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ABSOLUTELY MINIMIZING, THE INFINITY LAPLACIAN, AND ALL THAT: BASIC THEORY

These notes were accompanied by verbal elucidations during their presentation, but are intended to be readable, if abbreviated. They contain some material not fully presented in the accompanying lectures.

We minimize references herein, until the end, when comments are offered about some generalizations and other recent developments. The paper [5] is a source for information about the origins of the results we here take as currently common knowledge, from the community; see also [15]. Further references can be found via Google Scholar, for example, which provides 116 articles citing [5] the last time we checked, many of which have themselves then been cited in other articles. Or, even better, consult the about 180 articles citing Gunnar's kick off paper [4] on Google and the about 190 articles citing Bob's original uniqueness proof [20]. These last two papers were, imo, by far the most influential papers in stimulating the development of the theory we are about to describe. Another survey is available in [31], but it does not collect historical comments.

At the moment, there are no other sources quite like these notes, owing primarily to the incorporation here of new comparison arguments from S. Armstrong and C. Smart [2] and secondarily to our emphasis on the convexity criterion (see below), as motivated by [2]. There are probably a lot of typos, etc., and if you note some, please send them to me.

The notations following are for later reference by readers and attendees. We have attempted to minimize notation and definitions in these notes, and have used some notation different than in other sources, hopefully thereby making the lectures easier to follow.

Some notation:

- $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$, $\langle x, y\rangle$ is the Euclidean inner product of $y, x \in \mathbb{R}^n$.
- $B(x,r) = \{y : |y-x| < r\}, \overline{B}(x,r) = \{y : |y-x| \leq r\}$.
- $U, V, W,$ are bounded open subsets of $\mathbb{R}^n$, $K \subset \mathbb{R}^n$ need not be open.
- $V \ll U$ means $\bar{V}$ is a compact subset of U.
- $\overline{U}$, ∂U, are the closure and boundary of U;
- $\operatorname{dist}(x, K) = \inf_{y\in K} |x-y|$ is the distance from x to K.
- $U_r := \{x \in U : \operatorname{dist}(x, \partial U) > r\}$.
- $u^r(x) := \max_{\overline{B}(x,r)} u$, $u_r(x) := \min_{\overline{B}(x,r)} u$ for $x \in U_r$.
- $\operatorname{Lip}(u, K) := \sup\left\{\frac{|u(x)-u(y)|}{|x-y|} : x, y \in K, x \neq y\right\}$
- $|Du|(x) := \lim_{r\downarrow 0} \operatorname{Lip}(u, B(x,r))$.
- $Du = (u_{x_1}, \ldots, u_{x_n})$ is the gradient and $D^2u = ((u_{x_i,x_j}))$ is the Hessian matrix of u.

A "Sup Norm" Functional

$$\mathrm{Lip}(u,K) := \sup\left\{\frac{|u(x)-u(y)|}{|x-y|} : x,y\in K, x\neq y\right\}$$

The value ∞ is allowed.

A "sup norm" variational problem: Given

$$U \subset \mathbb{R}^n \text{ and boundary data } \ b : \partial U \to \mathbb{R},$$

with $\mathrm{Lip}(b : \partial U) < \infty$, define the class of "admissible functions"

$$\mathcal{A}_b := \left\{u \in C(\bar{U}) : u = b \text{ on } \ \partial U\right\}$$

and seek to solve the problem

$$\mathrm{MinLip} : \qquad u \in \mathcal{A}_b, \ \ \mathrm{Lip}(u,\bar{U}) = \min_{v\in\mathcal{A}_b} \mathrm{Lip}(v,\bar{U}).$$

Clearly $\mathrm{Lip}(u,\bar{U}) \geq \mathrm{Lip}(b,\partial U)$ for all $u \in \mathcal{A}_b$. Thus, if $u \in \mathcal{A}_b$ and

$$\mathrm{Lip}(u,\bar{U}) = L := \mathrm{Lip}(b,\partial U),$$

then u solves MinLip, and, for $y, z \in \partial U$,

$$\begin{cases} u(z) - L|x-z| = b(z) - L|x-z| \le u(x), \\ u(x) \le b(y) + L|x-y| = u(y) + L|x-y|. \end{cases}$$

Hence

$$\begin{aligned} \mathcal{M}^-(b)(x) &:= \max_{z \in \partial U}(b(z) - L|x-z|) \le u(x) \\ &\le \min_{y \in \partial U}(b(y) + L|x-y|) =: \mathcal{M}^+(b)(x). \end{aligned}$$

The $\mathcal{M}^-(b), \mathcal{M}^+(b)$ are called the McShane-Whitney extensions of b to $\bar{U}$. Clearly, $\mathcal{M}^{\pm}(b) = b$ on ∂U, $\mathrm{Lip}(\mathcal{M}^{\pm}(b), \bar{U}) = L$, and therefore $\mathcal{M}^-(b), \mathcal{M}^+(b)$ are minimal and maximal solutions of the problem MinLip.

