Review: Part I ~~Ch 4 (4.1 – 4.5), Ch 5 (5.1 – 5.3)~~.

Ch 4: 2^{nd} order linear diff. $eq^{\underline{n}}$s.

$$ay'' + by' + cy = f(t).$$

① homog. $f(t) = 0$.

char. $eq^{\underline{n}}$: $ar^2 + br + c = 0$

$\Rightarrow r_1, r_2$ roots.

a) distinct real roots $r_1 \neq r_2$

general $sol^{\underline{n}}$ $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

b). identical roots $r_1 = r_2$

$$y = c_1 e^{r_1 t} + c_2 t e^{r_2 t}$$

c). complex roots $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

$$\begin{matrix} c_1 \mathrm{Re}(e^{r_1 t}) \\ + c_2 \mathrm{Im}(e^{r_1 t}) \end{matrix} = y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$$

②. nonhomog. $f(t) \neq 0$.

general $sol^{\underline{n}}$ $y = y_h + y_p$

y_h solved as in 1.

y_p : method of undetermined coeff's.

What does $f(t)$ look like $\Rightarrow$ What y_p should look like.

a) $f(t) = (a_n t^n + \cdots + a_1 t + a_0) e^{kt}$

$$\Rightarrow \quad y_p = \begin{cases} (A_n t^n + \cdots + A_1 t + A_0) e^{kt} & \text{if } k \neq r_1, k \neq r_2 \\ t(A_n t^n + \cdots + A_0) e^{kt} & \text{if } k = \text{one but not both of } r_1, r_2 \\ t^2 (A_n t^n + \cdots + A_0) e^{kt} & \text{if } k = r_1 = r_2. \end{cases}$$

b). $f(t) = e^{kt}(a \cos \omega t + b \sin \omega t)$

$$\Rightarrow \quad y_p = \begin{cases} e^{kt}(A \cos \omega t + B \sin \omega t) & \text{if } k + i\omega \neq r_1, r_2. \\ t e^{kt}(A \cos \omega t + B \sin \omega t) & \text{if } k + i\omega = \text{one of } r_1, r_2. \end{cases}$$

<u>Appl^{n}</u>: Forced Oscillations

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos \omega_f t$$

① $b = 0$. <u>resonance</u> when $\omega_f = \sqrt{\frac{k}{m}}$

(when you need fudge factor "t")

② $b > 0$. $x = x_h + x_p$

x_h: transient sol^{n}. $\to 0$

x_p: steady-state sol^{n}.

Review:

Ch 5:

① Linear transf$^{\underline{n}}$s: $T: V \longrightarrow W$

s.t. $T(u+v) = T(u) + T(v)$ $\Rightarrow T(0)=0$

$T(ku) = k\,T(u)$

for all $u, v \in V$, scalar k.

② Linear transf$^{\underline{n}}$s $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are always given by a matrix:

$$T(x) = Ax \quad \text{w/.} \quad A = [T(e_1) | \cdots | T(e_n)]$$

② Image and rank, kernel and nullity

$\text{rank}(T) = \dim \text{Im}(T)$

$\text{nullity}(T) = \dim \ker(T)$.

For $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $T(x) = Ax$,

$\text{Im}(T)$ = span of column vectors of A

But to compute $\text{rank}(T)$, need $A \xrightarrow{\text{EROS}} \text{RREF}$ to find linearly indep. column vectors.

Similarly, to compute $\ker(T)$ and $\text{nullity}(T)$, use $A \xrightarrow{\text{EROS}} \text{RREF}$ to solve $Ax = 0$.

Review (Part II):

03/14/12

Ch 5: Linear Transf^{n}s

1). Eigenvalues & Eigenvectors $\quad A \quad n \times n$

$$\boxed{A\vec{v} = \lambda \vec{v}}, \qquad \vec{v} \neq 0.$$

$$\Leftrightarrow \quad (A - \lambda I)\vec{v} = 0 \Rightarrow \vec{v} \in \underbrace{\ker(A - \lambda I)}_{\text{Eigenspace}}$$

Finding e.v.'s

$$\Leftrightarrow \det(A - \lambda I) = 0.$$

Finding corresponding e-vectors

$$(A - \lambda I)\vec{v} = 0. \quad [A - \lambda I] \xrightarrow{EROS} \cdots$$

2). Diagonalization

$\lambda_1, \cdots, \lambda_n$ e.v.'s

$\vec{v}_1, \cdots, \vec{v}_n$ (full) set of linearly indep e.vectors

$\Rightarrow$ $P = [\vec{v}_1 | \cdots | \vec{v}_n]$ diagonalizes A:

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Very important.

Ch 6: Linear Systems of DE's:

$$\vec{x}' = A\vec{x} + \vec{f}(t).$$

1). Homogeneous $\vec{f}(t) = 0$: $\quad \vec{x}' = A\vec{x}$.

