240B   HW 5   Winter 2013

Due: Wednesday, 03/13.

1. Let (M, g) be Riemannian manifold and ∇ its Levi-Civita connection. Let $p \in M$. Fix an orthonormal basis $e_1, \cdots, e_n$ of T_pM. Recall that $\exp_p$ is a diffeomorphism from $B_r(0) \subset T_pM$ onto $B_r(p) \subset M$ for $r > 0$ sufficiently small. Fix such an r. Now for each e_i, define a vector field E_i on $B_r(p)$ by $E_i|_{\exp_p v} =$ the parallel transport of e_i along the geodesic $\gamma(t) = \exp_p(tv)$. Show that $E_1, \cdots, E_n$ is an orthonormal basis of T_qM for $q \in B_r(p)$ and $(\nabla_X E_i)(p) = 0$ for all vector fields X. In particular, $(\nabla_{E_i} E_j)(p) = 0$. Such a (nice) basis is called a geodesic frame at p.

2. Continue from 1. Show that for any smooth function f,

$$(\nabla f)(p) = \sum_{i=1}^{n} (E_i f)(p) E_i(p), \qquad (\Delta f)(p) = \sum_{i=1}^{n} (E_i(E_i f))(p).$$

Deduce that for any smooth function g,

$$\Delta(fg) = f\Delta g + g\Delta f + 2\langle \nabla f, \nabla g \rangle.$$

3. do Carmo p106: 7.

4. do Carmo p119: 1 **or** p152: 2.