However, in general, $\mathcal{M}^-(b)(x) < \mathcal{M}^+(b)(x)$ for lots of x's, and the solutions of the problem MinLip exist, but are not unique.

G. Aronsson proposed to improve the problem by adding additional conditions which might serve to make the solution better behaved, and for which there might be uniqueness. These conditions include

$$\text{AML}: \qquad \mathrm{Lip}(u,V)=\mathrm{Lip}(u,\partial V) \text{ for } V \ll U,$$

where "AML" stands for "absolutely minimizing Lipschitz." AML might roughly be thought of as not letting u locally behave worse than it might have to, as measured by the Lipschitz constant. Here we regard AML as a condition on a function $u : U \to \mathbb{R}$, untethered by any restrictions on its behavior near or on ∂U. We will answer all of the questions below in the affirmative:

- Are there any AML functions u which solve the problem MinLip?
- If so, are they unique?
- Is the property AML characterized by a pde?

We will also provide some information about:

- What other properties do AML functions have?

The theorems below provide the affirmative answers, and, in the current presentation, they are obtained via the properties stated in our first theorem.

We need two more bits of notation to state the theorem.

In Theorem 1 below, a cone function C is one of the form

$$C(x) = a|x - z|$$

where $a \in \mathbb{R}$. We call z above the vertex of C and write $z = \mathrm{ver}(C)$.

Next, for any function u, we can define

$$|Du|(x) := \lim_{r \downarrow 0} \mathrm{Lip}(u, B(x, r)). \tag{1}$$

The right hand side exists, if we allow the value ∞, as $\mathrm{Lip}(u, B(x, r))$ is nondecreasing in r. It is easy to see that

$$x \mapsto |Du|(x) \tag{2}$$

is upper semicontinuous for any u, and we leave this to the reader.

NOTE WELL THE ABUSE OF NOTATION: $|Du|(x)$ is defined for all $u : U \longrightarrow \mathbb{R}$ and $x \in U$, and does not require u to be differentiable anywhere. Moreover, even if u is locally lipschitz continuous, it is not true in general that $|Du|(x) = |Du(x)|$ even if u is differentiable at x, as in the 1-d example $u(x) = \sin(1/x)x^2$, $x \neq 0$, $u(0) = 0$, wherein $Du(0) = 0$, but $|Du|(0) = 1$.

IGNORE THIS FOR NOW: Later we remark that most of the proceedings are are valid if $|\cdot|$ is a general norm, pointing out any exceptions. In this regard, the dangerous notation $|Du|(x)$ should be replaced by $|Du|^*(x)$, where $|\cdot|^*$ is the norm dual to $|\cdot|$, in the general case, to convey the appropriate sense.

Theorem 1 (Equivalences)**.** *u is AML in U iff any one the equivalent conditions (3), (4), (5), and (6) below hold:*

Comparison with cones (aka ComCo):

$$(3) \qquad \begin{cases} \text{(i)}\ \ \max_{\bar{V}}(u-C) = \max_{\partial V}(u-C) \\ \text{(ii)}\ \ \min_{\bar{V}}(u-C) = \min_{\partial V}(u-C) \\ \forall\ V \ll U\ \&\ \text{cone function } C \ni \operatorname{ver}(C) \notin V \end{cases}$$

Convexity Criterion (aka ConvCri):

$$(4) \qquad \begin{cases} \text{(i)}\ r \mapsto u^r(x) := \max_{\bar{B}(x,r)} u \text{ is convex} \\ \text{(ii)}\ r \mapsto u_r(x) := \min_{\bar{B}(x,r)} u \text{ is concave} \\ \forall x \in U,\ \ 0 \le r < \operatorname{dist}(x, \partial U) \end{cases}$$

Infinity Harmonic (aka IH):

$$(5) \qquad \begin{cases} \text{(i)}\ \Delta_\infty u := \langle D^2 u Du, Du\rangle = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} \ge 0 \ \textit{in } U \\ \text{(ii)}\ \Delta_\infty u := \langle D^2 u Du, Du\rangle = \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} \le 0 \ \textit{in } U \end{cases}$$

Gradient Estimates (aka GradE):

$$(6) \qquad \begin{cases} \text{(i)}\ |Du|(x) \le \dfrac{u^r(x) - u(x)}{r} \\ \text{(ii)}\ |Du|(x) \le \dfrac{u(x) - u_r(x)}{r} \\ \forall x \in U,\ 0 \le r < \operatorname{dist}(x, \partial U) \end{cases}$$

The meaning of IH will be included later, but for those who know, pdes herein are understood in the viscosity sense.

