Ex for Nonhomogeneous Systems:

$$\dot{\underline{x}} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

We find the eigenvalues $\boxed{\lambda = 3}$, $\boxed{\lambda = -1}$. E-vectors associated w/ the e-values are

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ respectively.}$$

Thus $P = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$

$$P^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix}$$

$$\Rightarrow P^{-1} A P = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

So substituting $\underline{x} = P \underline{y}$, we get

$$P^{-1}\underline{x}' = \underline{y}' = P^{-1}A\underline{x} + P^{-1}f(t)$$

$$= P^{-1}AP\underline{y} + P^{-1}f(t)$$

$$= D\underline{y} + \begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{pmatrix}$$

$$\Rightarrow \underline{y}' = \begin{pmatrix} 3y_1 + \frac{1}{2}e^t \\ -y_2 + \frac{1}{2}e^t \end{pmatrix}$$

$$\Rightarrow y_1' - 3y_1 = \tfrac{1}{2}e^t$$

$$y_2' + y_2 = \tfrac{1}{2}e^t$$

$$\Rightarrow \left(y_1 e^{-3t}\right)' = \tfrac{1}{2}e^{-2t}$$

$$\left(y_2 e^t\right)' = \tfrac{1}{2}e^{2t}$$

$$\Rightarrow y_1 = -\tfrac{1}{4}e^t + c_1 e^{3t}$$

$$y_2 = \tfrac{1}{4}e^t + c_2 e^{-t}$$

By using Variation of Parameters.

Reverting Back to our sol'n $\underline{x}$, we get

$$\underline{x} = P\underline{y} = \begin{pmatrix} 1/4 & 1/2 \\ -1/4 & 1/2 \end{pmatrix} \begin{pmatrix} -1/4\, e^{t} + c_1 e^{3t} \\ 1/4\, e^{t} + c_2 e^{-t} \end{pmatrix}$$

$$\Rightarrow \underline{x} = \begin{pmatrix} -\frac{1}{16}e^{t} + \frac{c_1}{4}e^{3t} - \frac{1}{8}e^{t} + \frac{c_1}{2}e^{3t} \\ \frac{1}{16}e^{t} - \frac{c_1}{4}e^{3t} + \frac{1}{8}e^{t} + \frac{c_2}{2}e^{-t} \end{pmatrix}$$