qquad (8.2.25) \end{aligned}$$

for all $t \in [0, 1]$. Here π is the optimal transport from μ_0 to μ_1. If $N = \infty$, we say that (X, d, μ) satisfies the curvature dimension condition $\mathrm{CD}(K, \infty)$ if $H_\mu(\nu)$ is displacement K-convex.

As mentioned, the curvature dimension condition $\mathrm{CD}(K, N)$ for a metric measure space expresses the idea that some kind of Ricci curvature of the space is bounded from below by K and some kind of dimension of the space is bounded from above by N.

An immediate consequence of the definition is the scaling property: if (X, d, μ) satisfies $\mathrm{CD}(K, N)$, then $(X, c\, d, c'\mu)$ (c, c' positive constants) satisfies $\mathrm{CD}(K/c^2, N)$. Also, note that $\beta^{(t)}_{K,N}$ is increasing in K and decreasing in N. Hence, if (X, d, μ) satisfies $\mathrm{CD}(K, N)$, then (X, d, μ) also satisfies $\mathrm{CD}(K', N')$ for any $K' \leq K, N' \geq N$. Finally we remark that in the definition above, we would have needed to impose the condition

$$\mu(X \setminus B(x, \pi\sqrt{\frac{N-1}{K}})) = 0, \qquad \forall x \in X$$

if $K > 0$ and $N < \infty$ in order to stay inside the domain of $\beta^{(t)}_{K,N}(r)$. However, as we will see from the generalized Bonnet-Myers Theorem **??**, this is always the case.

Remark In the definition of Lott-Villani [], the condition $\mathrm{CD}(K, N)$ is defined in terms of entropy functional associated to certain class of convex functions (the so called displacement convexity class) for which the entropy used here is a special example. Moreover, they allow probability measures that are singular with respect to μ. For a large class of metric measure spaces, that is, nonbranching and proper, these two definitions turn out to be equivalent.

An important consequence of the condition $\mathrm{CD}(K, N)$ is obtained by specializing to delta measures and uniform distribution measures. This is the so called generalized Brunn-Minkowski inequality, which expresses the effect of Ricci curvature lower bound on the shape of the space in terms of the motions of the geodesics. Before stating the result, let us introduce the generalized Minkowski sum.

From now on, we assume that (X, d) is a complete, locally compact, separable length (or geodesic) space. For subsets $A, B \subset X$ and $t \in (0, 1)$, define

$$Z_t(A, B) = \{\gamma(t) \mid \gamma : [0, 1] \to X \text{ is a minimal geodesic}, \ \gamma(0) \in A, \gamma(1) \in B\}. \quad (8.2.26)$$

In other words, $Z_t(A,B)$ is the set of time t locations of all minimal geodesics starting from A and ending at B. For example, $Z_t(\{x\}, B(x,r)) = B(x,tr)$. We also introduce the notation

$$\beta_{A,B}(t,K,N) = \inf_{x\in A, y\in B} \beta^{(t)}_{K,N}(d(x,y))^{\frac{1}{N}}. \tag{8.2.27}$$

Theorem 8.2.11 (Generalized Brunn-Minkowski) *If a metric measure space (X,d,μ) satisfies CD(K,N), and A,B are measurable subsets of X, then,*
1). for $N\in(1,\infty)$, we have, for any $t\in(0,1)$,

$$\mu(Z_t(A,B))^{\frac{1}{N}} \geq (1-t)\beta_{A,B}(1-t,K,N)\mu(A)^{\frac{1}{N}} + t\beta_{A,B}(t,K,N)\mu(B)^{\frac{1}{N}};$$

2). for $N=\infty$, and $0<\mu(A),\mu(B)<\infty$, we have

$$\log\mu(Z_t(A,B)) \geq (1-t)\log\mu(A) + t\log\mu(B) + \frac{K}{2}(1-t)tW_2\left(\frac{\chi_A}{\mu(A)}\mu, \frac{\chi_B}{\mu(B)}\mu\right)^2$$

for any $t\in(0,1)$.

Remark In general the Minkowski sum of two measurable sets may not be measurable. In that case, we take the upper measure (?) of the set.
Proof: We show the case when $N\in(0,1)$, the other being similar. First of all, one can reduce to the case when both A,B are bounded; otherwise, one applies the bounded case to $A\cap B(x,R)$, $B\cap B(x,R)$ and take the limit $R\to\infty$. Now if both $\mu(A),\mu(B)=0$, the statement is trivial. Assume that $\mu(A)=0,\mu(B)>0$. Fix a point $x\in A$ and consider

$$\mu_0 = \delta_x, \qquad \mu_1 = \frac{\chi_B}{\mu(B)}\mu.$$

Then the minimal geodesic μ_t from μ_0 to μ_1 has supp$\mu_t\subset Z_t(A,B)$ (Cf. Proposition 8.1.14). Now CD(K,N) becomes

$$H_{N,\mu}(\mu_t) \leq -t\int_{X\times B} \beta^{(t)}_{K,N}(d(x,y))^{\frac{1}{N}}\mu(B)^{\frac{1}{N}}d\pi(x,y)$$

for the optimal transport π from μ_0 to μ_1. Since supp$\pi\subset A\times B$, we obtain

$$H_{N,\mu}(\mu_t) \leq -t\beta_{A,B}(t,K,N)\mu(B)^{\frac{1}{N}}.$$

On the other hand, Lemma 8.2.7 gives us

$$H_{N,\mu}(\mu_t) \geq -\mu(\mathrm{supp}\mu_t) \geq -\mu(Z_t(A,B))^{\frac{1}{N}}.$$

Combining the two gives us the generalized Brunn-Minkowski inequality in the case when $\mu(A)=0$.

