Solutions

147a Winter 2012

Homework 2

1.1 i. Recall that $\dot\gamma = (\frac{1}{2}(1+t)^{\frac{1}{2}}, -\frac{1}{2}(1-t)^{\frac{1}{2}}, \frac{1}{\sqrt{2}})$, and $\|\dot\gamma\| = 1$. Hence $\kappa(t) = \|\ddot\gamma(t)\|$, and since

$$\ddot\gamma(t) = (\frac{1}{4}(1+t)^{-\frac{1}{2}}, \frac{1}{2}(1-t)^{-\frac{1}{2}}, 0),$$

it follows that

$$\kappa(t) = \frac{1}{\sqrt{8(1-t^2)}}.$$

ii. Since $\dot\gamma = (-\frac{4}{5}\sin(t), -\cos(t), \frac{3}{5}\sin(t))$, and $\|\dot\gamma\| = 1$, it is again the case that $\kappa(t) = \|\ddot\gamma(t)\|$. The equation

$$\ddot\gamma = (-\frac{4}{5}\cos(t), \sin(t), \frac{3}{5}\cos(t)),$$

then implies that

$$\kappa(t) = \|\ddot\gamma(t)\| = 1.$$

iii. Since $\|\dot\gamma\| \neq 1$ the curvature is given by $\kappa(t) = \frac{\|\ddot\gamma\times\dot\gamma\|}{\|\dot\gamma\|^3}$. Now $\dot\gamma(t) = (1, \sinh(t))$ and $\ddot\gamma(t) = (0, \cosh(t))$. Therefore

$$\|\ddot\gamma \times \dot\gamma\| = \left\| \begin{matrix} 0 & \cosh(t) \\ 1 & \sinh(t) \end{matrix} \right\| = \cosh(t),$$

and

$$\kappa(t) = \text{sech}^2(t).$$

iv.

$$\kappa(t) = \frac{\|\ddot\gamma \times \dot\gamma\|}{\|\dot\gamma\|^3} = \frac{1}{3|\cos(t)\sin(t)|}.$$

This goes to ∞ when t is an integer multiple of $\pi/2$, at these time $(\cos^3 t, \sin^3 t) = (\pm1, 0), (0, \pm1)$.

1.2 Let $s(t)$ be the arclength from a fixed point p on the curve γ. Since γ is regular, by proposition 1.3.5 s is a smooth function of t. Since $\kappa(t) = \left\|\frac{d^2\gamma}{ds^2}(s(t))\right\|$ and $\|\cdot\| : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ is smooth $\kappa(t)$ can be given as a composition of smooth functions as long as $\frac{d^2\gamma}{ds^2} \neq (0,0)$. Of course $\frac{d^2\gamma}{ds^2} = (0,0)$ contradicts the assumption $\kappa(t) > 0$, showing that $\kappa(t)$ is indeed smooth.

For a counter example when $\kappa = 0$ one could use $\gamma(t) = (t, t^3)$, then $\dot\gamma(t) = (1, 3t^2)$, $\ddot\gamma(t) = (0, 6t)$, $\|\ddot\gamma \times \dot\gamma\| = \left\| \begin{matrix} 0 & 6t \\ 1 & 3t^2 \end{matrix} \right\| = 6|t|$, and $\kappa(t) = \frac{6|t|}{(1+9t^4)^{3/2}}$ is not differentiable at $t = 0$.

2.2 Assume that γ is unit speed parameterized then $\mathbf{t} = \dot\gamma$ is smooth. Since the counter clockwise rotation $r : \mathbb{R}^2 \to \mathbb{R}^2$ given by $r(x,y) = (-y, x)$ is smooth, $\mathbf{n}_s = r(\mathbf{t})$ is also smooth. Finally, the dot product $\_\cdot\_ : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is smooth so $\kappa_s = \ddot\gamma \cdot \mathbf{n}_s$ is smooth. Since any unit speed parameterization is smooth for regular curves this also proves the non-unit speed case by composing with such a reparameterization.