Some comments on the above:

- In each case, (i) is a "subsolution property" and (ii) is a "supersolution" property.
- Moreover, all the (i)'s are equivalent to each other and all the (ii)'s are equivalent to each other.
- In each case, u satisfies (i) iff $-u$ satisfies (ii) and each criterion is satisfied by u iff it is satisfied by $u +$ any constant. In particular, it suffices to prove the (i)'s are equivalent, for then the (ii)'s are equivalent.
- We call (3) (i) ComCo from above, and we call (3) (ii) ComCo from below. Similarly, we call (5) (i) infinity subharmonic, and we call 5 (ii) infinity superharmonic.

- Note that AML and ComCo involve an arbitrary test set $V \ll U$, while ConvCri and GradE only involve balls, and IH, being a pde, is purely local. It follows that AML and ComCo hold for all V if they hold when V is a ball. This will be evident in the proofs.
- All the (i)'s are meaningful for upper semicontinuous functions; likewise the (ii)'s and lower semicontinuous functions. However, any of the (i)'s (and hence the (ii)'s) imply that u is locally Lipschitz continuous. We take care of this in Lemma 2 immediately following, using ConvCri. In these notes, we often opt to work with ConvCri, and need to get used to this notation.

As we make our first use of ConvCri here, let us note well that

$$u^0 = u,\ (u^r)^s = u^{r+s},\ (u+v)^r \leq u^r + v^r,$$

for any old functions u, v, and that u^r is NOT a power of u. Also, convexity will usually be invoked in the form

$$(r, s) \mapsto \frac{u^r - u^s}{r - s} \text{ is nondecreasing in } r \text{ and in } s \text{ on } 0 \le s < r.$$

Proposition 2. *Let $u : U \to I\!R$ be upper semicontinuous and $r \mapsto u^r(x)$ be convex for $x \in U$, $0 \le r < \operatorname{dist}(x, \partial U)$. Then for $z \in U$, $0 < R < \operatorname{dist}(z, \partial U)$, $0 < 3r \le R$, we have*

$$\operatorname{Lip}(u, \overline{B}(z, r)) \le \frac{u^R(z) - u(z)}{R - 2r}. \tag{7}$$

Moreover, in consequence, if z is a local maximum point of u, then u is constant in a neighborhood of z. Hence if u has maximum point in some open connected $V \subset U$, then u is constant in V. In consequence, for any $r < \operatorname{dist}(z, \partial U)$, either u is constant in $B(r, z)$ or all points $w \in \overline{B}(z, r)$ for which $u^r(z) = u(w)$ satisfy $|z - w| = r$.

Proof. Let $x, y \in \overline{B}(z, r)$ and $x \neq y$. Under our assumptions, we have $u(x) \leq u^{|x-y|}(y)$, $0 < |x-y| \leq 2r \leq R - r$, $u(z) \leq u^r(y)$, and $u^{R-r}(y) \leq u^R(z)$, which combine with the convexity to yield

$$\frac{u(x)-u(y)}{|x-y|} \leq \frac{u^{|x-y|}(y)-u(y)}{|x-y|} \leq \frac{u^{R-r}(y)-u^r(y)}{R-2r} \leq \frac{u^R(z)-u(z)}{R-2r}. \tag{8}$$

As x, y can be interchanged, (7) holds. (Note that this makes perfect sense in a metric space, if we write "$d(x, y)$" instead of $|x - y|$.)

For the final assertion, note that if $u^R(z) = u(z)$, that is, z is a maximum point for u in $\overline{B}(z, R)$, then choosing $r = R/3$, we see that u is constant in $B(z, R/3)$. Thus the set of maximum points of u in V is, if nonempty, an open and closed and nonempty subset of V. As V is connected, it is all of V. □

Regarding the existence and uniqueness of AML functions which solve the problem MinLip, we have

Theorem 3. *Let $b \in C(\partial U)$. Then there exists exactly one function $u \in C(\bar{U}) \cap \mathrm{AML}$ such that $u = b$ on ∂U. Moreover,* $\mathrm{Lip}(u, \bar{U}) = \mathrm{Lip}(b, \partial U)$.

