Solutions

147a Winter 2012

Homework 3

2.1 Since $\mathbf{n}_s$ is a unit vector, $\mathbf{n}_s \cdot \mathbf{n}_s = 1$ and $2\dot{\mathbf{n}}_s \cdot \mathbf{n}_s = 0$. Since γ is a plane curve, it follows that $\dot{\mathbf{n}}_s = a\mathbf{t}$. Taking the derivative of $\mathbf{t} \cdot \mathbf{n}_s = 0$ gives the equation

$$\dot{\mathbf{t}} \cdot \mathbf{n}_s + \mathbf{t} \cdot \dot{\mathbf{n}}_s = \kappa_s \mathbf{n}_s \cdot \mathbf{n}_s + a\mathbf{t} \cdot \mathbf{t} = \kappa_s + a = 0.$$

This shows that a must be minus the signed curvature and $\dot{\mathbf{n}}_s = -\kappa_s \mathbf{t}$, as desired.

3.1 i. The curve $\gamma(t) = (\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}})$ is unit speed, thus

$$\mathbf{t} = \dot{\gamma} = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right).$$

The unit normal is given by

$$\kappa\mathbf{n} = \ddot{\gamma} = \left(\frac{1}{4}(1+t)^{-1/2}, \frac{1}{4}(1-t)^{-1/2}, 0\right),$$

showing that the curvature is

$$\kappa = \frac{1}{\sqrt{8(1-t^2)}},$$

and the unit normal is

$$\mathbf{n} = \left(\frac{1}{\sqrt{2}}(1-t)^{1/2}, \frac{1}{\sqrt{2}}(1+t)^{1/2}, 0\right).$$

The binormal is then

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} = \left(-\frac{1}{2}(1+t)^{1/2}, \frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right),$$

and the torsion is given by

$$-\tau\mathbf{n} = \dot{\mathbf{b}} = \left(-\frac{1}{4}(1+t)^{-1/2}, -\frac{1}{4}(1-t)^{-1/2}, 0\right).$$

Thus

$$\tau = \frac{1}{\sqrt{8(1-t^2)}},$$

and to check the Frenet-Serret formulas compute

$$\dot{\mathbf{n}} = \left(-\frac{1}{2\sqrt{2}}(1-t)^{-1/2}, \frac{1}{2\sqrt{2}}(1+t)^{-1/2}, 0\right),$$

and

$$
\begin{aligned}
-\kappa\mathbf{t}+\tau\mathbf{b} &= \frac{1}{\sqrt{8(1-t^2)}}\left[-\left(\frac{1}{2}(1+t)^{1/2},-\frac{1}{2}(1-t)^{1/2},\frac{1}{\sqrt{2}}\right)\right. \\
&\qquad\left.+\left(-\frac{1}{2}(1+t)^{1/2},\frac{1}{2}(1-t)^{1/2},\frac{1}{\sqrt{2}}\right)\right], \\
&= \frac{1}{\sqrt{8(1-t^2)}}\left(-(1+t)^{1/2},(1-t)^{1/2},0\right), \\
&= \frac{1}{2\sqrt{2}}\left(-(1-t)^{-1/2},(1+t)^{-1/2},0\right).
\end{aligned}
$$

ii. As before, when $\gamma(t)=(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t)$,

$$\mathbf{t}=\dot{\gamma}=\left(-\frac{4}{5}\sin t,-\cos t,\frac{3}{5}\sin t\right),$$

$$\kappa\mathbf{n}=\ddot{\gamma}=\left(-\frac{4}{5}\cos t,\sin t,\frac{3}{5}\cos t\right),$$

and the curvature is just $\kappa=1$. The binormal is

$$\mathbf{b}=\mathbf{t}\times\mathbf{n}=\left(-\frac{3}{5},0,-\frac{4}{5}\right),$$

and thus the torsion is $\tau=0$. The formula

$$\dot{\mathbf{n}}=-\mathbf{t},$$

is the final Frenet-Serret formula and is easily seen to be true.
Now γ lies in a plane normal to $\mathbf{b}$, and passes through the origin when $t=\frac{\pi}{2}$. Thus γ lies in the plane $3x+4z=0$. Its center is the midpoint of $\gamma(0)$ and $\gamma(\pi)$, which is

$$\frac{1}{2}\left[\left(\frac{4}{5},1,-\frac{3}{5}\right)+\left(\frac{-4}{5},1,\frac{3}{5}\right)\right]=(0,1,0).$$

From the curvature, the radius should be 1, to check the diamater is the length of

$$\gamma(\pi)-\gamma(0)=\left(\frac{-4}{5},1,\frac{3}{5}\right)-\left(\frac{4}{5},1,-\frac{3}{5}\right)=\left(-\frac{8}{5},0,\frac{6}{5}\right),$$

which is $\sqrt{\frac{64+36}{25}}=2$, as expected.

