(4.6) Clearly the patch $\boldsymbol{\sigma}$ covers the graph of f. Since f is smooth, $\boldsymbol{\sigma}$ is smooth. We then calculate $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-\frac{\partial f}{\partial u}, -\frac{\partial f}{\partial v}, 1)$, which is never zero, and so $\boldsymbol{\sigma}$ is in fact regular.

(4.7) Recall that the maps $\boldsymbol{\sigma}^x_\pm \colon U \to S^2$ are defined by

$$\boldsymbol{\sigma}^x_\pm(u,v) = (\pm\sqrt{1-u^2-v^2}, u, v),$$

where $U = \{(u,v)\colon u^2+v^2 < 1$, and similar definitions are given for $\boldsymbol{\sigma}^y_\pm$ and $\boldsymbol{\sigma}^z_\pm$. We will show that $\boldsymbol{\sigma}^x_+$ is regular, as the proof for the other coordinate patches is identical. We can compute

$$(\boldsymbol{\sigma}^x_+)_u \times (\boldsymbol{\sigma}^x_+)_v = \left(1, \frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}\right),$$

which is certainly never zero. Since we restrict to $u^2+v^2<1$, $\boldsymbol{\sigma}^x_+$ is smooth, and so is in fact regular.

To find the transition maps, first note that the images of $\boldsymbol{\sigma}^x_+$ and $\boldsymbol{\sigma}^x_-$ do not overlap, so there is no transition map between those two charts. Similarly there are no transition maps between $\boldsymbol{\sigma}^y_+$ and $\boldsymbol{\sigma}^y_-$ and between $\boldsymbol{\sigma}^z_+$ and $\boldsymbol{\sigma}^z_-$. Now, the transition map from $\boldsymbol{\sigma}^x_+$ to $\boldsymbol{\sigma}^y_\pm$ is the function $\Phi_\pm$ defined by $\Phi_\pm = (\boldsymbol{\sigma}^x_+)^{-1} \circ \boldsymbol{\sigma}^y_\pm$. This tells us that the domain of $\Phi_\pm$ must be the set $U' = \{(u,v) \in U \colon u > 0\}$, as these are the values for which $\boldsymbol{\sigma}^y_\pm$ gives a positive x value, and hence lie in the image of $\boldsymbol{\sigma}^x_+$. Denoting $(\widetilde{u}, \widetilde{v}) = \Phi_\pm(u,v)$, this means that $\boldsymbol{\sigma}^x_+(\widetilde{u}, \widetilde{v}) = \boldsymbol{\sigma}^y_\pm(u,v)$. Hence we see that $\widetilde{u} = \pm\sqrt{1-u^2-v^2}$ and $\widetilde{v} = v$. As $u^2+v^2<1$, the formulas for $\widetilde{u}$ and $\widetilde{v}$, and hence $\Phi_\pm$, are smooth.

Similarly, one could compute the other transition functions as:

$$\begin{aligned}
\boldsymbol{\sigma}^x_- &\to \boldsymbol{\sigma}^y_\pm : \Phi_\pm(u,v) = (\pm\sqrt{1-u^2-v^2}, v), u < 0\\
\boldsymbol{\sigma}^x_+ &\to \boldsymbol{\sigma}^z_\pm : \Phi_\pm(u,v) = (v, \pm\sqrt{1-u^2-v^2}), u > 0\\
\boldsymbol{\sigma}^x_- &\to \boldsymbol{\sigma}^z_\pm : \Phi_\pm(u,v) = (v, \pm\sqrt{1-u^2-v^2}), u < 0\\
\boldsymbol{\sigma}^y_+ &\to \boldsymbol{\sigma}^z_\pm : \Phi_\pm(u,v) = (u, \pm\sqrt{1-u^2-v^2}), v > 0\\
\boldsymbol{\sigma}^y_- &\to \boldsymbol{\sigma}^z_\pm : \Phi_\pm(u,v) = (u, \pm\sqrt{1-u^2-v^2}), v < 0.
\end{aligned}$$

To get the final maps, just take the inverse of the appropriate transition map above.