The uniqueness assertion follows from a more general comparison result.

Theorem 4 (Comparison Theorem)**.** *Let $u, v \in C(\bar{U})$, u satisfy any of the conditions (i), and v satisfy any of the conditions (ii), of Theorem 1. Then*

$$u - v \le \max_{\partial U}(u - v) \text{ in } U. \tag{9}$$

Important Remark: If we delete IH, which is specific to the Euclidean norm, from Theorem 1, all the results above hold if $|\cdot|$ is any norm. While there is a version of IH for any norm, it is not known at the moment that this variant is always equivalent to the other conditions.

We will provide fairly complete proofs of all the above, either in lecture and these notes, or in the notes alone, later. For the moment, we content ourselves with showing AML $\iff$ ComCo, as the demonstration is strikingly simple.

The following two slides contain, respectively,

PROOF: AML $\implies$ ComCo.

PROOF: ComCo $\implies$ AML.

Re AML $\implies$ ComCo, note that since both conditions are invariant under adding a constant, it suffices to show that AML implies ($u \le C$ on ∂V implies $u \le C$ in V if $\mathrm{ver}(C) \notin V$.)

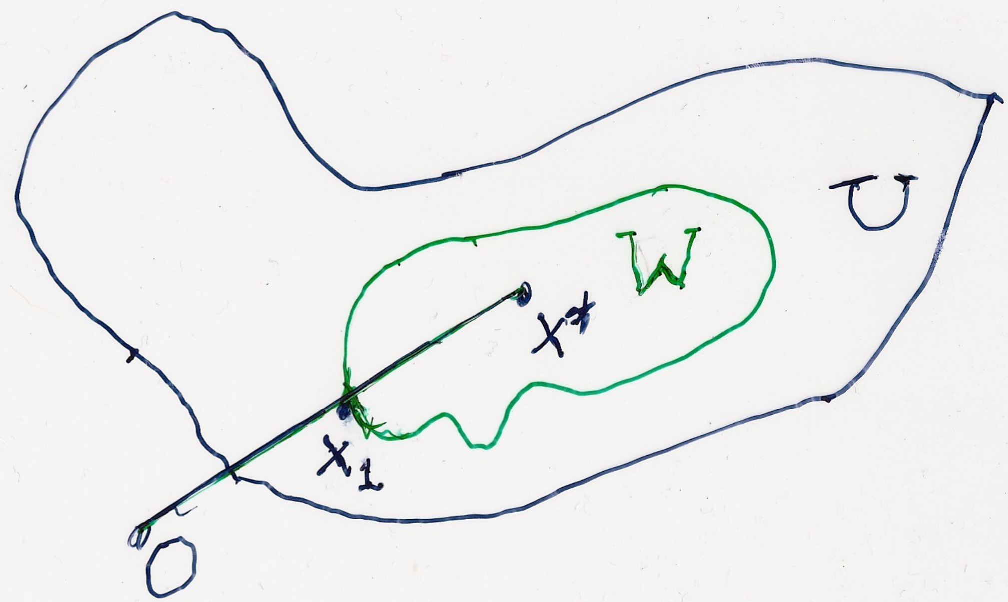
$u \in C(\bar{U}) \cap \text{AML} \implies u - C \leq \max_{\partial U}(u - C)$ if $\text{ver}(C) \notin U$.

Typical case: $C(x) = |x|, 0 \notin U$, $u \leq |x|$ on $\partial U \implies u \leq |x|$ in U.

If not, $\exists\, x^* \in U, \varepsilon > 0 \ni |x^*| + \varepsilon < u\,(x^*)$.

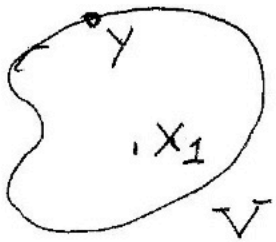
$W =$ component of $\{u(x) > |x| + \varepsilon\}$, $x^* \in W$, $W \ll U$,

$u(x) = |x| + \varepsilon$ on ∂U. Therefore: ♠ $\text{Lip}(u, W) = \text{Lip}(|x| + \varepsilon, \partial W) = 1$.



$$|x^* - x_1| = |x^*| - |x_1| = |x^*| + \varepsilon - (|x_1| + \varepsilon) < u(x^*) - u(x_1) (\Rightarrow\Leftarrow ♠)$$

Assume ComCo, $L := \mathrm{Lip}(u, \partial V)$, show $L = \mathrm{Lip}(u, \overline{V})$

