Solutions
147a Winter 2012
Homework 4

3.2 Assume that γ is paramaterized by arclength, then $\mathbf{t} = \dot{\gamma}$,

$$\dot{\mathbf{n}} = -\kappa\mathbf{t} + \tau\mathbf{b},$$

and since κ and τ are constant

$$\ddot{\mathbf{n}} = -(\kappa^2 + \tau^2)\mathbf{n}.$$

This implies that

$$\mathbf{n} = \mathbf{a}_1 \cos(\omega s) + \mathbf{a}_2 \sin(\omega s),$$

where $\omega = \sqrt{\kappa^2 + \tau^2}$ and $\mathbf{a}_1$ and $\mathbf{a}_2$ are some constant vectors. Since $\mathbf{n}$ is a unit vector it must be perpendicular to

$$\dot{\mathbf{n}} = -\mathbf{a}_1\omega \sin(\omega s) + \mathbf{a}_2\omega \cos(\omega s),$$

but

$$\mathbf{n} \cdot \dot{\mathbf{n}} = (\|\mathbf{a}_2\| - \|\mathbf{a}_1\|^2)\omega \cos(\omega s)\sin(\omega s) + \mathbf{a}_1 \cdot \mathbf{a}_2\omega(\cos^2(\omega s) - \sin^2(\omega s)),$$

which for $s = \frac{\pi}{4\omega}$ implies that $\|\mathbf{a}_2\| = \|\mathbf{a}_1\|^2 = 1$, and for $s = \frac{\pi}{2\omega}$ implies that $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$. Multiplying $\mathbf{n}$ by κ and integrating shows that

$$\mathbf{t} = \frac{\kappa}{\omega}(\mathbf{a}_1 \sin(\omega s) - \mathbf{a}_2 \cos(\omega s)) + \mathbf{a}_3,$$

where $\mathbf{a}_3$ is another constant vector, which must be perpendicular to $\mathbf{a}_1$ and $\mathbf{a}_2$ for $\mathbf{t}$ to be perpendicular to $\mathbf{n}$. Rewriting the second Frenet-Serret equation gives

$$\begin{aligned}
-\mathbf{a}_1\omega \sin(\omega s) + \mathbf{a}_2\omega \cos(\omega s) &= \frac{\kappa^2}{\omega}(-\mathbf{a}_1 \sin(\omega s) + \mathbf{a}_2 \cos(\omega s)) - \kappa\mathbf{a}_3 + \tau\mathbf{b},\\
\omega^2(-\mathbf{a}_1 \sin(\omega s) + \mathbf{a}_2 \cos(\omega s)) &= \kappa^2(-\mathbf{a}_1 \sin(\omega s) + \mathbf{a}_2 \cos(\omega s)) - \kappa\omega\mathbf{a}_3 + \tau\omega\mathbf{b},\\
\frac{\omega\tau^2}{\kappa}(\mathbf{a}_3 - \mathbf{t}) &= -\kappa\omega\mathbf{a}_3 + \tau\omega\mathbf{b},\\
\tau^2\mathbf{a}_3 - \tau^2\mathbf{t} &= -\kappa^2\mathbf{a}_3 + \tau\kappa\mathbf{b},\\
\mathbf{a}_3 &= \frac{\tau}{\omega^2}(\tau\mathbf{t} + \kappa\mathbf{b}).
\end{aligned}$$

Finally choosing constant unit vectors $\mathbf{c}_1 = -\mathbf{a}_1$, $\mathbf{c}_2 = -\mathbf{a}_2$, and $\mathbf{c}_3 = \frac{1}{\omega}(\tau\mathbf{t} + \kappa\mathbf{b})$, the unit tangent can be written as

$$\mathbf{t} = \frac{\kappa}{\omega}(-\mathbf{c}_1 \sin(\omega s) + \mathbf{c}_2 \cos(\omega s)) + \frac{\tau}{\omega}\mathbf{c}_3,$$

and integrating gives the curve

$$\gamma(s) = \frac{\kappa}{\omega}(\mathbf{c}_1 \cos(\omega s) + \mathbf{c}_2 \sin(\omega s)) + \frac{\tau}{\omega}\mathbf{c}_3 s + \mathbf{d}.$$

This shows that γ is a rotation and translation, and possibly a reflection (if $\mathbf{c}_1 \times \mathbf{c}_2 = -\mathbf{c}_3$) of the helix

$$\left(\frac{\kappa}{\omega}\cos(\omega s), \frac{\kappa}{\omega}\sin(\omega s), \frac{\tau}{\omega}s\right).$$

3.4 If γ lies on a sphere centered at $\mathbf{p}$ with radius r, then $\|\gamma - \mathbf{p}\|^2 = r^2$, and taking a derivative yields $\gamma \cdot \mathbf{t} = 0$. Thus

$$\gamma(t) - \mathbf{p} = \alpha\mathbf{n} + \beta\mathbf{b},$$

and

$$\begin{aligned}\dot\gamma(t) &= \dot\alpha\mathbf{n} + \alpha(-\kappa\mathbf{t} + \tau\mathbf{b}) + \dot\beta\mathbf{b} - \beta\tau\mathbf{n},\\ &= -\alpha\kappa\mathbf{t} + (\dot\alpha - \beta\tau)\mathbf{n} + (\alpha\tau + \dot\beta)\mathbf{b}.\end{aligned}$$

