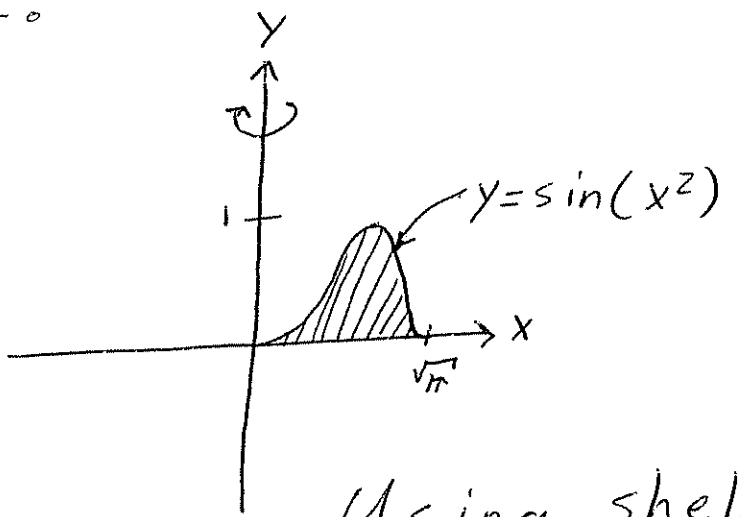
1.



Using shells

$$\text{Volume} = 2\pi \int_a^b h\, r\, dx = 2\pi \int_0^{\sqrt{\pi}} \sin(x^2)\, x\, dx$$

Use a u-sub

Let $u = x^2$ If $x=0$, then $u=0$

$du = 2x\,dx$ If $x=\sqrt{\pi}$, then $u=\pi$

$dx = \dfrac{du}{2x}$

$$2\pi \int_0^{\sqrt{\pi}} \sin(x^2)\, x\, dx = 2\pi \int_0^{\pi} \sin(u)\, x\, \frac{du}{2x}$$

$$= \pi \int_0^{\pi} \sin(u)\, du$$

$$= \pi(-\cos(u))\Big|_0^{\pi}$$

$$= \pi(-(-1)-(-(1))) = \boxed{2\pi}$$

2 a)

$$\text{Ave. Val.} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \cos^4(x)\sin^3(x)\,dx$$

Recall: $\sin^2(x) = 1 - \cos^2(x)$

$$= \frac{1}{\pi}\int_0^{\pi} \cos^4(x)\left(1-\cos^2(x)\right)(\sin x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left(\cos^4(x) - \cos^6(x)\right)\sin x\,dx$$

Use a u-sub.

Let $u = \cos x$ | If $x=0$, then $u=1$.

$du = -\sin x\,dx$ | If $x=\pi$, then $u=-1$.

$dx = \frac{-du}{\sin x}$

$$\frac{1}{\pi}\int_0^{\pi}(\cos^4(x) - \cos^6(x)\sin(x)\,dx = \frac{1}{\pi}\int_1^{-1}(u^4-u^6)\sin(x)\cdot\frac{-du}{\sin(x)}$$

$$= \frac{-1}{\pi}\int_1^{-1} u^4 - u^6\,du$$

$$= \frac{-1}{\pi}\left(\frac{u^5}{5}-\frac{u^7}{7}\right)\Big|_1^{-1} = \frac{-1}{\pi}\left(\left(\frac{-1}{5}+\frac{1}{7}\right)-\left(\frac{1}{5}-\frac{1}{7}\right)\right) = \boxed{\frac{4}{35\pi}}$$

2 b)

$$\text{Ave. Val.} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{3-1}\int_1^3 r^4 \ln(r)\,dr$$

Use By Parts

$u = \ln(r) \qquad dv = r^4\,dr$

$du = \frac{1}{r}\,dr \qquad v = \frac{r^5}{5}$

$$\frac{1}{2}\int_1^3 r^4 \ln(r)\,dr = \frac{1}{2}\left(\frac{1}{5}r^5\ln(r)\Big|_1^3 - \int_1^3 \left(\frac{1}{r}\right)\left(\frac{r^5}{5}\right)dr\right)$$

$$= \frac{1}{2}\left(\frac{1}{5}r^5\ln(r)\Big|_1^3 - \int_1^3 \frac{r^4}{5}\,dr\right)$$

$$= \frac{1}{2}\left(\frac{1}{5}r^5\ln(r) - \frac{1}{25}r^5\right)\Big|_1^3$$

$$= \frac{1}{10}r^5\left(\ln(r) - \frac{1}{5}\right)\Big|_1^3$$

$$= \frac{1}{10}\left(3^5\left(\ln(3) - \frac{1}{5}\right) - \frac{1}{5}\right)$$

3a)

$$\int (\ln(x))^2\, dx$$

Use By Parts.