Now assume that both A,B are bounded and of positive measure. Set

$$\mu_0 = \frac{\chi_A}{\mu(A)}\mu, \qquad \mu_1 = \frac{\chi_B}{\mu(B)}\mu$$

in CD(K, N) yields

$$H_{N,\mu}(\mu_t) \leq -(1-t)\beta_{A,B}(1-t, K, N)\mu(A)^{\frac{1}{N}} - t\beta_{A,B}(t, K, N)\mu(B)^{\frac{1}{N}}.$$

Again, using Lemma 8.2.7 and supp$\mu_t \subset Z_t(A, B)$, we arrive at the desired inequality. ■

8.3 Stability of N-Ricci Lower Bound under Convergence

A prime feature of the curvature dimension condition is its stability under the measured Gromov-Hausdorff convergence. The usefulness of this comes partly from the following precompactness.

Theorem 8.3.1 *Let $(M_i, g_i, vol_{g_i}, x_i)$ be a sequence of (pointed) complete Riemannian manifolds with $\mathrm{Ric}_{g_i} \geq K$, $\dim M_i \leq N$ for some $K \in \mathbb{R}$, $N \in \mathbb{N}$. Then a subsequence converges to a pointed proper metric measure space (X, d, μ, x) in the sense of measured (pointed) Gromov-Hausdorff convergence.*

The stability of the curvature dimension condition is proved in [].

Theorem 8.3.2 *Let $(M_i, g_i, vol_{g_i}, x_i)$ be a sequence of pointed proper metric measure spaces satisfying CD(K, N) for some $K \in \mathbb{R}$, $N \in (1, \infty]$ and converges to a pointed proper metric measure space (X, d, μ, x) in the sense of measured (pointed) Gromov-Hausdorff convergence. If $0 < \mu(B(x, r)) < \infty$, $\forall x \in X$, $r \in (0, \infty)$, then (X, d, μ, x) satisfies CD(K, N).*

8.4 Geometric and Analytical Consequences

As we have seen, an immediate consequence of the curvature dimension condition is the generalized Brunn-Minkowski inequality. In this section we develop geometric and analytic consequence of the curvature dimension condition, mainly from the generalized Brunn-Minkowski inequality.

First, the generalized Bishop-Gromov volume comparison theorem.

Theorem 8.4.1 (Generalized Bishop-Gromov) *If a metric measure space (X, d, μ) satisfies CD(K, N), then*

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq \frac{\int_0^R \mathrm{sn}_k(t)^{N-1} dt}{\int_0^r \mathrm{sn}_k(t)^{N-1} dt}$$

with $k = \frac{K}{N-1}$, for all $x \in X$ and $0 < r < R$ (and $R \leq \pi/\sqrt{k}$ if $K > 0$).

Proof: Let $A(x;r_1,r_2) = B(x,r_2) \setminus B(x,r_1)$ denote the annulus with radius $r_2 > r_1$. Then for any $t \in [0,1]$, $Z_t(\{x\}, A(x;r_1,r_2)) \subset A(x;tr_1,tr_2)$. Hence by the generalized Brunn-Minkowski inequality,

$$\mu(A(x;tr_1,tr_2)) \geq t^N \beta_t\, \mu(A(x;r_1,r_2)),$$

where

$$\beta_t = \inf_{y\in A(x;r_1,r_2)} \beta^{(t)}_{K,N}(d(x,y)) = \inf_{d\in[r_1,r_2]} \left(\frac{\mathrm{sn}_k(td)}{t\,\mathrm{sn}_k(d)}\right)^{N-1}.$$

Hence

$$\mu(A(x;tr_1,tr_2)) \geq t\frac{\inf_{d\in[r_1,r_2]} h(td)}{\sup_{d\in[r_1,r_2]} h(d)}\mu(A(x;r_1,r_2)), \tag{8.4.1}$$

where we have denoted $h(t) = \mathrm{sn}_k(t)^{N-1}$.

