Solutions
Math 147A
Winter 2012
Homework 6

2.7 This surface is the collection of points that are a distance a from the curve γ.
By the Frenet-Serret equations

$$\begin{aligned}\boldsymbol{\sigma}_s &= \mathbf{t} + a((-\kappa\mathbf{t} + \tau\mathbf{b})\cos\theta - \tau\mathbf{n}\sin\theta),\\ &= (1 - a\kappa\cos\theta)\mathbf{t} - a\tau\sin\theta\mathbf{n} + a\tau\cos\theta\mathbf{b}.\end{aligned}$$

The theta derivative can be computed directly

$$\boldsymbol{\sigma}_\theta = -a\sin\theta\mathbf{n} + a\cos\theta\mathbf{b}.$$

Thus

$$\boldsymbol{\sigma}_s \times \boldsymbol{\sigma}_\theta = -a\cos\theta(1 - a\kappa\cos\theta)\mathbf{n} - a\sin\theta(1 - a\kappa\cos\theta)\mathbf{b},$$

which is only zero if $a\kappa\cos\theta = 1$. This will never happen if $\kappa < a^{-1}$ as this implies $a\kappa < 1$ and thus $a\kappa\cos\theta < 1$.

4.3 First, using the chain rule, note that

$$\left.\frac{d}{dt}\right|_{t=t_0} F(\gamma(t)) = \nabla F(\mathbf{p}) \cdot \dot{\gamma}(t_0),$$

but since γ is contained in the surface $\mathcal{S}$, $\dot{\gamma}$ is tangent to $\mathcal{S}$. Thus

$$\nabla F \cdot \dot{\gamma} = \nabla_{\mathcal{S}} F \cdot \dot{\gamma} + (\nabla F \cdot \mathbf{N})(\dot{\gamma} \cdot \mathbf{N}) = \nabla_{\mathcal{S}} F \cdot \dot{\gamma},$$

where $\mathbf{N}$ is some unit vector normal to $\mathcal{S}$.

If $\nabla_{\mathcal{S}} F = 0$ then the above calculation shows that for every curve, γ, in $\mathcal{S}$ passing through $\mathbf{p}$ at $t = t_0$, the function $F \circ \gamma$ has a local maximum or minimum at t_0. However it could be a local maximum for some curves and a local minimum for other curves, in which case F restricted to the surface $\mathcal{S}$ would have neither a local minimum nor a local maximum at $\mathbf{p}$, but rather a saddle point.

1.1 i. Since

$$\boldsymbol{\sigma}_u = (\cosh u \sinh v, \cosh u \cosh v, \cosh u)$$

and

$$\boldsymbol{\sigma}_u = (\sinh u \cosh v, \sinh u \sinh v, 0),$$

$$\begin{aligned} E &= \|\boldsymbol{\sigma}_u\|^2 = & 2\cosh^2 u \cosh^2 v,\\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = & \frac{1}{2}\sinh 2u \sinh 2v,\\ G &= \|\boldsymbol{\sigma}_v\|^2 = & \sinh^2 u \cosh 2v.\end{aligned}$$

Thus the first fundamental form is given by

$$2\cosh^2 u \cosh^2 v du^2 + \sinh 2u \sinh 2v du dv + \sinh^2 u \cosh 2v dv^2.$$

ii. Since

$$\boldsymbol{\sigma}_u = (1, 1, 2u)$$

and

$$\boldsymbol{\sigma}_u = (-1, 1, 2v),$$

$$\begin{aligned} E &= \|\boldsymbol{\sigma}_u\|^2 = & 2 + 4u^2, \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = & 4uv, \\ G &= \|\boldsymbol{\sigma}_v\|^2 = & 2 + 4v^2. \end{aligned}$$

Thus the first fundamental form is given by

$$(2 + 4u^2)du^2 + 8uvdudv + (2 + 4v^2)dv^2.$$

iii. Since

$$\boldsymbol{\sigma}_u = (\sinh u, \cosh u, 0)$$

and

$$\boldsymbol{\sigma}_u = (0, 0, 1),$$

$$\begin{aligned} E &= \|\boldsymbol{\sigma}_u\|^2 = & \cosh 2u, \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = & 0, \\ G &= \|\boldsymbol{\sigma}_v\|^2 = & 1. \end{aligned}$$

Thus the first fundamental form is given by

$$\cosh 2u du^2 + dv^2.$$

iv. Since

$$\boldsymbol{\sigma}_u = (1, 0, 2u)$$

and

$$\boldsymbol{\sigma}_u = (0, 1, 2v),$$

$$\begin{aligned} E &= \|\boldsymbol{\sigma}_u\|^2 = & 1 + 4u^2, \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = & 4uv, \\ G &= \|\boldsymbol{\sigma}_v\|^2 = & 1 + 4v^2. \end{aligned}$$