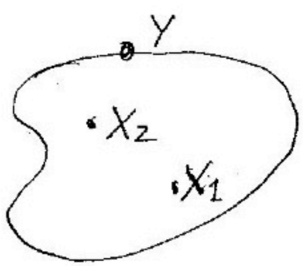


$$u(y) - L|x-y| \le u(x) \le u(y) + L|x-y|$$

True for $x \in \partial V \Rightarrow$ for $x \in \overline{V}$

$$\Rightarrow \quad |u(x_1) - u(y)| \le L|x_1 - y| \text{ for } y \in \partial V$$

$$\therefore \; L = \mathrm{Lip}(u, (\partial V) \cup \{x_1\})$$
$$= \mathrm{Lip}(u, \partial(V \setminus \{x_1\}))$$



Apply with V replaced by $V \setminus \{x_1\}$

$$\Longrightarrow |u(x_2) - u(x_1)| \le L|x_2 - x_1|$$

Remark 5. Between them, the previous two slides establish the final assertion of Theorem 3, for they show that $u \in C(\bar{U}) \cap \mathrm{AML}$ implies $\mathrm{Lip}(u, \bar{U}) = \mathrm{Lip}(u, \partial U)$.

The proof of AML $\iff$ IH, as given later, is simple (well, not as dramatically so as the preceding) and direct, and is given in the form ComCo $\iff$ IH. However, it leaves mysterious how Gunnar Aronsson found the operator Δ_∞. We segue into another line of the theory with the answer to this question.

In convex sets V, we have

$$\mathrm{Lip}(u, V) = \| |Du| \|_{L^\infty(V)} \tag{10}$$

and the AML condition includes: $\forall$ convex $V \ll U$ and $v \in C(\bar{V})$ with $u = v$ on ∂V,

$$\begin{aligned} \| |Du| \|_{L^\infty(V)} &= \lim_{p\to\infty} \| |Du| \|_{L^p(V)} \le \mathrm{Lip}(u, \partial V) = \\ \mathrm{Lip}(v, \partial V) &\le \mathrm{Lip}(v, V) = \| |Dv| \|_{L^\infty(V)} = \lim_{p\to\infty} \| |Dv| \|_{L^p(V)}. \end{aligned} \tag{11}$$

Moreover, if u_p minimizes $\||Du|\|_{L^p(V)}$ among functions satisfying $v = u$ on ∂V, then it satisfies the Euler equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(|Du_p|^{p-2}(u_p)_{x_i}\right)$$
$$= (p-2)|Du_p|^{p-4} \left(\frac{2}{p-2}|Du_p|^2 \Delta u_p + \Delta_\infty u_p\right) = 0.$$

Assuming $|Du_p| \neq 0$ and $u_p \to u$ in the limit as $p \to \infty$, one formally finds $\Delta_\infty u = 0$.

We state then the problem corresponding to MinLip when then functional Lip is replaced by

$$\mathrm{Grad}_\infty(u, V) := \||Du|\|_{L^\infty(V)}, \tag{12}$$

that is

$$\mathrm{MinG}_\infty : \qquad u \in \mathcal{A}_b, \quad \mathrm{Grad}_\infty(u, U) = \min_{v \in \mathcal{A}_b} \mathrm{Grad}_\infty(v, U).$$

If U is convex, we have noted that the two problems coincide, so there is existence for MinG_∞ (if $\mathrm{Lip}(b, \partial U) < \infty$) via the McShane-Whitney extensions, but also nonuniqueness.

In general, when U is not convex, at the level of their statements, the problems MinLip and MinG_∞ are not equivalent. Moreover, they have different sets of minimizers and different minimum values.

Thus, looking at (11), we are teased by the idea of defining a property analogous to AML:

$$\mathrm{AMG} : \mathrm{Grad}_\infty(u, V) \leq \mathrm{Grad}_\infty(v, V) \text{ if } V \ll U \,\&\; u = v \text{ on } \partial V.$$

It is a little bit of a surprise that

$$\mathrm{AML} \iff \mathrm{AMG}.$$

A simple tweak of the proof that AML $\implies$ ComCo siffices to show that AMG $\implies$ ComCo, so AMG $\implies$ AML.

The implication AML $\implies$ AMG is not so simple to establish, but it is true. This fact is linked to the equivalences of AML, ComCo, ConvCri, IH, which are themselves equivalent, with GradE, and the previous observation that, via Theorem 1 and the proofs of its parts, that it suffices to use balls as test sets in defining AML.