Choose an integer L sufficiently large so that $t_L = (r/R)^{1/L} < 1$. Then by (8.4.1),

$$\begin{aligned}
\mu(B(x,r)\setminus\{x\}) &= \sum_{l=1}^{\infty} \mu(A(x;t_L^l r, t_L^{l-1} r)) \\
&\geq \sum_{l=1}^{\infty} t_L^{l-1} \frac{\inf_{d\in[t_L r,r]} h(t_L^{l-1}d)}{\sup_{d\in[t_L r,r]} h(d)}\mu(A(x;t_L r,r)).
\end{aligned}\tag{8.4.2}$$

On the other hand,

$$\begin{aligned}
\mu(A(x;r,R)) &= \sum_{l=1}^{L} \mu(A(x;t_L^{l-L} r, t_L^{l-L-1} r)) \\
&\leq \sum_{l=1}^{L} t_L^{l-L-1} \frac{\sup_{d\in[t_L^{l-L}r,t_L^{l-L-1}r]} h(d)}{\inf_{d\in[t_L^{l-L}r,t_L^{l-L-1}r]} h(t_L^{L+1-l}d)}\mu(A(x;t_L r,r)) \\
&= \frac{R}{r}\frac{\sum_{l=1}^{L} t_L^{l-1} \sup_{d\in[t_L R,R]} h(t_L^{l-1}d)}{\inf_{d\in[t_L r,r]} h(d)}\mu(A(x;t_L r,r))
\end{aligned}\tag{8.4.3}$$

again by (8.4.1).

Combining (8.4.2) with (8.4.3) gives us

$$\mu(B(x,r)\setminus\{x\}) \geq \frac{r}{R}\frac{\inf_{d\in[t_L r,r]} h(d)}{\sup_{d\in[t_L r,r]} h(d)} \frac{\sum_{l=1}^{\infty} t_L^{l-1} \inf_{d\in[t_L r,r]} h(t_L^{l-1}d)}{\sum_{l=1}^{L} t_L^{l-1} \sup_{d\in[t_L R,R]} h(t_L^{l-1}d)}\mu(A(x;r,R)).$$

Thus,

$$\begin{aligned}
&\mu(B(x,r)\setminus\{x\}) \sum_{l=1}^{L} (t_L^{l-1} - t_L^l) R \sup_{d\in[t_L^l R, t_L^{l-1}R]} h(d) \\
&= (1-t_L) R \mu(B(x,r)\setminus\{x\}) \sum_{l=1}^{L} t_L^{l-1} \sup_{d\in[t_L R,R]} h(t_L^{l-1}d)
\end{aligned}$$

$$\geq (1-t_L)r\frac{\inf_{d\in[t_Lr,r]}h(d)}{\sup_{d\in[t_Lr,r]}h(d)}\mu(A(x;r,R))\sum_{l=1}^{\infty}t_L^{l-1}\inf_{d\in[t_Lr,r]}h(t_L^{l-1}d)$$
$$=\mu(A(x;r,R))\frac{\inf_{d\in[t_Lr,r]}h(d)}{\sup_{d\in[t_Lr,r]}h(d)}\sum_{l=1}^{\infty}(t_L^{l-1}-t_L^l)r\inf_{d\in[t_L^lr,t_L^{l-1}r]}h(d).$$

Taking $L\longrightarrow\infty$, we arrive at

$$\mu(B(x,r)\setminus\{x\})\int_r^R \mathrm{sn}_k(t)^{N-1}dt\geq \mu(A(x;r,R))\int_0^r \mathrm{sn}_k(t)^{N-1}dt.$$

Or equivalently,

$$\mu(A(x;r,R))\leq\frac{\int_r^R \mathrm{sn}_k(t)^{N-1}dt}{\int_0^r \mathrm{sn}_k(t)^{N-1}dt}\mu(B(x,r)\setminus\{x\}).$$

Letting $R\longrightarrow r^+$, we deduce that $\mu(\partial B(x,r))=0$. In particular, $\mu(\{y\})=0$ for all $y\neq x$.

This implies that if $\mu(\{x\})\neq 0$, then $X=\{x\}$ is a single point. Otherwise, $\mu(\{x\})=0$, and $\mu(B(x,r))=\mu(B(x,r)\setminus\{x\})$. Plug this back into the above inequality we have an inequality that is equivalent to the generalized Bishop-Gromov volume comparison. ■

An immediate consequence here is that the volume doubling constant

$$\sup_{x\in X,r\leq R}\frac{\mu(B(x,2r))}{\mu(B(x,r))} \tag{8.4.4}$$

is bounded for each $R\in(0,\infty)$. Thus X must be proper.

Corollary 8.4.2 *If a metric measure space (X,d,μ) satisfies CD(K,N), then X is proper.*

The second major consequence of the curvature dimension condition is the generalized Bonnet-Myers Theorem.

Theorem 8.4.3 (Generalzied Bonnet-Myers) *If a metric measure space (X,d,μ) satisfies CD(K,N) with $K>0$ and $1<N<\infty$, then*

$$\mathrm{diam}(X)\leq\pi\sqrt{\frac{N-1}{K}}.$$

Furthermore, each $x\in X$ has at most one point of distance $\pi\sqrt{\frac{N-1}{K}}$ from x.

Proof: By rescaling we can assume that $K=N-1$. For the first part of the statement, suppose that there exist $x,y\in X$ such that $d(x,y)>\pi$. Set $\delta=d(x,y)-\pi>0$ and take a minimal geodesic $\gamma:\ [0,\pi+\delta]\longrightarrow X$ from x to y.Without loss of generality, we can assume that $\delta<\pi/2$.

