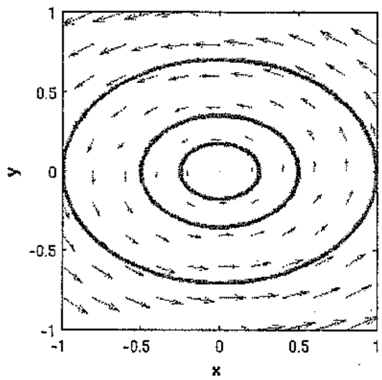
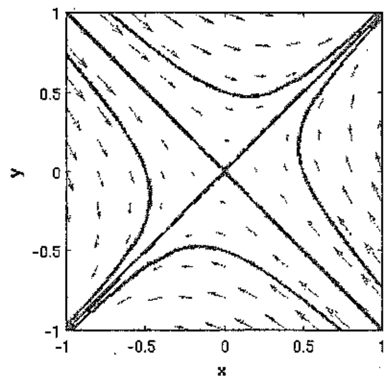
Match the following systems of differential equations with their respective phase portraits. Below each portrait, classify the stability of the system, and explain your answer.

____ $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$ ____ $\mathbf{x}' = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \mathbf{x}$

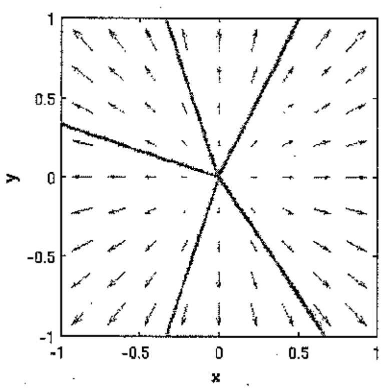
____ $\mathbf{x}' = \begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix} \mathbf{x}$ ____ $\mathbf{x}' = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \mathbf{x}$



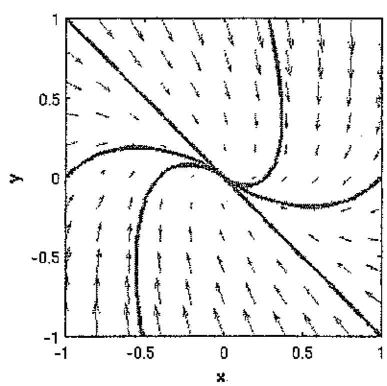
A



B



C



D

Summary

In extending phase-plane analysis to nonlinear systems, we encounter multiple equilibrium points and limit cycles. Analysis is facilitated by studying nullclines and the regions into which they separate the plane.

7.1 Problems

Review of Classifications *For each system in Problems 1–5, determine the dependent variables, given that t is the independent variable, and the parameters; determine whether the system is autonomous or nonautonomous, linear or nonlinear, and (if linear) homogeneous or nonhomogeneous.*

1. $x' = x + ty$
   $y' = 2x + y + \gamma \sin t$

2. $u' = 3u + 4v$
   $v' = -2u + \sin t$

3. $x_1' = \kappa x_2$
   $x_2' = -\sin x_1$

4. $p' = q$
   $q' = pq - \sin t$

5. $S' = -rSI$
   $I' = rSI - \gamma I$
   $R' = \gamma I$

Verification Review *Show that the system in each of Problems 6–9 is satisfied by the given set of functions.*

6. $x' = x$
   $y' = y$
   $\{x = e^t,\ y = e^t\}$

7. $x' = y$
   $y' = -x$
   $\{x = \sin t,\ y = \cos t\}$

8. $x' = y + t$
   $y' = -2x + 3y + 5$
   $\left\{x = -\frac{3}{2}t + \frac{3}{4},\ y = -t - \frac{3}{2}\right\}$

9. $x' = x$
   $y' = 2z$
   $z' = -2y$
   $\{x = 0,\ y = \sin 2t,\ z = \cos 2t\}$

A Habit to Acquire *For each of the nonlinear systems in Problems 10–13, make a graph of the nullclines with arrows on and between them showing the direction of solutions. Identify each equilibrium and label it stable or unstable. Overlay these sketches on the phase portraits in Fig. 7.1.1(a)–(d) to clarify the behaviors of the trajectories. Then for each graph write a paragraph to describe the behaviors.*

10. $x' = y$
    $y' = x(x-1)$

11. $x' = xy$
    $y' = y - 3\cos x$

12. $x' = x^2 - y + 2$
    $y' = y + 2x$

13. $x' = 1 + y + x^2$
    $y' = y/2 - 3x$

Phase Portraits from Nullclines *For the nonlinear systems in Problems 14–19, determine the equilibrium solutions, if any, and sketch the h- and v-nullclines, drawing appropriate arrows on each and in the regions between, to indicate the direction of the solution curves. If the system has equilibrium points, determine if they are stable or unstable. Add some typical solutions and write a description of their behaviors. ~~Identify any limit cycles.~~*

14. $x' = xy$
    $y' = y - x^2 + 1$

15. $x' = y - \ln|x|$
    $y' = x - \ln|y|$

16. $x' = y + x(1 - x^2 - y^2)$
    $y' = -x + y(1 - x^2 - y^2)$

17. $x' = 1 - x^2 - y^2$
    $y' = x$

18. $x' = y - x^2 + 1$
    $y' = y + x^2 - 1$

19. $x' = |x| - y - 1$
    $y' = |x| + y - 1$

Equilibria for Second-Order DEs *For the differential equations in Problems 20–25, find and classify the constant solutions as follows:*

(a) Rewrite the second-order equation as a system of two first-order equations.

(b) Draw the nullclines for the first-order system, labeled with appropriate arrows, and find the equilibria.

(c) Deduce whether the equilibrium points of the nonlinear system are stable, thereby determining the stability of the constant solutions of the second-order equation.

(d) Identify any periodic solutions and state whether they are limit cycles.

20. $x'' + (x^2 - 1)x' + x = 0$

21. $\theta'' + (g/L)\sin\theta = 0$

22. $x'' - \dfrac{x}{x-1} = 0$

23. $\ddot{x} + \dot{x}^2 + x^2 = 0$