Once we know that AML $\implies$ AMG, the theorem below follows from Theorems (3) and (4):

Theorem 6. *Let $b \in C(\partial U)$. Then there exists exactly one function $u \in C(\bar{U}) \cap$ AMG such that $u = b$ on ∂U. Moreover, if $u, v \in C(\bar{U}) \cap$ AMG, then*

$$u - v \leq \max_{\partial U}(u - v) \text{ in } U. \tag{13}$$

The assertion that if $u \in$ AMG $\cap\, C(\bar{U})$, $u = b$ on ∂U, then u solves the problem MinG_∞ is not included above, but follows from proofs below.

After some orientation, we next present two technical tools used in proving AML $\implies$ AMG.

First, here is a way to think about these tools. If $u \in C^2(U) \cap$ AML, then for $x \in U$ we can define $\gamma(t)$ on a maximal interval of existence $[0, T)$ by

$$\dot{\gamma} = Du(\gamma(t)), \;\; \gamma(0) = x. \tag{14}$$

Then, by calculus, and the (as yet unproven) equivalence of AML and IH,

$$\begin{aligned} \frac{d}{dt}|Du(\gamma(t))|^2 &= 2\langle D^2u(\gamma(t))\dot{\gamma}(t), Du(\gamma(t))\rangle \\ &= 2\langle D^2u(\gamma(t))Du(\gamma(t)), Du(\gamma(t))\rangle = 2(\Delta_\infty u)(\gamma(t)) = 0, \end{aligned}$$

so $|Du(\gamma(t))|$ is constant and

$$\frac{d}{dt}u(\gamma(t)) = \langle Du(\gamma(t)), \dot{\gamma}(t)\rangle = |Du(\gamma(t))|^2.$$

is constant. If $Du(x) = 0$, then $\gamma(t) \equiv x$ satisfies satisfies the assertions of Proposition 8 below, while if $Du(x) \neq 0$, then making γ unit speed does the same job.

However, perhaps the most informative example of an AML function is Gunnar Aronsson's famous

$$u(x,y) = |x|^{4/3} - |y|^{4/3} \tag{15}$$

in $\mathbb{R}^2$. This function is not C^2, (14) does not have solutions along which the length of the gradient is constant for any initial values, but among its solutions are ones which, when made unit speed, satisfy the assertions of Proposition 8 below. See also Remark 10 below.

Lemma 7 (Increasing Gradient)**.** *Let u satisfy ConvCri (i),* $x_0 \in U$, $\mathrm{dist}(x_0, \partial U) < r$*. Let $u^r(x_0) = u(x_r)$ where $|x_r - x_0| = r$. Then $|Du|(x_r) \geq |Du|(x_0)$.*

Lemma 7 is proved following Lemma 17 below.

Via an Euler type approximation, Lemma 7 parlays into

Proposition 8. *Let u satisfy* ConvCri *(i) in U, $x \in U$. Then there is a $T > 0$ and Lipschitz continuous curve $\gamma : [0, T) \to U$ with the following properties:*

$$
\begin{aligned}
&\text{(i) } \gamma(0) = x,\\
&\text{(ii) } |\dot{\gamma}(t)| \leq 1 \ \textit{a.e. on } [0, T),\\
&\text{(iii) } |Du|(\gamma(t)) \geq |Du|(x) \ \textit{on } [0, T),\\
&\text{(iv) } u(\gamma(t)) \geq u(x) + t|Du|(x) \ \textit{on } [0, T),\\
&\text{(v) } t \mapsto u(\gamma(t)) \ \textit{is convex on } [0, T),\\
&\text{(vi) } \textit{either } T = \infty \textit{ or } T < \infty \textit{ and } \gamma(T) := \lim_{t \uparrow T} \gamma(t) \in \partial U.
\end{aligned}
\tag{16}
$$

We won't prove Proposition 8 in these notes; it is a straightforward consequence of Lemma 7. See [15], Proposition 6.2. To work with Proposition 8, it helps to know:

Lemma 9. *Let* $\xi : [0,T] \to U$ *be an absolutely continuous curve and* $\max_{0\le t\le 1} |Du|(\xi(t)) < \infty$. *Then* $t \mapsto u(\xi(t))$ *is absolutely continuous and*

$$|\frac{d}{dt}u(\xi(t))| \le |Du|(\xi(t))|\dot{\xi}(t)| \text{ a.e. on } [0,1] \tag{17}$$

where $\dot{\xi}(t) = d\xi(t)/dt$.