Alternatively use propositions 2.2.1 and 2.2.3 to show that κ_s is the derivative of a smooth function and is therefore smooth.

2.5 Let $s(t)$ be a unit speed parameterization of γ then

$$\frac{d\gamma^\lambda}{ds} = \mathbf{t} - \lambda\kappa_s\mathbf{t} = (1 - \lambda\kappa_s)\mathbf{t}.$$

This shows that γ^λ is regular as long as $\lambda\kappa_s \neq 1$, and its unit tangent vector and normal vectors coincide with those of γ. Now let $s^\lambda(s)$ be a unit speed paramaterization for γ^λ, then its inverse $s(s^\lambda)$ has derivative $\frac{ds}{ds^\lambda} = \frac{1}{|1-\lambda\kappa_s|}$, and the curvature of γ^λ is

$$\kappa_s^\lambda = \frac{d\mathbf{t}}{ds^\lambda} \cdot \mathbf{n}_s = \frac{d\mathbf{t}}{ds}\frac{ds}{ds^\lambda} \cdot \mathbf{n}_s = \frac{\kappa_s}{|1 - \lambda\kappa_s|}.$$

2.6 The three points, $\gamma(s_0), \gamma(s_0 \pm \delta s)$ are on a circle of radius r and center ϵ provided

$$r^2 = \|\gamma(s_0) - \epsilon\|^2 = \|\gamma(s_0 + \delta s) - \epsilon\|^2 = \|\gamma(s_0 - \delta s) - \epsilon\|^2.$$

There will always be a unique r and ϵ solving these equations provided δs is non-zero, and in the case of a closed curve, small enough to guarantee $\gamma(s_0 + \delta s) \neq \gamma(s_0 - \delta s)$. Let $a(s_0, \delta s)$ and $b(s_0, \delta s)$ be given by

$$\epsilon - \gamma(s_0) = a(s, \delta s)\mathbf{t} + b(s, \delta s)\mathbf{n}_s.$$

Then it must be that

$$\begin{aligned}
r^2 &= \|\gamma(s_0 + \delta s) - \epsilon\|^2, \\
&= \|(\gamma(s_0 + \delta s) - \gamma(s_0)) - (\epsilon - \gamma(s_0))\|^2, \\
&= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 - 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [\epsilon - \gamma(s_0)] + \|\epsilon - \gamma(s_0)\|^2, \\
&= \|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 - 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [\epsilon - \gamma(s_0)] + r^2.
\end{aligned}$$

Which, in turn, implies that

$$\|\gamma(s_0 + \delta s) - \gamma(s_0)\|^2 = 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [a(s_0, \delta s)\mathbf{t} + b(s_0, \delta s)\mathbf{n}_s].$$

The left hand side of this is on the order of δs^2, so dividing by δs and taking δs to zero yields.

$$\dot{\gamma}(s_0) \cdot \lim_{\delta s \to 0} (a(s_0, \delta s)\mathbf{t} + b(s_0, \delta s)\mathbf{n}_s) = \lim_{\delta s \to 0} a(s_0, \delta s) = 0.$$

Note that

$$\begin{aligned}
0 &= \|\gamma(s_0 + \delta s) - \epsilon\|^2 + \|\epsilon - \gamma(s_0 - \delta s)\|^2 - 2\|\gamma(s_0) - \epsilon\|^2, \\
&= \|(\gamma(s_0 + \delta s) - \gamma(s_0)) - (\epsilon - \gamma(s_0))\|^2 + \|(\gamma(s_0) - \gamma(s_0 - \delta s)) + (\epsilon - \gamma(s_0))\|^2 \\
&\quad - 2\|\gamma(s_0) - \epsilon\|^2, \\
&= \|(\gamma(s_0 + \delta s) - \gamma(s_0))\|^2 - 2[\gamma(s_0 + \delta s) - \gamma(s_0)] \cdot [\epsilon - \gamma(s_0)] \\
&\quad + \|(\gamma(s_0) - \gamma(s_0 - \delta s))\|^2 + 2[\gamma(s_0) - \gamma(s_0 - \delta s)] \cdot [\epsilon - \gamma(s_0)],
\end{aligned}$$