(4.8) Restrict to the region $R = \{(r,\theta) \in (0,\infty) \times \mathbb{R}\}$ so that $\boldsymbol{\sigma}$ is injective and defined on an open region (this is because r^2 is not injective in general). It is easy to then check that $\boldsymbol{\sigma}$ maps onto the part of the hyperbolic cylinder with $z > 0$. Another possible parametrization we could have used is $\widetilde{\boldsymbol{\sigma}}(u,v) = (u, v, u^2 - v^2)$ defined on the region $\{(u,v)\colon u^2 - v^2 > 0\}$, coming from the setup from Exercise 4.6. Setting $r = \sqrt{u^2-v^2}$ and $\theta = \cosh^{-1}(\frac{u}{\sqrt{u^2-v^2}})$, and noting that since $u^2 - v^2 > 0$, the expressions for r and θ are well-defined and in fact smooth in u and v, we see that $\widetilde{\boldsymbol{\sigma}}$ is a reparametrization of $\boldsymbol{\sigma}$.

To parametrize the region $z < 0$, we can use the two analogous parametrizations

$$\begin{aligned}
\boldsymbol{\sigma}(r,\theta) &= (r\sinh\theta, r\cosh\theta, -r^2)\\
\widetilde{\boldsymbol{\sigma}}(u,v) &= (u, v, u^2 - v^2)
\end{aligned}$$

on the regions $\{(r,\theta) \in (-\infty, 0) \times \mathbb{R}\}$ and $\{(u,v)\colon u^2 - v^2 < 0\}$. The reparametrization is then given by $r = \sqrt{v^2 - u^2}$ and $\theta = \cosh^{-1}(\frac{v}{\sqrt{v^2-u^2}})$.

(4.12) Let Σ be a smooth surface and let $\boldsymbol{\sigma}$ be a regular surface patch. Let $\mathbf{a}$ be a vector and A be an invertible linear transformation, and define $\boldsymbol{\sigma}_1 = \boldsymbol{\sigma} + \mathbf{a}$, $\boldsymbol{\sigma}_2 = A \circ \boldsymbol{\sigma}$. Since translations and linear transformations are both smooth, $\boldsymbol{\sigma}_i$ are both smooth. We then calculate $(\boldsymbol{\sigma}_1)_u = \boldsymbol{\sigma}_u$, $(\boldsymbol{\sigma}_1)_v = \boldsymbol{\sigma}_v$, $(\boldsymbol{\sigma}_2)_u = A \circ \boldsymbol{\sigma}_u$, and $(\boldsymbol{\sigma}_2)_v = A \circ \boldsymbol{\sigma}_v$. As A is invertible, these shows that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are regular. Doing this for the atlas of Σ then shows us that we get an atlas for the surfaces Σ_1 and Σ_2 coming from translation by $\mathbf{a}$ and the linear transformation A, respectively.

(4.13) (i) $-2x + 2y + z = 0$.
(ii) $-2x - 2y + z = 0$.

(4.14) Each point of the propellor will sweep out a helix, and we've already seen that a helix with radius v is parametrized by $u \mapsto (v\cos u, v\sin u, \lambda u)$ for some constant λ. Hence letting v vary, we get the given parametrization of the helicoid. Next, we calculate the standard unit normal of $\boldsymbol{\sigma}$ as $\mathbf{N} = (\lambda^2 + v^2)^{-1/2}(-\lambda \sin u, \lambda \cos u, v)$. The angle this makes with the z-axis is θ, where θ is given by $\cos\theta = \frac{v}{(\lambda^2+v^2)^{1/2}}$. Hence $\cot\theta = \pm\frac{v}{\lambda}$. But the distance from the z-axis is given by v, so $\cot\theta$ is proportional to the distance from the z-axis.

