Solutions
Math 147A
Winter 2012
Homework 5

2.1 Since u, v, and $f(u,v)$ are all smooth, the surface patch

$$\boldsymbol{\sigma}(u,v) = (u, v, f(u,v)),$$

is smooth. To check regularity compute

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1, 0, f_u(u,v)),\\ \boldsymbol{\sigma}_v &= (0, 1, f_v(u,v)),\end{aligned}$$

and

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-f_u, -f_v, 1),$$

which is never zero.

2.2 To check that the surface patch

$$\boldsymbol{\sigma}_+^x(u,v) = (\sqrt{1-u^2-v^2}, u, v),$$

is regular compute it partial derivatives

$$\begin{aligned}\boldsymbol{\sigma}_{+u}^x &= \left(\frac{-u}{\sqrt{1-u^2-v^2}}, 1, 0\right),\\ \boldsymbol{\sigma}_{+v}^x &= \left(\frac{-v}{\sqrt{1-u^2-v^2}}, 0, 1\right),\end{aligned}$$

and

$$\boldsymbol{\sigma}_{+u}^x \times \boldsymbol{\sigma}_{+v}^x = \left(1, \frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}\right),$$

which is never zero. Similarly, the other surface patches are also regular.

The transition function

$$\boldsymbol{\Phi}(\tilde{u},\tilde{v}) = (\boldsymbol{\sigma}_+^x)^{-1} \circ \boldsymbol{\sigma}_+^z(\tilde{u},\tilde{v}) = (\tilde{v}, \sqrt{1-\tilde{u}^2-\tilde{v}^2})$$

is smooth. The other 23 transition functions also consist of one of the coordinates and $\sqrt{1-\tilde{u}^2-\tilde{v}^2}$ thus are also smooth.

2.3 i. The map $\boldsymbol{\sigma}$ is injective as $(u,v,uv) = (x,y,xy)$ implies $(u,v) = (x,y)$. Since

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1, 0, v),\\ \boldsymbol{\sigma}_v &= (0, 1, u),\\ \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (-v, -u, 1),\end{aligned}$$

this is a regular surface patch.

ii. The map $\boldsymbol{\sigma}$ is injective as $(u, v^2, v^3) = (x, y^2, y^3)$ implies $u = x$ and $v^3 = y^3$, hence $(u, v) = (x, y)$. However,

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1, 0, 0),\\ \boldsymbol{\sigma}_v &= (0, 2v, 3v^2),\\ \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (0, -3v^2, 2v),\end{aligned}$$

showing this surface patch is not regular on the line $v = 0$.

iii. The map $\boldsymbol{\sigma}$ is not injective as $\boldsymbol{\sigma}(0, 1) = (0, 1, 1) = \boldsymbol{\sigma}(-1, 1)$. Also,

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1 + 2u, 0, 0),\\ \boldsymbol{\sigma}_v &= (0, 1, 2v),\\ \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (0, -2v(1 + 2u), 1 + 2u)\end{aligned}$$

showing this map is not regular on the line $u = -1/2$.

4.1 The normal vector is $\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v(1, 1)$. Since

i.

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1, 0, 2u),\\ \boldsymbol{\sigma}_v &= (0, 1, -2v),\\ \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (-2u, 2v, 1),\end{aligned}$$

which is $(-2, 2, 1)$ at $(1, 1, 0)$, thus the tangent plane is given by

$$-2x + 2y + z = 0$$

ii. The normal vector is $\boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_\theta(1, 0)$. Since

$$\begin{aligned}\boldsymbol{\sigma}_r(1, 0) &= (1, 0, 2),\\ \boldsymbol{\sigma}_\theta(1, 0) &= (0, 1, 0),\\ \boldsymbol{\sigma}_r \times \boldsymbol{\sigma}_\theta(1, 0) &= (-2, 0, 1)\end{aligned}$$

which is $(-2, 2, 1)$ at $(1, 1, 0)$, thus the tangent plane is given by

$$-2x + z = 0$$

4.2 Let $\tilde{\boldsymbol{\sigma}}(\tilde{u}, \tilde{v}) = \boldsymbol{\sigma}(u, v)$, then

$$\begin{aligned}\boldsymbol{\sigma}_u &= \frac{\partial \tilde{u}}{\partial u}\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial u}\tilde{\boldsymbol{\sigma}}_{\tilde{v}},\\ \boldsymbol{\sigma}_v &= \frac{\partial \tilde{u}}{\partial v}\tilde{\boldsymbol{\sigma}}_{\tilde{u}} + \frac{\partial \tilde{v}}{\partial v}\tilde{\boldsymbol{\sigma}}_{\tilde{v}}.\end{aligned}$$

This shows that $\operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} \subset \operatorname{span}\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$. The reverse containment is also true as

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}_{\tilde{u}} &= \frac{\partial u}{\partial \tilde{u}}\boldsymbol{\sigma}_u + \frac{\partial v}{\partial \tilde{u}}\boldsymbol{\sigma}_v,\\ \tilde{\boldsymbol{\sigma}}_{\tilde{v}} &= \frac{\partial u}{\partial \tilde{v}}\boldsymbol{\sigma}_u + \frac{\partial v}{\partial \tilde{v}}\boldsymbol{\sigma}_v.\end{aligned}$$

Thus $\operatorname{span}\{\boldsymbol{\sigma}_u, \boldsymbol{\sigma}_v\} = \operatorname{span}\{\tilde{\boldsymbol{\sigma}}_{\tilde{u}}, \tilde{\boldsymbol{\sigma}}_{\tilde{v}}\}$.