implies that

$$\|(\gamma(s_0 + \delta s) - \gamma(s_0))\|^2 + \|(\gamma(s_0) - \gamma(s_0 - \delta s))\|^2$$

and

$$2[\gamma(s_0 + \delta s) - 2\gamma(s_0) + \gamma(s_0 - \delta s)] \cdot [a(s_0, \delta s)\mathbf{t} + b(s_0, \delta s)\mathbf{n}_s]$$

are equal. Dividing by δs^2 and taking δs to zero yields

$$2\|\dot{\gamma}(s_0)\|^2 = 2\ddot{\gamma}(s_0) \cdot \lim_{\delta s \to 0} b(s, \delta s)\mathbf{n}_s.$$

Since γ is unit speed, this is just

$$2 = 2\kappa_s \lim_{\delta s \to 0} b(s, \delta s).$$

Finally, this shows that if $\kappa_s \neq 0$ then ϵ converges as δs goes to zero, and it converges to

$$\epsilon(s_0) = \gamma(s_0) + \frac{1}{\kappa_s}\mathbf{n}_s.$$

2.7 The tangent to ϵ is

$$\dot{\epsilon} = \mathbf{t} - \frac{\dot{\kappa}_s}{\kappa_s^2}\mathbf{n}_s - \mathbf{t} = -\frac{\dot{\kappa}_s}{\kappa_s^2}\mathbf{n}_s.$$

Therefore since $\dot{\kappa}_s > 0$ the arclength is

$$s^\epsilon(s) = \int_{s_0}^{s} \frac{\dot{\kappa}_s}{\kappa_s^2} ds' = -\frac{1}{\kappa_s} + C.$$

Since the unit tangent to ϵ is $\mathbf{t}^\epsilon = -\mathbf{n}_s$, the signed unit normal is $\mathbf{n}_s^\epsilon = \mathbf{t}$. Thus the signed curvature of ϵ is

$$\kappa_s^\epsilon = -\frac{d\mathbf{n}_s}{ds}\frac{ds}{ds^\epsilon} \cdot \mathbf{t} = \frac{\kappa_s^3}{\dot{\kappa}_s}.$$

For the *cycloid*, $\gamma(t) = a(t - \sin t, 1 - \cos t)$, the first and second derivative are

$$\dot\gamma(t) = a(1 - \cos t, \sin t) \qquad \text{and} \qquad \ddot\gamma(t) = a(\sin t, \cos t).$$

Thus the signed unit normal is

$$\mathbf{n}_s = \frac{1}{\|\dot\gamma\|} a(-\sin t, 1 - \cos t),$$

and the curvature is

$$\kappa = \frac{a^2}{\|\dot\gamma\|^3} \left\| \begin{matrix} \sin t & \cos t \\ 1 - \cos t & \sin t \end{matrix} \right\| = \frac{a^2}{\|\dot\gamma\|^3}(1 - \cos(t)).$$

The signed curvature is always negative so

$$\frac{1}{\kappa_s}\mathbf{n}_s = \frac{\|\dot\gamma\|^2}{a(1 - \cos(t))}(\sin t, -1 + \cos t).$$

Since $\|\dot\gamma\|^2 = a^2(2 - 2\cos t)$ this is just

$$\frac{1}{\kappa_s}\mathbf{n}_s = 2a(\sin t, -1 + \cos t),$$

and

$$\epsilon(t) = a(t - \sin t, 1 - \cos t) + 2a(\sin t, -1 + \cos t) = a(t + \sin t, -1 + \cos(t).$$

Using the reparameterization $t = \tilde t - \pi$ this becomes

$$\epsilon(\tilde t) = a(\tilde t - \sin \tilde t, -1 - \cos \tilde t),$$

which is just $\gamma(\tilde t) - (0, 2)$.