Summary

In extending phase-plane analysis to nonlinear systems, we encounter multiple equilibrium points and limit cycles. Analysis is facilitated by studying nullclines and the regions into which they separate the plane.

7.1 Problems

Review of Classifications *For each system in Problems 1–5, determine the dependent variables, given that t is the independent variable, and the parameters; determine whether the system is autonomous or nonautonomous, linear or nonlinear, and (if linear) homogeneous or nonhomogeneous.*

1. $x' = x + ty$
$y' = 2x + y + \gamma \sin t$

2. $u' = 3u + 4v$
$v' = -2u + \sin t$

3. $x_1' = \kappa x_2$
$x_2' = -\sin x_1$

4. $p' = q$
$q' = pq - \sin t$

5. $S' = -rSI$
$I' = rSI - \gamma I$
$R' = \gamma I$

Verification Review *Show that the system in each of Problems 6–9 is satisfied by the given set of functions.*

6. $x' = x$
$y' = y$
$\{x = e^t,\ y = e^t\}$

7. $x' = y$
$y' = -x$
$\{x = \sin t,\ y = \cos t\}$

8. $x' = y + t$
$y' = -2x + 3y + 5$
$$\left\{x = -\frac{3}{2}t + \frac{3}{4},\ y = -t - \frac{3}{2}\right\}$$

9. $x' = x$
$y' = 2z$
$z' = -2y$
$\{x = 0,\ y = \sin 2t,\ z = \cos 2t\}$

A Habit to Acquire *For each of the nonlinear systems in Problems 10–13, make a graph of the nullclines with arrows on and between them showing the direction of solutions. Identify each equilibrium and label it stable or unstable. Overlay these sketches on the phase portraits in Fig. 7.1.1(a)–(d) to clarify the behaviors of the trajectories. Then for each graph write a paragraph to describe the behaviors.*

10. $x' = y$
$y' = x(x-1)$

11. $x' = xy$
$y' = y - 3\cos x$

12. $x' = x^2 - y + 2$
$y' = y + 2x$

13. $x' = 1 + y + x^2$
$y' = y/2 - 3x$

Phase Portraits from Nullclines *For the nonlinear systems in Problems 14–19, determine the equilibrium solutions, if any, and sketch the h- and v-nullclines, drawing appropriate arrows on each and in the regions between, to indicate the direction of the solution curves. If the system has equilibrium points, determine if they are stable or unstable. Add some typical solutions and write a description of their behaviors. Identify any limit cycles.*

14. $x' = xy$
$y' = y - x^2 + 1$

15. $x' = y - \ln|x|$
$y' = x - \ln|y|$

16. $x' = y + x(1 - x^2 - y^2)$
$y' = -x + y(1 - x^2 - y^2)$

17. $x' = 1 - x^2 - y^2$
$y' = x$

18. $x' = y - x^2 + 1$
$y' = y + x^2 - 1$

19. $x' = |x| - y - 1$
$y' = |x| + y - 1$

Equilibria for Second-Order DEs *For the differential equations in Problems 20–25, find and classify the constant solutions as follows:*

(a) *Rewrite the second-order equation as a system of two first-order equations.*

(b) *Draw the nullclines for the first-order system, labeled with appropriate arrows, and find the equilibria.*

(c) *Deduce whether the equilibrium points of the nonlinear system are stable, thereby determining the stability of the constant solutions of the second-order equation.*

~~(d) *Identify any periodic solutions and state whether they are limit cycles.*~~

20. $x'' + (x^2 - 1)x' + x = 0$

21. $\theta'' + (g/L)\sin\theta = 0$

22. $x'' - \dfrac{x}{x-1} = 0$

23. $\ddot{x} + \dot{x}^2 + x^2 = 0$

39. $x' = 1 + x$
$y' = (1 + x)\sqrt{y}$

40. $x' = x/y$
$y' = x - y/x$

41. **Hamiltonian for the Harmonic Oscillator** Hamiltonian[5] mechanics is based on the Hamiltonian function $H(p, q)$, representing the total energy in terms of the **generalized coordinate** p and **generalized momentum** q. (Newtonian mechanics focuses on forces.) The **Hamiltonian system** is then defined by

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}.$$

For the undamped mass-spring system with mass m, spring constant k and displacement x, we let $q = x$ and $p = m\dot{x}$ (the momentum).

(a) Show that the kinetic energy of the mass is $\dfrac{p^2}{2m}$.

(b) Show that the total energy is $H(p, q) = \dfrac{p^2}{2m} + \dfrac{kq^2}{2}$.

(c) Derive the corresponding Hamiltonian system.

Computer Lab: Phase-Plane Analysis *For the system in each of Problems 42–46, use appropriate software to carry out the following investigation:*

(a) Draw a vector field.

(b) Draw sample solution curves.

(c) Determine the equilibrium points.

(d) Determine the stability behavior of the equilibrium points.

(e) Discuss the long-term behavior of the system.

(f) Identify any periodic solutions and state whether they are limit cycles.

42. $x' = x(x - y)$
$y' = y(1 - y)$

43. $x' = x - x^2$
$y' = -y$

44. $x' = 1 - |x|$
$y' = x - y$

45. $x' = x(2 - x - y)$
$y' = -y$

46. $x' = x + y - x^3$
$y' = -x$

47. $x' = \sin(xy)$
$y' = \cos(x + y)$

48. **Computer Lab: Graphing in Two Dimensions** Do IDE Lab 17 to help answer the following questions, which become even more important for nonlinear DEs than for linear DEs: What do second-order differential equations have in common with systems of two first-order equations? Why are phase planes and vector fields so important? How do they relate to $x(t)$ and $y(t)$ time series? What information can you squeeze out of the nullclines?

49. **Computer Lab: The Glider** If you've ever played with a balsa-wood glider, you know that it flies in a wavy path if you throw it gently and does loop-the-loops if you throw it hard. Do IDE Lab 19 to see how this is all explained by nonlinear phase-plane analysis.[6]

50. **Computer Lab: Nonlinear Oscillators** A child on a swing asks to be started as high as you can, with only a single initial push. Small-angle assumptions no longer hold, so this is a case of an unforced nonlinear oscillator. Work IDE Lab 20 to answer the question of whether a loop-the-loop is a possible outcome.

51. **Suggested Journal Entry** Discuss the distinction between quantitative and qualitative methods in the analysis of differential equations and systems. Contrast the advantages and limitations of each approach.

7.2 Linearization

SYNOPSIS: *We will study the behavior of solutions of an autonomous nonlinear* 2×2 *system near an equilibrium point by analyzing a related linear system called the linearization. This merger of linear algebra and calculus makes it possible to classify the stability behavior of equilibria and limit cycles for many nonlinear systems.*

[5]Named for William Rowan Hamilton. (See Sec. 3.5.)

[6]Model development and analysis by Steven Strogatz.