This shows that $\alpha = -\frac{1}{\kappa}$, $\beta = \frac{\dot\alpha}{\tau} = \frac{\dot\kappa}{\tau\kappa^2}$, and $\dot\beta = \frac{\tau}{\kappa}$. This is the desired equation

$$\frac{\tau}{\kappa} = \frac{d}{ds}\left(\frac{\dot\kappa}{\tau\kappa^2}\right).$$

To prove the converse, let $\rho = 1/\kappa$ and $\sigma = 1/\tau$, the equation becomes

$$\frac{\rho}{\sigma} = -\frac{d}{ds}(\dot\rho\sigma).$$

Multiplying by $2\dot\rho\sigma$ and moving both terms to the left gives

$$2\rho\dot\rho + 2\dot\rho\sigma\frac{d}{ds}(\dot\rho\sigma) = 0.$$

Integrating gives the equation

$$\rho^2 + (\dot\rho\sigma)^2 = r^2,$$

for some constant r. Now let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ be the unit tangent, normal and binormal vectors of γ and consider the spherical curve given by

$$\mu(t) = \rho\mathbf{n} + \dot\rho\sigma\mathbf{b}.$$

This has derivative

$$\begin{aligned}\dot\mu(t) &= \dot\rho\mathbf{n} + \rho(-\kappa\mathbf{t} + \tau\mathbf{b}) - \rho/\sigma\mathbf{b} - \dot\rho\sigma\tau\mathbf{n},\\ &= -\mathbf{t}.\end{aligned}$$

Thus $\gamma(t) + \mu(t) = \mathbf{p}$ for some constant $\mathbf{p}$. Showing that γ is a spherical curve, as it is a translation of a spherical curve.

For *Viviani's curve*

$$\gamma(t) = \left(\cos^2 t - \frac{1}{2}, \sin t\cos t, \sin t\right),$$

the derivatives are

$$\begin{aligned}\dot\gamma(t) &= (-\sin 2t, \cos 2t, \cos t),\\ \ddot\gamma(t) &= (-2\cos 2t, -2\sin 2t, -\sin t),\\ \dddot\gamma(t) &= (4\sin 2t, -4\cos 2t, -\cos t).\end{aligned}$$

Thus

$$\begin{aligned}\|\dot\gamma(t)\|^2 &= 1 + \cos^2 t\\ \dot\gamma(t) \times \ddot\gamma(t) &= (\sin t(\cos(2t) + 2), -2\cos^3 t, 2),\\ \|\dot\gamma(t) \times \ddot\gamma(t)\|^2 &= 5 + 3\cos^2 t,\\ (\dot\gamma(t) \times \ddot\gamma(t)) \cdot \dddot\gamma(t) &= 6\cos t,\end{aligned}$$

and

$$\kappa = \frac{\sqrt{5+3\cos^2 t}}{(1+\cos^2 t)^{\frac{3}{2}}},$$
$$\frac{d\kappa}{ds} = \frac{6\sin t\cos t(2+\cos^2 t)}{\sqrt{5+3\cos^2 t}(1+\cos^2 t)^3},$$
$$\tau = \frac{6\cos(t)}{5+3\cos^2 t},$$
$$\frac{\dot{\kappa}}{\tau\kappa^2} = \frac{\sin t(2+\cos^2 t)}{\sqrt{5+3\cos^2 t}}.$$

Finally, taking the derivative yields

$$\frac{d}{ds}\frac{\dot{\kappa}}{\tau\kappa^2} = 6\cos t\left(\frac{1+\cos^2 t}{5+3\cos^2 t}\right)^{\frac{3}{2}} = \frac{\tau}{\kappa}.$$

3.6 Let $b_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$ then these satisfy

$$\dot{b}_{ij} = \sum_k (a_{ik}\mathbf{v}_k \cdot \mathbf{v}_j + \mathbf{v}_i \cdot (a_{jk}\mathbf{v}_k)) = \sum_k (a_{ik}b_{kj} + b_{ik}a_{jk}).$$

For the matrices $A = (a_{ij})$ and $B = (b_{ij})$ this can be written succinctly as

$$\dot{B} = AB + BA^T = AB - BA,$$

while the initially orthonormal condition becomes

$$B(s_0) = I,$$

where I is the identity matrix. Now, $B = I$ is a constant solution to this system of equations. Uniqueness of solutions to systems of ODE's then implies that $B(s) = I$ for all s, thus the vectors remain orthonormal.

In fact, the only initial conditions which will give constant solutions for any skew symmetric matrix are $B(s_0) = cI$ for a constant c.