Thus the first fundamental form is given by

$$(1 + 4u^2)du^2 + 8uvdudv + (1 + 4v^2)dv^2.$$

1.4 i. Let $\mathbf{v} = a\boldsymbol{\sigma}_{\tilde{u}} + b\boldsymbol{\sigma}_{\tilde{v}}$, then

$$\begin{aligned} \mathbf{v} &= a\left(\frac{\partial u}{\partial \tilde{u}}\boldsymbol{\sigma}_u + \frac{\partial v}{\partial \tilde{u}}\boldsymbol{\sigma}_v\right) + b\left(\frac{\partial u}{\partial \tilde{v}}\boldsymbol{\sigma}_u + \frac{\partial v}{\partial \tilde{v}}\boldsymbol{\sigma}_v\right), \\ &= \left(a\frac{\partial u}{\partial \tilde{u}} + b\frac{\partial u}{\partial \tilde{v}}\right)\boldsymbol{\sigma}_u + \left(a\frac{\partial v}{\partial \tilde{u}} + b\frac{\partial v}{\partial \tilde{v}}\right)\boldsymbol{\sigma}_v. \end{aligned}$$

Thus

$$du(\mathbf{v}) = a\frac{\partial u}{\partial \tilde{u}} + b\frac{\partial u}{\partial \tilde{v}} = \frac{\partial u}{\partial \tilde{u}}d\tilde{u}(\mathbf{v}) + \frac{\partial u}{\partial \tilde{v}}d\tilde{v}(\mathbf{v}),$$

showing that $du = \frac{\partial u}{\partial \tilde{u}} d\tilde{u} + \frac{\partial u}{\partial \tilde{v}} d\tilde{v}$. Similarly

$$dv(\mathbf{v}) = a\frac{\partial v}{\partial \tilde{u}} + b\frac{\partial v}{\partial \tilde{v}} = \frac{\partial v}{\partial \tilde{u}} d\tilde{u}(\mathbf{v}) + \frac{\partial v}{\partial \tilde{v}} d\tilde{v}(\mathbf{v}),$$

showing that $dv = \frac{\partial v}{\partial \tilde{u}} d\tilde{u} + \frac{\partial v}{\partial \tilde{v}} d\tilde{v}$.

ii. The first fundamental form is given by

$$\begin{pmatrix} d\tilde{u} & d\tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

Substituting

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix} \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix} = J \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix},$$

from part (i) shows that

$$\begin{pmatrix} d\tilde{u} & d\tilde{v} \end{pmatrix} \begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix} = \begin{pmatrix} d\tilde{u} & d\tilde{v} \end{pmatrix} J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J \begin{pmatrix} d\tilde{u} \\ d\tilde{v} \end{pmatrix}.$$

Thus

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = J^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} J.$$

1.5 $(i) \Rightarrow (ii)$

Assume $E_v = G_u = 0$ then

$$E_v = 2\boldsymbol{\sigma}_{uv} \cdot \boldsymbol{\sigma}_u = 0,$$

and

$$G_u = 2\boldsymbol{\sigma}_{uv} \cdot \boldsymbol{\sigma}_v = 0.$$

Thus $\boldsymbol{\sigma}_{uv}$ must be normal to the surface.

$(ii) \Rightarrow (iii)$

Assume that $\boldsymbol{\sigma}_{uv}$ is normal to the surface and consider the parallelogram

$$\boldsymbol{\sigma}([u_0, u_1] \times [v_0, v_1]),$$

the length of the $v = v_0$ side is given by

$$\int_{u_0}^{u_1} \|\boldsymbol{\sigma}_u(u, v_0)\| du.$$

However this does not depend on v as

$$\frac{d}{dv} \int_{u_0}^{u_1} \|\boldsymbol{\sigma}_u(u, v)\| du = \int_{u_0}^{u_1} \frac{\boldsymbol{\sigma}_{uv} \cdot \boldsymbol{\sigma}_u}{\|\boldsymbol{\sigma}_u(u, v)\|} du = 0.$$

Thus the length of the $v = v_0$ and $v = v_1$ side must be equal, and similarly for the $u = u_0$ and $u = u_1$ side.

$(iii) \Rightarrow (i)$

Assume that the length of a paramater curve $u \mapsto \boldsymbol{\sigma}(u, v)$,

$$\int_{u_0}^{u_1} \sqrt{E(u, v)} du,$$

does not depend on v, then

$$\frac{d}{dv}\int_{u_0}^{u_1}\sqrt{E(u,v)}du=\int_{u_0}^{u_1}\frac{E_v(u,v)}{2\sqrt{E(u,v)}}du=0.$$

Thus

$$0=\lim_{u_1\to u_0}\frac{1}{u_1-u_0}\int_{u_0}^{u_1}\frac{E_v(u,v)}{2\sqrt{E(u,v)}}du=\frac{E_v(u_0,v)}{2\sqrt{E(u_0,v)}}.$$

Since this is true for all u_0 it must be that $E_v=0$. A similar argument using the v-parameter curves will show that $G_u=0$.