For any $\epsilon \in (0,\delta)$, let $t = (\pi-\delta-\epsilon)/\pi$. Then $0 < t < 1$. Now the generalized Brunn-Minkowski inequality yields

$$\begin{aligned}
\frac{\mu(Z_t(\{\gamma(\delta+\epsilon)\}, B(y,\epsilon))}{\mu(B(y,\epsilon))} &\geq t^N \inf_{r\in(\pi-2\epsilon,\pi)} \left(\frac{\sin(t\,r)}{t\,\sin r}\right)^{N-1} \\
&= t\left(\frac{\sin(t(\pi-2\epsilon))}{\sin(\pi-2\epsilon)}\right)^{N-1} \\
&\geq \frac{\pi-\delta-\epsilon}{\pi}\left(\frac{\sin(\delta+\epsilon)}{\sin(2\epsilon)}\right)^{N-1},
\end{aligned}$$

where we have used that $t(\pi-2\epsilon) < t\pi = \pi-\delta-\epsilon$. This implies that

$$\lim_{\epsilon\longrightarrow 0} \frac{\mu(Z_t(\{\gamma(\delta+\epsilon)\}, B(y,\epsilon))}{\mu(B(y,\epsilon))} = \infty.$$

On the other hand, for any $x' \in Z_t(\{\gamma(\delta+\epsilon)\}, B(y,\epsilon))$, $x' = \eta(t)$ for a minimal geodesic $\eta :\ [0,1] \longrightarrow X$ such that $\eta(0) = \gamma(\delta+\epsilon)$, $\eta(1) = z \in B(y,\epsilon)$. Then

$$d(\gamma(\delta+\epsilon),\eta(t)) = t\,d(\gamma(\delta+\epsilon),z) < t[d(\gamma(\delta+\epsilon),y)+\epsilon] = \pi-\delta-\epsilon.$$

$$\begin{aligned}
d(x,\eta(t)) &\geq d(x,z) - d(z,\eta(t)) > \pi+\delta-\epsilon-(1-t)d(\gamma(\delta+\epsilon),z) \\
&> \pi+\delta-\epsilon-(1-t)\pi = \pi-2\epsilon.
\end{aligned}$$

This means that

$$Z_t(\{\gamma(\delta+\epsilon)\}, B(y,\epsilon)) \subset B(\gamma(\delta+\epsilon),\pi-\delta-\epsilon)\setminus B(x,\pi-2\epsilon) \subset A(x:\pi-2\epsilon,\pi).$$

Now the generalized Bishop-Gromov volume comparison applied repeatedly gives us that

$$\begin{aligned}
\mu(Z_t(\{\gamma(\delta+\epsilon)\}, B(y,\epsilon)) &\leq \mu(A(x:\pi-2\epsilon,\pi)) \\
&\leq \frac{\int_{\pi-2\epsilon}^{\pi} \sin^{N-1} r dr}{\int_0^{\pi-2\epsilon} \sin^{N-1} r dr}\mu(B(x,\pi-2\epsilon)) \\
&= \frac{\int_0^{2\epsilon} \sin^{N-1} r dr}{\int_0^{\pi-2\epsilon} \sin^{N-1} r dr}\mu(B(x,\pi-2\epsilon)) \\
&\leq \mu(B(x,2\epsilon)) \\
&\leq 2^N \mu(B(x,\epsilon)).
\end{aligned}$$

Here, in the last step we have applied the generalized Bishop-Gromov with $K = 0$. It follows then that

$$\lim_{\epsilon\longrightarrow 0} \frac{\mu(B(x,\epsilon))}{\mu(B(y,\epsilon))} = \infty,$$

which is clearly a contradiction as we can switch x and y.

For the second part of the theorem, assume otherwise and let $y \neq z \in X$ such that $d(x,y) = d(x,z) = \pi$. Then, for any $r \in (0, \pi/2)$,

$$\begin{aligned}
\mu(B(x,r)) &\geq \frac{\int_0^r \sin^{N-1} r dr}{\int_0^{\pi-r} \sin^{N-1} r dr} \mu(B(x, \pi - r)) \\
&= \frac{\int_{\pi-r}^{\pi} \sin^{N-1} r dr}{\int_0^{\pi-r} \sin^{N-1} r dr} \mu(B(x, \pi - r)) \\
&\geq \mu(A(x; \pi - r, \pi)).
\end{aligned}$$

On the other hand, $B(y,r) \subset X \setminus B(x, \pi - r)$ and $X = B(x,\pi) \cup \partial B(x,\pi)$ (by the first part). Hence $\mu(B(y,r)) \leq \mu(A(x; \pi - r, \pi))$ since $\mu(\partial B(x,\pi)) = 0$. Therefore, we deduce that

$$\mu(B(x,r)) \geq \mu(B(y,r)).$$

Exchanging x and y (and applying the same argument to x, z), we obtain that, for any $0 < r < \pi/2$,

$$\mu(B(x,r)) = \mu(B(y,r)) = \mu(B(z,r)) = \mu(A(x; \pi - r, \pi)).$$

Now take $\epsilon < \frac{1}{2} d(y,z)$ $(\leq \pi/2)$. Then

$$\begin{aligned}
2\mu(B(x,\epsilon)) &= \mu(B(y,\epsilon)) + \mu(B(z,\epsilon)) = \mu(B(y,\epsilon) \cup B(z,\epsilon)) \\
&\leq \mu(A(x; \pi - \epsilon, \pi)) = \mu(B(x,\epsilon))
\end{aligned}$$

which is a contradiction. ∎

There is also a weak version of the Cheng's Maximal Diameter Theorem. Recall that a metric space (X,d) is non-branching if for any $z, x_0, x_1, x_2 \in X$ with $d(x_0,x_1) = d(x_0,x_2) = 2d(z,x_1) = 2d(z,x_2)$ we have $x_1 = x_2$.