Proof. $^{(**)}$ (starred material will not be presented in detail in lecture, but is written out for you here)
First, if $|Du|(x) < \infty$ and $\varepsilon > 0$, it follows from the definition of $|Du|(x)$ that there is an $r_\varepsilon > 0$ such that $\mathrm{Lip}(u, \overline{B}(x, r_\varepsilon)) \le |Du|(x) + \varepsilon$. It follows then from our assumptions u is Lipschitz continuous on a neighborhood of $\xi([0,1])$; therefore $u(\xi(t))$ is absolutely continuous. When $u(\xi(t))$ and $\xi(t)$ are both

differentiable at t, for $h > 0$ we have

$$
\begin{aligned}
\frac{|u(\xi(t+h)) - u(\xi(t))|}{h} &= \frac{|u(\xi(t) + h\dot{\xi}(t) + \mathrm{o}(h)) - u(\xi(t))|}{h} \\
&\leq \frac{|u(\xi(t) + h\dot{\xi}(t)) - u(\xi(t))|}{h} + \frac{\mathrm{o}(h)}{h} \\
&\leq \mathrm{Lip}(u, \overline{B}(\xi(t), h|\dot{\xi}(t)))|\dot{\xi}(t)| + \frac{\mathrm{o}(h)}{h} \to |Du|(\xi(t))|\dot{\xi}(t)|
\end{aligned}
$$

as $h \downarrow 0$. $\square$

PROOF: AML $\implies$ AMG.

The first thing we observe is that if $u : V \longrightarrow \mathbb{R}$ is locally Lipschitz continuous, then

$$
\sup_{x \in V} |Du|(x) = \text{essential } \sup_{x \in V} |Du(x)|. \tag{18}
$$

Recall our abuse of notation: on the left is the sup of an everywhere defined upper-semicontinuous function, and on the right is the L^∞ norm of a function defined a.e. As to why (18) holds, if you believed (10) for convex V, and who doesn't, you see that (18) holds with "$\le$" in place of "$=$". It is also easy to see that if u is differentiable at x, then $\mathrm{Lip}(u, B(x,r)) \ge |Du(x)| - \mathrm{o}(1)$ as $r \downarrow 0$.

Suppose now $V \ll U, u$ satisfies ConvCri (i) and $u = v$ on ∂V. Suppose that

$$\sup_{x\in V} |Du|(x) = \text{essential } \sup_V |Du(x)| > \text{essential } \sup_V |Dv(x)| = \sup_{x\in V} |Dv|(x), \tag{19}$$

that is, AMG fails. Then there exists $x_0 \in V$ and $\varepsilon > 0$ such that

$$|Du|(x_0) \ge |Dv|(x) + \varepsilon \text{ for } \ x \in V. \tag{20}$$

Let γ be as in Proposition 8. Clearly $|Du|(x_0) > 0$, so (iv) of (16) and the fact that u is bounded on V imply that the second alternative of (vi) holds. Let

$$y_0 := \gamma(T) := \lim_{t\uparrow T} \gamma(t) \in \partial V.$$

We show that then $u(x_0) \ge v(x_0)$ is impossible. Indeed, we then have

$$u(y_0) = u(\gamma(T)) \ge u(x_0) + T|Du|(x_0). \tag{21}$$

On the other hand, by (20),

$$v(y_0) \leq v(x_0) + \int_0^T |Dv|(\gamma(t))|\dot{\gamma}(t)|\,dt \leq v(x_0) + T(|Du|(x_0) - \varepsilon).$$

Combining this with (21) and $u(y_0) = v(y_0)$, as follows from $y_0 \in \partial V$, we find that

$$u(x_0) \leq v(x_0) - T\varepsilon,$$

contradicting $u(x_0) \geq v(x_0)$. If u also satisfies ConvCri (ii), then we apply this result to $-u, -v$ (and (19) is invariant under this substitution), to find that (20) cannot hold at a point where $u(x_0) \leq v(x_0)$ either. □

Remark 10. Returning to the example (15), it is C^1, and even if (14) does not have unique solutions, it always has a solution on $[0, \infty)$ which satisfies $|\dot{\gamma}(t)| \geq |Du(x)|$ for $0 \leq t$. You can see this by direct calculations by hand. With slight modifications to the argument above, this observation allows one to verify that (15) is AMG.

We turn to the proofs of the as yet unproven assertions.