3.3 Since $\mathbf{t}$ makes a constant angle, θ, with $\mathbf{a}$, it follows that $\mathbf{t}\cdot\mathbf{a}=\cos(\theta)$. Noting that $\mathbf{a}$ and θ are constant and taking the derivative of this equation yields $\kappa\mathbf{n}\cdot\mathbf{a}=0$. which shows that $\mathbf{n}$ and $\mathbf{a}$ are perpendicular. Let $\mathbf{b}\cdot\mathbf{a}=c$ then

$$\mathbf{a}=\cos(\theta)\mathbf{t}+c\mathbf{b}.$$

Since $\mathbf{a}$ is a unit vector $\|\mathbf{a}\|^2=\cos^2(\theta)+c^2=1$. This shows that $c=\pm\sin(\theta)$. The derivative of $\mathbf{n}\cdot\mathbf{a}=0$ is

$$\dot{\mathbf{n}}\cdot\mathbf{a}=-\kappa\mathbf{t}\cdot\mathbf{a}+\tau\mathbf{b}\cdot\mathbf{a}=-\kappa\cos(\theta)\pm\tau\sin(\theta)=0.$$

This gives the desired result, $\tau=\pm\kappa\cot(\theta)$.

For $\lambda \neq 0$, let $\theta = \arctan\left(\frac{1}{\lambda}\right)$ and $\mathbf{a} = \cos(\theta)\mathbf{t} + \sin(\theta)\mathbf{b}$, then γ will be a generalized helix provide $\mathbf{a}$ is fixed. This is evident from its derivative and the equation $\kappa = \tan(\theta)\tau$,

$$\dot{\mathbf{a}} = \cos(\theta)\kappa\mathbf{n} - \sin(\theta)\tau\mathbf{b} = (\cos(\theta)\tan(\theta)\tau - \sin(\theta)\tau)\mathbf{n} = 0.$$

For $\lambda = 0$ the torsion is zero and $\mathbf{b}$ is a constant vector with constant angle $\frac{\pi}{2}$ with the unit tangent.

The standard helix $(a\cos t, a\cos t, bt)$ has tangent $(-a\sin t, a\cos t, b)$ which has constant angle $\theta = \arccos(b/\sqrt{a^2+b^2})$ with the unit vector $(0,0,1)$. Thus it is also a generalized helix.

(1) Since $\gamma(t) = (e^t\cos t, e^t\sin t, e^t)$ is not unit speed, use $\kappa = \frac{\|\dot\gamma\times\ddot\gamma\|}{\|\dot\gamma\|^3}$ and $\tau = \frac{(\dot\gamma\times\ddot\gamma)\cdot\dddot\gamma}{\|\dot\gamma\times\ddot\gamma\|^2}$. Computing

$$\begin{aligned}
\dot\gamma &= (e^t(\cos t - \sin t), e^t(\sin t + \cos t), e^t),\\
\ddot\gamma &= (-2e^t\sin t, 2e^t\cos t, e^t),\\
\dddot\gamma &= (-2e^t(\sin t + \cos t), 2e^t(\cos t - \sin t), e^t),
\end{aligned}$$

and

$$\begin{aligned}
\|\dot\gamma\|^2 &= 3e^{2t},\\
\dot\gamma\times\ddot\gamma &= (e^{2t}(\sin t - \cos t), -e^{2t}(\cos t + \sin t), 2e^{2t}),\\
(\dot\gamma\times\ddot\gamma)\cdot\dddot\gamma &= 2e^{3t},
\end{aligned}$$

then yields

$$\begin{aligned}
\kappa &= \frac{\sqrt{2}}{3}e^{-t},\\
\tau &= \frac{2}{3}e^{-t}.
\end{aligned}$$

(2) Again $\gamma(t) = (\cosh t, \sinh t, t)$ is not unit speed, thus $\kappa = \frac{\|\dot\gamma\times\ddot\gamma\|}{\|\dot\gamma\|^3}$ and $\tau = \frac{(\dot\gamma\times\ddot\gamma)\cdot\dddot\gamma}{\|\dot\gamma\times\ddot\gamma\|^2}$. Computing

$$\begin{aligned}
\dot\gamma &= (\sinh t, \cosh t, 1),\\
\ddot\gamma &= (\cosh t, \sinh t, 0),\\
\dddot\gamma &= (\sinh t, \cosh t, 0),
\end{aligned}$$

and

$$\begin{aligned}
\|\dot\gamma\|^2 &= 2\cosh^2 t,\\
\dot\gamma\times\ddot\gamma &= (-\sinh t, \cosh t, -1),\\
(\dot\gamma\times\ddot\gamma)\cdot\dddot\gamma &= 1,
\end{aligned}$$

then yields

$$\begin{aligned}
\kappa &= \frac{1}{2}\mathrm{sech}^2 t,\\
\tau &= \frac{1}{2}\mathrm{sech}^2 t.
\end{aligned}$$