The following result is shown in [].

Theorem 8.4.4 *Assume that the metric measure space (X,d,μ) is non-branching and satisfies $CD(N-1,N)$ with $1 < N < \infty$. If further $\mathrm{diam}(X) = \pi$, then X is homeomorphic to a (spherical) suspension of a metric space.*

Remark While the spherical suspension cannot be improved in the presence of singularity, it is an open question whether the homeomorphism can be improved to an isometry.

Proof: Fix $x_N, x_S \in X$ with $d(x_N, x_S) = \pi$. Then the previous proof gives

$$\mu(B(x_N, r)) + \mu(B(x_S, \pi - r)) = \mu(X), \tag{8.4.5}$$

for all $r \in (0,\pi)$.

Claim: For any $z \in X$, there is a unique minimal geodesic from x_N to x_S passing through z.

Granted, we define

$$Y = \{\ \gamma \mid \gamma \text{ unit speed minimal geodesic from } x_N \text{ to } x_S\ \},$$

and

$$d_Y(\gamma_1, \gamma_2) = \sup_{0 \le t \le \pi} d_X(\gamma_1(t), \gamma_2(t)).$$

Then one readily verifies that

$$\begin{aligned} \Psi : \quad \Sigma Y \quad &\longrightarrow X \\ (\gamma, t) \quad &\longrightarrow \gamma(t) \end{aligned}$$

is a homeomorphism.

Thus it remains to prove the claim. First of all, note that the non-branching condition implies that if $z \in X$ with $d(x_N, z) + d(x_S, z) = \pi$, then there is a unique minimal geodesic from x_N to x_S passing through z. Now for any $z \in X$ with $r = d(x_N, z) > 0$, let $\gamma(t)$ be a minimal geodesic from z to x_S. Let z' be the last point on γ lying inside $\overline{B(x_N, r)}$. Then $d(x_N, z') = r$.

On the other hand, we must have $d(x_S, z') = \pi - r$. If not, one would find a point on γ which lies outside both $\overline{B(x_N, r)}$ and $\overline{B(x_S, \pi - r)}$. Since a small ball always carries positive measure, this would contradict to (8.4.5).

Therefore $\gamma(t)$ passes through x_N if $z' \neq z$ by the first observation. But this contradicts to the condition that $\mathrm{diam}(X) = \pi$. ■

8.5 Cheeger-Colding

From comparison theorems, various quantities like the volume, the diameter, the first Betti number, and the first eigenvalue are bounded by the corresponding quantity of the model. When equality occurs one has the rigid case. In Section 5 we discuss many rigidity and stability results for nonnegative and positive Ricci curvature. The Ricci curvature lower bound gives very good control on the fundamental group and the first Betti number of the manifold; this is covered in Section 6 (see also the very recent survey article by Shen-Sormani [121] for more elaborate discussion).

Many of the results in this article are covered in the very nice survey articles [150, 31], where complete proofs are presented. We benefit greatly from these two articles. Some materials here are adapted directly from [31] and we are very grateful to Jeff Cheeger for his permission. We also benefit from [59, 32] and the lecture notes [137] of a topics course I taught at UCSB. I would also like to thank Jeff Cheeger, Xianzhe Dai, Karsten Grove, Peter Petersen, Christina Sormani, and William Wylie for reading earlier versions of this article and for their helpful suggestions.

From comparison theorems, various quantities are bounded by that of the model. When equality occurs one has the rigid case. In this section we concentrate on the rigidity and stability results for nonnegative and positive Ricci curvature. See Section 4 for rigidity and stability under Gromov-Hausdorff convergence and a general lower bound.

The simplest rigidity is the maximal volume. From the equality of volume comparison (1.4.6), we deduce that if M^n has $\mathrm{Ric}_M \geq n-1$ and $\mathrm{Vol}_M =$

$\mathrm{Vol}(S^n)$, then M^n is isometric to S^n. Similarly if M^n has $\mathrm{Ric}_M \geq 0$ and $\lim_{r\to\infty} \frac{\mathrm{Vol}B(p,r)}{\omega_n r^n} = 1$, where $p \in M$ and ω_n is the volume of the unit ball in $\mathbb{R}^n$, then M^n is isometric to $\mathbb{R}^n$.