(4.15) Let $\widetilde{\boldsymbol{\sigma}}(\widetilde{u}, \widetilde{v}) = \boldsymbol{\sigma}(u,v)$ be a reparametrization of $\boldsymbol{\sigma}$. Then we calculate

$$\widetilde{\boldsymbol{\sigma}}_{\widetilde{u}} = \frac{\partial u}{\partial \widetilde{u}}\boldsymbol{\sigma}_u + \frac{\partial v}{\partial \widetilde{u}}\boldsymbol{\sigma}_v$$
$$\widetilde{\boldsymbol{\sigma}}_{\widetilde{v}} = \frac{\partial u}{\partial \widetilde{v}}\boldsymbol{\sigma}_u + \frac{\partial v}{\partial \widetilde{v}}\boldsymbol{\sigma}_v.$$

Thus $\widetilde{\boldsymbol{\sigma}}_{\widetilde{u}}$ and $\widetilde{\boldsymbol{\sigma}}_{\widetilde{v}}$ are in the span of $\boldsymbol{\sigma}_u$ and $\boldsymbol{\sigma}_v$, and so too are linear combinations of $\widetilde{\boldsymbol{\sigma}}_{\widetilde{u}}$ and $\widetilde{\boldsymbol{\sigma}}_{\widetilde{v}}$.

(4.16) Let $\gamma(t) = (x(t), y(t), z(t))$ be a curve in S. Then $f(\gamma(t)) = 0$, and differentiating, we see that

$$0 = \nabla f \cdot \gamma'.$$

Hence ∇f is perpendicular to the tangent of γ. Since the tangent space of S is the collection of all tangent vectors to curves, this shows that ∇f is perpendicular to every tangent plane of S.

Now, note that since ∇f is never zero, we can define $\mathbf{N} = \|\nabla f\|^{-1}\nabla f$, which is a smooth choice of a normal vector over the whole of S. Thus we need to show that this implies that S is orientable (the book states this as a fact, and leaves it to the reader to prove, so we give the proof). To see that this is the case, let $\mathcal{A}$ be the maximal atlas of S, and let $\mathcal{A}' \subset \mathcal{A}$ consist of those surface charts which have standard unit normal in the direction of $\mathbf{N}$. We need to show this is still an atlas, which will follow so long as we can show it covers S. But this is obvious, since if $\boldsymbol{\sigma} \in \mathcal{A}$, then $-\boldsymbol{\sigma} \in \mathcal{A}$, where $-\boldsymbol{\sigma}(u,v) = \boldsymbol{\sigma}(-u,v)$ is the same chart as $\boldsymbol{\sigma}$ except with the opposite orientation.

(4.18) As this is a surface of revolution, we use Example 4.13 to write down an atlas consisting of two surface patchs:

$$\boldsymbol{\sigma}(u,v) = (\cosh v \cos u, \cosh v \sin u, v) \text{ for } u \in \mathbb{R} \text{ and } v \in (-\pi, \pi)$$
$$\boldsymbol{\sigma}(u,v) = (\cosh v \cos u, \cosh v \sin u, v) \text{ for } u \in \mathbb{R} \text{ and } v \in (0, 2\pi)$$

As any angle in $[0, 2\pi]$ lies in one of the surface patches, these cover the catenoid, and so we indeed have an atlas.

(4.19) One can check that $\|\boldsymbol{\sigma}\| = 1$, showing that $\boldsymbol{\sigma}$ parametrizes part of the unit sphere. We then note that $\boldsymbol{\sigma}$ is also the parametrization of the surface of revolution obtained from rotating the curve $t \mapsto (\operatorname{sech} t, 0, \tanh t)$ about the z-axis. This curve is regular, and since $\operatorname{sech} > 0$, the curve never touches the z-axis. Thus, by Example 4.13, $\boldsymbol{\sigma}$ is a regular surface patch.

Parallels are then the curves in the surface which come from rotating a fixed point about the z-axis; that is, by setting v to be constant. Meridians are the curves in the surface which are the images of the original curve after rotating through a fixed angle; that is, by setting u to be constant.

(5.1) Let g denote the first fundamental form. Then we have

(i) $g = 2\cosh^2 u \; du^2 + 4 \sinh u \cosh u \sinh v \cosh v \; dudv + \sinh^2 u \; dv^2$. This surface is a quadric cone (see Proposition 4.6).

(ii) $g = (2+4u^2)du^2 + 8uv \; dudv + (2+4v^2)dv^2$. This is a paraboloid of revolution (see Example 4.13).

(iii) $g = (\cosh^2 u + \sinh^2 u)du^2 + dv^2$. This is a hyperbolic cylinder (see Exercise 4.8).

(iv) $g = (1+4u^2)du^2 + 8uv \; dudv + (1+4v^2)dv^2$. This is a paraboloid of revolution.