$u = (\ln(x))^2 \qquad dv = dx$

$du = \dfrac{2\ln(x)}{x} \qquad v = x$

$$\int (\ln(x))^2\, dx = x(\ln(x))^2 - \int 2\ln(x)\, dx$$

Use By Parts Again

$u = 2\ln(x) \qquad dv = dx$

$du = \dfrac{2}{x} \qquad v = x$

$$\int (\ln(x))^2 dx = x(\ln(x))^2 - \left(2x\ln(x) - \int 2\, dx\right)$$

$$= x(\ln(x))^2 - 2x\ln x + 2x + C$$

3 b)

$$\int \tan^{-1}(x)\,dx$$

Use By Parts

$u = \tan^{-1}(x)$ $\qquad dv = dx$

$du = \frac{1}{x^2+1}\,dx$ $\qquad v = x$

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$$

Use u-sub

Let $u = x^2+1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{u}\left(\frac{du}{2x}\right)$$

$$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{du}{u}$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|u| + C$$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C$$

3 c)

$$\int e^{2t}\sin(3t)\,dt$$

Use By Parts

$u = e^{2t}$ $\quad dv = \sin(3t)\,dt$

$du = 2e^{2t}dt$ $\quad v = -\frac{1}{3}\cos(3t)$

$$\int e^{2t}\sin(3t)\,dt = -\frac{1}{3}e^{2t}\cos(3t) - \int\left(-\frac{1}{3}\right)\cos(3t)\,2e^{2t}dt$$

$$= -\frac{1}{3}e^{2t}\cos(3t) + \frac{2}{3}\int e^{2t}\cos(3t)\,dt$$

Use By Parts Again

$u = e^{2t}$ $\quad dv = \cos(3t)\,dt$

$du = 2e^{2t}dt$ $\quad v = \frac{1}{3}\sin(3t)$

$$\int e^{2t}\sin(3t)\,dt = -\frac{1}{3}e^{2t}\cos(3t) + \frac{2}{3}\left(\frac{1}{3}e^{2t}\sin(3t) - \int\frac{1}{3}\sin(3t)\,2e^{2t}dt\right.$$

$$\int e^{2t}\sin(3t)\,dt = -\frac{1}{3}e^{2t}\cos(3t) + \frac{2}{9}e^{2t}\sin(3t) - \frac{4}{9}\int e^{2t}\sin(3t)\,dt$$

$$\int e^{2t}\sin(3t) = \frac{9}{13}\left(-\frac{1}{3}e^{2t}\cos(3t) + \frac{2}{9}e^{2t}\sin(3t)\right) + C$$

3d)

$$\int \frac{x^3}{x^2-2x-8}\,dx$$

Use Long Div.

$$
\begin{array}{r l}
 & \phantom{x^3+0x^2+{}} x + 2 \\
x^2-2x-8 & \overline{\big)\, x^3 + 0x^2 + 0x + 0} \\
 & -\left(x^3 - 2x^2 - 8x\right) \\
\hline
 & \phantom{x^3+{}} 2x^2 + 8x + 0 \\
 & -\left(2x^2 - 4x - 16\right) \\
\hline
 & \phantom{x^3+2x^2+{}} 12x + 16
\end{array}
$$

$$\int \frac{x^3}{x^2-2x-8}\,dx = \int x + 2 + \frac{12x+16}{x^2-2x-8}$$

$$= \frac{x^2}{2} + 2x + 4\int \frac{3x+4}{x^2-2x-8}\,dx$$

$$= \frac{x^2}{2} + 2x + 4\int \frac{3x+4}{(x-4)(x+2)}\,dx$$

Use Part. Frac. Ex.