From the equality of the area of geodesic ball (the first quantity in (1.4.5)) we get another volume rigidity: volume annulus implies metric annulus. This is first observed in [34, Section 4], see also [32, Theorem 2.6]. For the case of nonnegative Ricci curvature, this result says that if $\mathrm{Ric}_{M^n} \geq 0$ on the annulus $A(p, r_1, r_2)$, and

$$\frac{\mathrm{Vol}(\partial B(p,r_1))}{\mathrm{Vol}(\partial B(p,r_2))} = \frac{r_1^{n-1}}{{r_2}^{n-1}},$$

then the metric on $A(p, r_1, r_2)$ is of the form, $dr^2 + r^2\tilde{g}$, for some smooth Riemannian metric $\tilde{g}$ on $\partial B(p, r_1)$.

By Myers' theorem (see Theorem 1.2.3) when Ricci curvature has a positive lower bound the diameter is bounded by the diameter of the model. In the maximal case, using an eigenvalue comparison (see below) Cheng [45] proved that if M^n has $\mathrm{Ric}_M \geq n-1$ and $\mathrm{diam}_M = \pi$, then M^n is isometric to S^n. This result can also be directly proven using volume comparison [122, 150].

Applying the Bochner formula (1.1.1) to the first eigenfunction Lichnerowicz showed that if M^n has $\mathrm{Ric}_M \geq n-1$, then the first eigenvalue $\lambda_1(M) \geq n$ [82]. Obata showed that if $\lambda_1(M)$ then M^n is isometric to S^n [100].

From these rigidity results (the equal case), we naturally ask what happens in the almost equal case. Many results are known in this case. For volume we have the following beautiful stability results for positive and nonnegative Ricci curvatures [35].

Theorem 8.5.1 (Volume Stability, Cheeger-Colding, 1997) *There exists $\epsilon(n) > 0$ such that*
(i) if a complete Riemannian manifold M^n has $\mathrm{Ric}_M \geq n-1$ and $\mathrm{Vol}_M \geq (1-\epsilon(n))\mathrm{Vol}(S^n)$, then M^n is diffeomorphic to S^n;
(ii) if a complete Riemannian manifold M^n has $\mathrm{Ric}_M \geq 0$ and for some $p \in M$, $\mathrm{Vol}B(p,r) \geq (1-\epsilon(n))\,\omega_n r^n$ for all $r > 0$, then M^n is diffeomorphic to $\mathbb{R}^n$.

This was first proved by Perelman [104] with the weaker conclusion that M^n is homeomorphic to S^n (contractible resp.).

The analogous stability result is not true for diameter. In fact, there are manifolds with $\mathrm{Ric} \geq n-1$ and diameter arbitrarily close to π which are not homotopic to sphere [3, 102]. This should be contrasted with the sectional curvature case, where we have the beautiful Grove-Shiohama diameter sphere theorem [65], that if M^n has sectional curvature $K_M \geq 1$ and $\mathrm{diam}_M > \pi/2$ then M is homeomorphic to S^n. Anderson showed that the stability for the splitting theorem (Theorem 2.1.5) does not hold either [6].

By work of Cheng and Croke [45, 49], if $\mathrm{Ric}_M \geq n-1$ then diam_M is close to π if and only if $\lambda_1(M)$ is close to n. So the naive version of the stability for $\lambda_1(M)$ does not hold either. However from the work of [46, 35, 109] we have the following modified version.

Theorem 8.5.2 (Colding, Cheeger-Colding, Petersen) *There exists $\epsilon(n) > 0$ such that if a complete Riemannian manifold M^n has $\mathrm{Ric}_M \geq n-1$, and radius $\geq \pi - \epsilon(n)$ or $\lambda_{n+1}(M) \leq n + \epsilon(n)$, then M^n is diffeomorphic to S^n.*

Here $\lambda_{n+1}(M)$ is the $(n+1)-$th eigenvalue of the Laplacian. The above condition is natural in the sense that for S^n the radius is π and the first eigenvalue is n with multiplicity $n+1$. Extending Cheng and Croke's work Petersen showed that if $\mathrm{Ric}_M \geq n-1$ then the radius is close to π if and only if $\lambda_{n+1}(M)$ is close to n.

The stability for the first Betti number, conjectured by Gromov, was proved by Cheeger-Colding in [35]. Namely there exists $\epsilon(n) > 0$ such that if a complete Riemannian manifold M^n has $\mathrm{Ric}_M(\mathrm{diam}_M)^2 \geq -\epsilon(n)$ and b_1, then M is diffeomorphic to T^n. The homeomorphic version was first proved in [47].

Although the direct stability for diameter does not hold, Cheeger-Colding's breakthrough work [34] gives quantitative generalizations of the diameter rigidity results, see Section **??**.

Although the analogous stability results for maximal diameter in the case of positive/nonnegative Ricci curvature do not hold, Cheeger-Colding's significant work [34] provides quantitative generalizations of Cheng's maximal diameter theorem, Cheeger-Gromoll's splitting theorem (Theorem 2.1.5), and the volume annulus implies metric annulus theorem in terms of Gromov-Hausdroff distance. These results have important applications in extending rigidity results to the limit space.

The following version (not assuming $E(p) = 0$, but without the sharp estimate) is from [31, Theorem 9.1].