$\Rightarrow$ general sol^{n}s

(1a) $$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

provided $\vec{v}_1, \cdots, \vec{v}_n$ full set linearly indep e-vectors

(1b) Short of eigenvectors $\longrightarrow$ $\underbrace{\text{generalized eigenvector}}_{\text{fudge factor }+}$

$\lambda_1 = \lambda_2$, $\vec{v}_1$ only linearly indep e-vector

one sol$^{\underline{n}}$ $\quad e^{\lambda_1 t}\vec{v}_1$

need one more : $\quad t e^{\lambda_1 t}\vec{v}_1 + e^{\lambda_1 t}\vec{u}$

fudge factor

generalized eigenvector

w/. $\vec{u}$ solved from

$$\boxed{(A - \lambda_1 I)\vec{u} = \vec{v}_1}$$

$[A - \lambda_1 I \mid \vec{v}_1] \xrightarrow{\text{EROs}} \cdots$

$\Rightarrow$ general sol$^{\underline{n}}$

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2\left(t e^{\lambda_1 t}\vec{v}_1 + e^{\lambda_1 t}\vec{u}\right)$$

$$+ c_3 e^{\lambda_3 t}\vec{v}_3 + \cdots + c_n e^{\lambda_n t}\vec{v}_n .$$

(1c) complex eigenvalues

$$\lambda_1 = \alpha + i\beta \Rightarrow \lambda_2 = \bar{\lambda}_1 = \alpha - i\beta$$

$$\vec{v}_1 = \vec{p} + i\vec{q} \Rightarrow \vec{v}_2 = \overline{\vec{v}_1} = \vec{p} - i\vec{q}$$

general sol$^{\underline{n}}$

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\bar{\lambda}_1 t}\overline{\vec{v}_1} + c_3 e^{\lambda_3 t}\vec{v}_3 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

complex form

real form $\quad = c_1 \operatorname{Re}(e^{\lambda_1 t}\vec{v}_1) + c_2 \operatorname{Im}(e^{\lambda_1 t}\vec{v}_1) + \cdots$

2). Nonhomog. $\vec{f}(t) \neq 0$.

(2a) $$\vec{x}' = A\vec{x} + \vec{f}(t)$$ undetermined coeff's.

$\vec{f}(t)$ very special: exp. f^n's entries / polynomial " | does not coincide w/ homog. part

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t).$$

$$P^{-1}AP = D$$

(2b) decoupling: $\vec{x} = P\vec{u} \Longrightarrow$

$$\vec{u}' = D\vec{u} + P^{-1}\vec{f}(t) = \begin{bmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

decouples into n eq^n's (indep. of each other)

$$\left\{\begin{array}{l} u_1' = \lambda_1 u_1 + g_1(t) \rightsquigarrow u_1(t) \\ \vdots \\ u_n' = \lambda_n u_n + g_n(t) \xrightarrow{\text{integrating factors}} u_n(t) \end{array}\right\}$$

(2c) Use matrix exp.

$$\vec{x}' - A\vec{x} = \vec{f}(t)$$

$$\underbrace{e^{-tA}(\vec{x}' - A\vec{x})}_{=(e^{-tA}\vec{x})'} = e^{-tA}\vec{f}(t)$$

$$(e^{-tA}\vec{x})' \Rightarrow e^{-tA}\vec{x} = \int_0^t e^{-sA}\vec{f}(s)\,ds + \vec{C}$$

i.e. $$\vec{x}(t) = e^{tA}\vec{C} + e^{tA}\int_0^t e^{-sA}\vec{f}(s)\,ds$$

But needs to compute e^{tA}.

2d matrix exponential

$$e^{tA} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

Basic Properties:

- $(e^{tA})' = A e^{tA} = e^{tA} A$
- $e^{tA}\big|_{t=0} = I$
- $(e^{tA})^{-1} = e^{-tA}$
- For $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ diagonal, $e^{tD} = \begin{bmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{bmatrix}$
- If P invertible and diagonalizes A:

  $P^{-1}AP = D \Rightarrow A = PDP^{-1}$

Hence

$$e^{tA} = e^{t(PDP^{-1})} = P e^{tD} P^{-1}$$

$$= P \begin{bmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{bmatrix} P^{-1}$$

short of eigenvectors

- If $A = \lambda I + N$ and N is nilpotent: $N^k = 0$ for some k.

$$\Rightarrow e^{tA} = e^{t(\lambda I)} \cdot e^{tN} = \begin{bmatrix} e^{t\lambda} & & \\ & \ddots & \\ & & e^{t\lambda} \end{bmatrix} \cdot \left(I + tN + \cdots + \frac{t^{k-1}}{(k-1)!}N^{k-1}\right)$$

3). Stability : $\vec{x}' = A\vec{x}$

$\vec{x} = 0$ is an equilibrium

Stable if ALL nearby trajectories/sol^{n}s stay bounded.

unstable if some nearby trajectories/sol^{n}s go away.

Completely determined by eigenvalues of A (and eigenvectors)

3a All eigenvalues are negative or have negative real parts $\Longrightarrow$ stable

3b One eigenvalue is positive or has positive real part $\Longrightarrow$ unstable

3C Borderline case: all eigenvalues ≤ 0 and one is 0 or ~~has~~ purely imaginary.

Case 1: $\lambda_1 = 0$ has full set of linearly indep. eigenvectors $\Longrightarrow$ stable

Case 2: $\lambda_1 = 0$ does not have full set of linearly indep. eigenvectors $\Longrightarrow$ unstable.

Case 3: $\lambda_1 = i\beta$ purely imaginary $\Longrightarrow$ stable

Ch 7 : Nonlinear System of DE's

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$ autonomous system.

1). Phase Portraits

v-nullcline, h-nullcline, equilibrim

sketch phase portraits

2). Stability of equilibrim

Stable if ALL nearby trajectories stay close.

unstable if SOME nearby trajectories wander away.

3). Stability via Linearyation

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$ (x_e, y_e) equilibrim

3a All eigenvalues of $J(x_e, y_e) < 0$ or real parts $< 0 \implies (x_e, y_e)$ is stable.

3b One eigenvalue of $J(x_e, y_e) > 0$ or real part $> 0 \implies (x_e, y_e)$ is unstable.

3c Borderline case : can not be determined.