$$\frac{3x+4}{(x-4)(x+2)} = \frac{A}{(x-4)} + \frac{B}{(x+2)}$$

$$3x+4 = A(x+2) + B(x-4)$$

If $x=-2$, $3(-2)+4 = A(0) + B(-2-4)$

$$-2 = -6B$$

$$B = \frac{1}{3}$$

If $x = 4$, $3(4) + 4 = A(4+2) + B(0)$

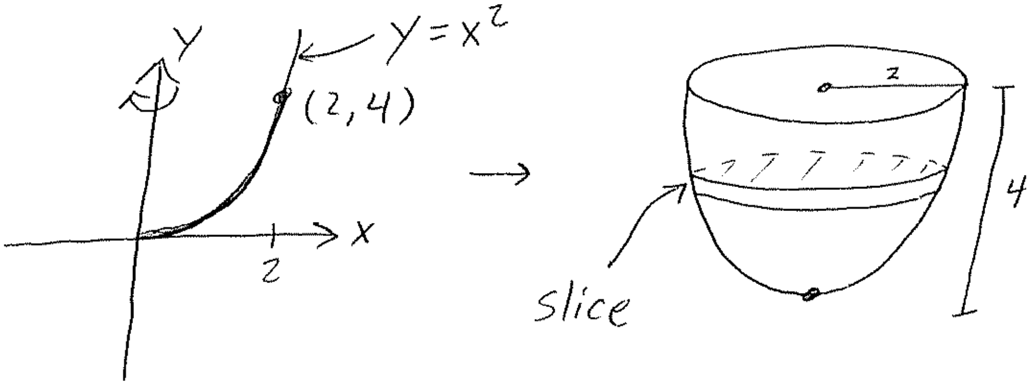
$$16 = 6A$$

$$A = \frac{8}{3}$$

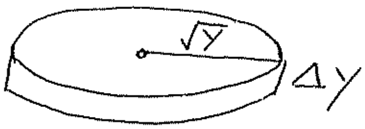
$$\int \frac{x^3}{x^2 - 2x - 8}\,dx = \frac{x^2}{2} + 2x + 4\left(\int \frac{\frac{8}{3}}{x-4}\,dx + \int \frac{\frac{1}{3}}{x+2}\,dx\right)$$

$$= \frac{x^2}{2} + 2x + 4\left(\frac{8}{3}\ln|x-4| + \frac{1}{3}\ln|x+2|\right) + C.$$

4. Assume the units of distance are in meters.



First find the work done to pull a single "slice" of water out of the tank.

Slice: 

Vol. of Slice: $\pi(\sqrt{y})^2 \Delta y \text{ m}^3 = \pi y \Delta y \text{ m}^3$

Mass of Slice: $(1000 \text{ kg/m}^3)\, \pi y \Delta y \text{ m}^3 = 1000\pi y \Delta y \text{ kg}$

Force of Slice: $(9.8 \text{ m/s}^2)(1000\pi y \Delta y \text{ kg}) = 9800\pi y \Delta y \text{ N}$

Work done to pull slice out of tank: (Force)(distance)

$$= (9800\pi y \Delta y \text{ N})(4-y)$$

$$= 9800\pi y(4-y)\Delta y \text{ J}$$

Work done to empty tank: $\int_0^4 9800\pi y(4-y)\,dy$

4 (cont.)

$$\int_0^4 9800\pi y(4-y)\,dy = 9800\pi\int_0^4 4y - y^2\,dy$$

$$= 9800\pi\left(2y^2 - \frac{y^3}{3}\right)\Big|_0^4$$

$$= 9800\pi\left(2(4)^2 - \tfrac{1}{3}(4)^3 - \left(2(0)^2 - \tfrac{1}{3}(0)^3\right)\right)$$

$$= 9800\pi\left(32 - \frac{64}{3}\right)$$

$$= \boxed{9800\pi\left(\frac{32}{3}\right)\text{ J}}$$

5a) $\displaystyle\int_0^2 \frac{2x+1}{x^2+x-2}\,dx$

Since $x^2+x-2=(x+2)(x-1)$ and $2x+1 \neq 0$ when $x=1$, then $\dfrac{2x+1}{x^2+x-2}$ has an infinite discontinuity at $x=1$.

$$\int_0^2 \frac{2x+1}{x^2+x-2}\,dx = \lim_{a\to 1^-}\int_0^a \frac{2x+1}{x^2+x-2}\,dx + \lim_{b\to 1^+}\int_b^2 \frac{2x+1}{x^2+x-2}\,dx$$

Use Part. Frac. Ex.