Theorem 8.5.3 (Excess Estimate, Abresch-Gromoll, 1990) *If M^n has $\mathrm{Ric}_M \geq -(n-1)\delta$, and for $p \in M$, $s(p) \geq L$ and $E(p) \leq \epsilon$, then on $B(p, R)$, $E \leq \Psi = \Psi(\delta, L^{-1}, \epsilon |\, n, R)$, where Ψ is a nonnegative constant such that for fixed n and R Ψ goes to zero as $\delta, \epsilon \to 0$ and $L \to \infty$.*

This can be interpreted as a weak almost splitting theorem. Cheeger-Colding generalized this result tremendously by proving the following almost splitting theorem [34], see also [31].

Theorem 8.5.4 (Almost Splitting, Cheeger-Colding, 1996) *With the same assumptions as Theorem 8.5.3, there is a length space X such that for some ball, $B((0,x), \frac{1}{4}R) \subset \mathbb{R} \times X$, with the product metric, we have*

$$d_{GH}\left(B(p, \frac{1}{4}R), B((0,x), \frac{1}{4}R)\right) \leq \Psi.$$

Note that X here may not be smooth, and the Hausdorff dimension could be smaller than $n-1$. Examples also show that the ball $B(p, \frac{1}{4}R)$ may not have the topology of a product, no matter how small δ, ϵ, and L^{-1} are [6, 92].

The proof is quite involved. Using the Laplacian comparison, the maximum principle, and Theorem 8.5.3 one shows that the distance function $b_i = d(x, y_i) - d(p, y_i)$ associated to p and y_i is uniformly close to $\mathbf{b}_i$, the harmonic function with same values on $\partial B(p,R)$. From this, together with the lower bound for the smallest eigenvalue of the Dirichlet problem on $B(p,R)$ (see Theorem 1.6.2) one shows that $\nabla b_i, \nabla \mathbf{b}_i$ are close in the L_2 sense. In particular $\nabla \mathbf{b}_i$ is close to 1 in the L_2 sense. Then applying the Bochner formula to $\mathbf{b}_i$ multiplied with a cut-off function with bounded Laplacian one shows that $|\mathrm{Hess}\mathbf{b}_i|$ is small in the L_2 sense in a smaller ball. Finally, in the most significant step, by using the segment inequality (1.6.4), the gradient estimate (2.2.5) and the information established above one derives a quantitative version of the Pythagorean theorem, showing that the ball is close in the Gromov-Hausdorff sense to a ball in some product space; see [34, 31].

An immediate application of the almost splitting theorem is the extension of the splitting theorem to the limit space.

Theorem 8.5.5 (Cheeger-Colding, 1996) *If M_i^n has $\mathrm{Ric}_{M_i} \geq -(n-1)\delta_i$ with $\delta_i \to 0$ as $i \to \infty$, converges to Y in the pointed Gromov-Hausdorff sense, and Y contains a line, then Y is isometric to $\mathbb{R} \times X$ for some length space X.*

Similarly, one has almost rigidity in the presence of finite diameter (with simpler a proof) [34, Theorem 5.12]. As a special consequence, we have that if M_i^n has $\mathrm{Ric}_{M_i} \geq (n-1)$, $\mathrm{diam}_{M_i} \to \pi$ as $i \to \infty$, and converges to Y in the Gromov-Hausdorff sense, then Y is isometric to the spherical metric suspension of some length space X with $\mathrm{diam}(X) \leq \pi$. This is a kind of stability for diameter.

Along the same lines (with more complicated technical details) Cheeger and Colding [34] have an almost rigidity version for the volume annulus implies metric annulus theorem (see Section 5). As a very nice application to the asymptotic cone, they showed that if M^n has $\mathrm{Ric}_M \geq 0$ and has Euclidean volume growth, then every asymptotic cone of M is a metric cone.

As we have seen, understanding the structure of the limit space of manifolds with lower Ricci curvature bound often helps in understanding the structure of the sequence. Cheeger-Colding made significant progress in understand the regularity and geometric structure of the limit spaces [35, 36, 37]. On the other hand Menguy constructed examples showing that the limit space could have infinite topology in an arbitrarily small neighborhood [92]. In [126, 127] Sormani-Wei showed that the limit space has a universal cover.

Let (Y^m, y) (Hausdorff dimension m) be the pointed Gromov-Hausdorff limit of a sequence of Riemannian manifolds (M_i^n, p_i) with $\mathrm{Ric}_{M_i} \geq -(n-1)$. Then $m \leq n$ and Y^m is locally compact. Moreover Cheeger-Colding [35] showed that if $m = \dim Y < n$, then $m \leq n-1$.

The basic notion for studying the infinitesimal structure of the limit space Y is that of a tangent cone.

