240B HW 4 Winter 2013

Due: Friday, 02/29.

1. Let (M, g) be Riemannian manifold and ∇ its Levi-Civita connection. For a smooth function f on M, define its Hessian to be the linear operator

$$\text{Hess}_f: \ \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

via

$$\text{Hess}_f(X) = \nabla_X \nabla f,$$

where ∇f is the gradient of f (defined in the first homework assignment). Clearly Hess_f is tensorial and hence can be viewed as a linear map on each of the tangent spaces:

$$\text{Hess}_f: \ T_pM \longrightarrow T_pM.$$

Define the Laplacian to be

$$\Delta f = \text{tr}(\text{Hess}_f),$$

where the trace is the usual trace on linear maps. Derive a formula for Δf in local coordinates. (Remark: some of us, including me, prefer to put a minus sign in the definition of the Laplacian, thus making it *posiitive*!)

2. Let (M, g) be a Riemannian manifold and f a smooth function on M. For any geodesic $\gamma(t)$, compute the first and second derivative of $f(\gamma(t))$ in terms of the gradient and Hessian of f. Hence conclude that f has nonnegative Hessian iff $f(\gamma(t))$ is convex for any geodesic γ.

3. do Carmo p84: 10.

4. Show that there is a complete Riemannian metric on any manifold. (Our manifolds are assumed to be Hausdorff and second countable.) (**Hint**: from Topology, you can find a nested sequence of open sets U_k such that the closure $\overline{U}_k$ is compact, $\overline{U}_k \subset U_{k+1}$, and $\cup_k U_k = M$. Use Hopf-Rinow to guide you to such a metric.)