$$\frac{2x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$2x+1 = A(x-1) + B(x+2)$$

If $x=1$, $2(1)+1 = A(0) + B(3)$

$$3 = 3B$$
$$B = 1$$

If $x=-2$, $2(-2)+1 = A(-2-1) + B(0)$

$$-3 = -3A$$
$$A = 1$$

5a) (cont.)

$$\int_0^2 \frac{2x+1}{x^2+x-2}\,dx = \lim_{a\to 1^-} \int_0^a \frac{1}{x+2} + \frac{1}{x-1}\,dx$$

$$+ \lim_{b\to 1^+} \int_b^2 \frac{1}{x+2} + \frac{1}{x-1}\,dx$$

$$= \lim_{a\to 1^-} \ln|x+2| + \ln|x-1| \Big|_0^a$$

$$+ \lim_{b\to 1^+} \ln|x+2| + \ln|x-1| \Big|_b^2$$

$$= \lim_{a\to 1^-} \ln|a+2| + \ln|a-1| - (\ln(2) + \ln(1))$$

$$+ \lim_{b\to 1^+} \ln(4) + \ln(1) - (\ln|2+b| + \ln|b-1|)$$

Since $\lim\limits_{a\to 1^-} \overbrace{\ln|a+2|}^{\ln(3)} + \overbrace{\ln|a-1|}^{-\infty} - (\ln(2) + \ln(1)) = -\infty$,

then $\boxed{\int_0^2 \frac{2x+1}{x^2+x-2}\,dx \text{ diverges.}}$

5 b) $$\int_0^{\infty} t e^{-2t} dt = \lim_{a\to\infty} \int_0^a t e^{-2t} dt$$

Use By Parts.

$u = t \qquad dv = e^{-2t} dt$

$du = dt \qquad v = \frac{-1}{2} e^{-2t}$

$$\lim_{a\to\infty} \int_0^a t e^{-2t} dt = \lim_{a\to\infty} \left(-\tfrac{1}{2} t e^{-2t} \Big|_0^a - \int_0^a -\tfrac{1}{2} e^{-2t} dt \right)$$

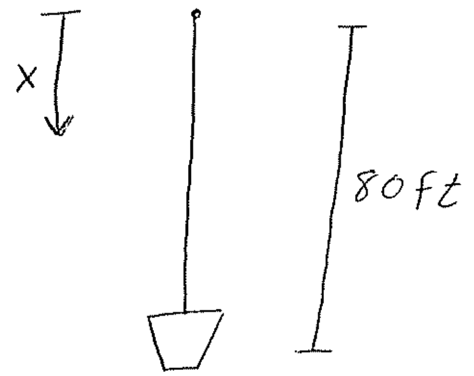
$$= \lim_{a\to\infty} \left(-\tfrac{1}{2} t e^{-2t} - \tfrac{1}{4} e^{-2t} \right) \Big|_0^a$$

$$= \lim_{a\to\infty} \left(-\tfrac{1}{2} \cdot \frac{a}{e^{2a}} - \frac{1}{4e^{2a}} \right) - \left(-\tfrac{1}{2} 0 e^0 - \tfrac{1}{4} e^0 \right)$$

$$= \lim_{a\to\infty} \left(\cancel{\frac{-a}{2e^{2a}}}^{\,0} - \cancel{\frac{1}{4e^{2a}}}^{\,0} + \frac{1}{4} \right)$$

$$= \boxed{\frac{1}{4}}$$

6.



Let x be the distance down the well measured in ft.

weight of bucket at bottom: $40+4=44$ lb.

Time it takes to pull bucket up: $\frac{80\text{ ft}}{2\text{ ft/s}}=40$ s.

weight of bucket at top: $44-(.1\text{ lb/s})(40\text{s})=40$ lb.

Since water is leaking out at a constant rate there is a linear relationship between the weight of the bucket and x.

Let $w(x)$ = weight of bucket when the bucket is x feet from the top of the well.

We know $w(0)=40$ and $w(80)=44$.

So, $w(x)=40+\frac{1}{20}x$.

The work done to pull bucket up $=\int_0^{80} 40+\frac{1}{20}x\,dx$

$$=40x+\frac{1}{40}x^2\Big|_0^{80}$$

$$=(3200+160)-(0+0)$$

$$=\boxed{3360\text{ ft}\cdot\text{lb}}$$