Solutions

147a Winter 2012

Homework 1

1.1 (t^2, t^4) is not a paramaterization of $y = x^2$ as it misses the left half of the parabola.

1.2 Many different paramaterizations are possible.

i. $(\tan(t), \sec(t))$ $\quad t \in (-\pi/2, \pi/2)$, $t \in (\pi/2, 3\pi/2)$

ii. $(2\cos(t), 3\sin(t))$ $\quad t \in [0, 2\pi)$.

1.4 i. $\dot{\gamma}(t) = (-2\cos(t)\sin(t), 2\cos(t)\sin(t))$

ii. $\dot{\gamma}(t) = (e^t, 2t)$

1.7 Let a circle of radius a start touching the origin and roll along the x axis at constant unit angular velocity. Then the curve traced by the center of the circle is, $\alpha(t) = (at, a)$. If $\gamma(t)$ is the curve traced by the point starting at the origin then $\beta(t) = \gamma(t) - \alpha(t)$ is the vector pointing from the center of the circle to this point. Since the circle is moving clockwise with consant unit angular velocity and the initial vector is $(0, -a)$ it can be seen that $\beta(t) = (-a\sin(t), -a\cos(t))$. Thus

$$\gamma(t) = \beta(t) + \alpha(t) = (at - a\sin(t), a - a\cos(t)).$$

2.1 The tangent vector to the curve $\gamma(t) = (t, \cosh(t))$ is

$$\dot{\gamma}(t) = (1, \sinh(t)).$$

Since $(0, 1)$ corresponds to $t = 0$ the arclength after time t is

$$\begin{aligned} s(t) &= \int_0^t \|\dot{\gamma}(t')\| dt' \\ &= \int_0^t \sqrt{1 + \sinh^2(t')} dt' \\ &= \int_0^t \cosh(t') dt' \\ &= \sinh(t). \end{aligned}$$

2.2 i. $\dot{\gamma}(t) = (\frac{1}{2}\sqrt{1+t}, -\frac{1}{2}\sqrt{1-t}, \frac{1}{\sqrt{2}})$

$\|\dot{\gamma}(t)\| = \sqrt{\frac{1}{4}(1+t) + \frac{1}{4}(1-t) + \frac{1}{2}} = 1$

ii. $\dot{\gamma}(t) = (-\frac{4}{5}\sin(t), -\cos(t), \frac{3}{5}\sin(t))$

$\|\dot{\gamma}(t)\| = \sqrt{\frac{16}{25}\sin^2(t) + \cos^2(t) + \frac{9}{25}\sin^2(t)} = 1$

3.1 i. The tangent to the curve, $\dot{\gamma} = (-2\cos(t)\sin(t), 2\cos(t)\sin(t))$ is zero at all integer multiples of $\pi/2$.

ii. This curve is regular as it avoids the zeros from part (i). An obvious unit speed paramaterization of the line $y = 1 - x$ is $\gamma(s) = (1 - \sqrt{2}s, \sqrt{2}s)$.

iii. $\dot{\gamma}(t) = (1, \cosh(t))$ is never zero thus this curve is regular. Recall from (2.1) that

$$s(t) = \sinh(t)$$

. Therefore a unit speed paramterization would be

$$\gamma(s) = (\sinh^{-1}(s), \cosh(\sinh^{-1}(s))).$$

3.4 i. Let $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ be a reparamaterization of $\gamma : (\alpha, \beta) \to \mathbb{R}^n$. Then there is a smooth bijection $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ with smooth inverse, ϕ^{-1}, such that

$$\tilde{\gamma} = \gamma \circ \phi.$$

Composing with ϕ^{-1} gives the equation

$$\tilde{\gamma} \circ \phi^{-1} = \gamma.$$

Since $\phi^{-1} : (\alpha, \beta) \to (\tilde{\alpha}, \tilde{\beta})$ is smooth with smooth inverse, ϕ, this shows that γ is a reparamaterization of $\tilde{\gamma}$.

ii. Let $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^n$ be a reparamaterization of $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, and let $\hat{\gamma} : (\hat{\alpha}, \hat{\beta}) \to \mathbb{R}^n$ be a reparamaterization of $\tilde{\gamma}$. Then there are smooth bijections $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ and $\psi : (\hat{\alpha}, \hat{\beta}) \to (\tilde{\alpha}, \tilde{\beta})$ with smooth inverses, ϕ^{-1} and ψ^{-1} such that

$$\tilde{\gamma} = \gamma \circ \phi \qquad \text{and} \qquad \hat{\gamma} = \tilde{\gamma} \circ \psi.$$

Substituting the first into the second and using the associativity of function composition gives the equation

$$\hat{\gamma} = (\gamma \circ \phi) \circ \psi = \gamma \circ (\phi \circ \psi).$$

Now $\phi \circ \psi$ is smooth, as a composition of smooth functions, and bijective, as a composition of bijections. Furthermore its inverse $(\phi \circ \psi)^{-1} = \psi^{-1} \circ \phi^{-1}$ is also a composition of smooth functions, and thus is smooth itself. This shows that $\hat{\gamma}$ is a reparamateriztion of γ. Q.E.D.