Definition 8.5.6 *A tangent cone, Y_y, at $y \in (Y^m, d)$ is the pointed Gromov-Hausdorff limit of a sequence of the rescaled spaces $(Y^m, r_i d, y)$, where $r_i \to \infty$ as $i \to \infty$.*

By Gromov's precompactness theorem (Theorem 4.0.5), every such sequence has a converging subsequence. So tangent cones exist for all $y \in Y^m$, but might depend on the choice of convergent sequence. Clearly if M^n is a Riemannian manifold, then the tangent cone at any point is isometric to $\mathbb{R}^n$. Motivated by this one defines [35]

Definition 8.5.7 *A point, $y \in Y$, is called k-regular if for some k, every tangent cone at y is isometric to $\mathbb{R}^k$. Let $\mathcal{R}_k$ denote the set of k-regular points and $\mathcal{R} = \cup_k \mathcal{R}_k$, the regular set. The singular set, $Y \setminus \mathcal{R}$, is denoted $\mathcal{S}$.*

Let μ be a renormalized limit measure on Y as in (4.0.3). Cheeger-Colding showed that the regular points have full measure [35].

Theorem 8.5.8 (Cheeger-Colding, 1997) *For any renormalized limit measure μ, $\mu(\mathcal{S}) = 0$, in particular, the regular points are dense.*

Furthermore, up to a set of measure zero, Y is a countable union of sets, each of which is bi-Lipschitz equivalent to a subset of Euclidean space [37].

Definition 8.5.9 *A metric measure space, (X, μ), is called μ-rectifiable if $0 < \mu(X) < \infty$, and there exists $N < \infty$ and a countable collection of subsets, A_j, with $\mu(X \setminus \cup_j A_j) = 0$, such that each A_j is bi-Lipschitz equivalent to a subset of $\mathbb{R}^{l(j)}$, for some $1 \le l(j) \le N$ and in addtion, on the sets A_j, the measures μ and and the Hausdorff measure $\mathcal{H}^{l(j)}$ are mutually absolutely continous.*

Theorem 8.5.10 (Cheeger-Colding, 2000) *Bounded subsets of Y are μ-rectifiable with respect to any renormalized limit measure μ.*

At the singular points, the structure could be very complicated. Following a related earlier construction of Perelman [106], Menguy constructed 4-dimensional examples of (noncollapsed) limit spaces with, $\mathrm{Ric}_{M_i^n} > 1$, for which there exists point so that any neighborhood of the point has infinite second Betti number [92]. See [35, 91, 93] for examples of collapsed limit space with interesting properties.

Although we have very good regularity results, not much topological structure is known for the limit spaces in general. E.g., is Y locally simply connected? Although this is unknown, using the renormalized limit measure and the existence of regular points, together with δ-covers, Sormani-Wei [126, 127] showed that the universal cover of Y exists. Moreover when Y is compact, the fundamental group of M_i has a surjective homomorphism onto the group of deck transforms of Y for all i sufficiently large.

When the sequence has the additional assumption that

$$\mathrm{Vol}(B(p_i, 1)) \ge v > 0, \tag{8.5.1}$$

the limit space Y is called noncollapsed. This is equivalent to m. In this case, more structure is known.

Definition 8.5.11 *Given $\epsilon > 0$, the ϵ-regular set, $\mathcal{R}_\epsilon$, consists of those points y such that for all sufficiently small r,*

$$d_{GH}(B(y, r), B(0, r)) \le \epsilon r,$$

where $0 \in \mathbb{R}^n$.

Clearly $\mathcal{R} = \cap_\epsilon \mathcal{R}_\epsilon$. Let $\mathring{\mathcal{R}}_\epsilon$ denote the interior of $\mathcal{R}_\epsilon$.

Theorem 8.5.12 (Cheeger-Colding 1997, 2000) *There exists $\epsilon(n) > 0$ such that if Y is a noncollapsed limit space of the sequence M_i^n with $\mathrm{Ric}_{M_i} \geq -(n-1)$, then for $0 < \epsilon < \epsilon(n)$, the set $\mathring{\mathcal{R}}_\epsilon$ is $\alpha(\epsilon)$-bi-Hölder equivalent to a smooth connected Riemannian manifold, where $\alpha(\epsilon) \to 1$ as $\epsilon \to 0$. Moreover,*

$$\dim(Y \setminus \mathring{\mathcal{R}}_\epsilon) \leq n-2. \tag{8.5.2}$$

In addition, for all $y \in Y$, every tangent cone Y_y at y is a metric cone and the isometry group of Y is a Lie group.

This is proved in [35, 36].

If, in addition, Ricci curvature is bounded from two sides, we have stronger regularity [2].

Theorem 8.5.13 (Anderson, 1990) *There exists $\epsilon(n) > 0$ such that if Y is a noncollapsed limit space of the sequence M_i^n with $|\mathrm{Ric}_{M_i}| \leq n-1$, then for $0 < \epsilon < \epsilon(n)$, $\mathcal{R}_\epsilon = \mathcal{R}$. In particular the singular set is closed. Moreover, $\mathcal{R}$ is a $C^{1,\alpha}$ Riemannian manifold, for all $\alpha < 1$. If the metrics on M_i^n are Einstein, $\mathrm{Ric}_{M_i^n} = (n-1)Hg_i$, then the metric on $\mathcal{R}$ is actually C^∞.*

Many more regularity results are obtained when the sequence is Einstein, Kähler, has special holonomy, or has bounded L^p-norm of the full curvature tensor, see [7, 33, 38, 41], especially [32] which gives an excellent survey in this direction. See the recent work [42] for Einstein 4-manifolds with possible collapsing.

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