PROOF$^{(**)}$: ComCo $\Longrightarrow$ ConvCri (starred material will not be presented in detail in lecture, but is written out for you here)

(starred material will not be presented in detail in lecture, but is written out for you here)

The inequality below holds for $x \in \partial B(y,r)$, so by ComCo it holds for $x \in B(y,r)$:

$$u(x) - u(y) \le \frac{u^r(y) - u(y)}{r}|x-y| \text{ for } |x-y| \le r < \operatorname{dist}(y, \partial U). \tag{22}$$

Therefore, maxing on $x \in \bar{B}(y,s)$,

$$\frac{u^s(y) - u(y)}{s} \le \frac{u^r(y) - u(y)}{r} \text{ for } 0 < s \le r < \operatorname{dist}(y, \partial U),$$

and then

$$u^s(y) \le \frac{s}{r}u^r(y) + \frac{r-s}{r}u(y) \text{ so for } \gamma \ge 0$$

$$u^{s+\gamma}(y) \le \frac{s}{r}u^{r+\gamma}(y) + \frac{r-s}{r}u^{\gamma}(y)$$

$$\text{and note that } s+\gamma = \frac{s}{r}(r+\gamma) + \frac{r-s}{r}\gamma.$$

$\square$

PROOF OF: ConvCri $\implies$ ComCo

In fact, we will prove much more. When we get to it, we use without further comment that cones satisfy ConvCri . Indeed, if $C(x) = |x - z|$ then

$$\begin{cases} C^r(x) = \max_{\bar{B}(x,r)} C = |x - z| + r, \\ C_r(x) = \min_{\bar{B}(x,r)} C = |x - z| - r. \end{cases}$$

so long as $z \notin B(x, r)$. It follows that any cone function satisfies both (i) and (ii) of ConvCri.

Theorem 11 (NEW! Elemental Comparison Theorem of Charley and Scott). *Let* $u, v \in C(U), \varepsilon > 0$ *and*

$$U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}. \tag{23}$$

Assume that

$$\begin{cases} \text{(i) } u - u_\varepsilon \le u^\varepsilon - u, \\ \text{(ii) } v^\varepsilon - v \le v - v_\varepsilon, \end{cases} \tag{24}$$

on U_ε. *Then*

$$\sup_U (u - v) = \sup_{U \setminus U_\varepsilon} (u - v). \tag{25}$$

Moreover, if $u, v \in C(\bar{U})$, *and* (24) *holds for all* $0 \le \varepsilon \le \varepsilon_0$ *for some* $\varepsilon_0 > 0$, *then*

$$u - v \le \max_{\partial U} (u - v) \ \text{in } U. \tag{26}$$

Before proving this result, let us show what it has to do with ConvCri $\Longrightarrow$ ComCo, and derive some other consequences.

Lemma 12. *Let $w \in C(U)$ and $r \mapsto w^r(x)$ be convex for $x \in U_r$. Then for $0 < \varepsilon \le r$, w^r satisfies*

$$(w^r)^\varepsilon - w^r \ge w^r - (w^r)_\varepsilon \text{ on } (U_r)_\varepsilon. \tag{27}$$

Proof. By the assumed convexity,

$$(w^r)^\varepsilon - w^r = w^{r+\varepsilon} - w^r \ge w^r - w^{r-\varepsilon} \text{ on } (U_r)_\varepsilon.$$

Now use the easily checked inequality $(w^r)_\varepsilon \ge w^{r-\varepsilon}$, which holds for any old function w, to conclude that (27), that is, $(w^r)^\varepsilon - w^r \ge w^r - (w^r)_\varepsilon$, holds. $\square$

Corollary 13. *Let $u, v \in C(\bar{U})$. If $r \mapsto u^r(x)$(resp., $v_r(x)$) is convex (resp., concave), for $x \in U$, $0 \le r < \operatorname{dist}(x, \partial U)$, then*

$$u - v \le \max_{\partial U}(u - v). \tag{28}$$

In consequence, if $u \in C(U)$ satisfies ConvCri *(i) in U, then it satisfies* ComCo *(i).*

Proof. By Lemma 12, if $r > 0$, (24) holds with u^r, v_r in place of u, v and U_r in place of U provided that $\varepsilon \leq r$. Thus (26) applies to yield

$$u^r - v_r \leq \max_{\partial U_r}(u^r - v_r).$$

The result follows on letting $r \downarrow 0$.

For the final assertion, let $V \ll U$ and let C be a cone function with $\mathrm{ver}(C) \notin V$. We may apply the result already proved with V in place of U and C in place of v to conclude that $\max_{\bar{V}}(u - C) = \max_{\partial V}(u - C)$.

$\square$

Corollary 14. *If $u, v \in C(\bar{U})$ are* AML *in U, then*

$$u - v \leq \sup_{\partial U}(u - v).$$

Proof. As we know that AML $\implies$ ComCo $\implies$ ConvCri, we may apply Corollary 13. $\square$

Remark 15. In particular, the uniqueness assertion of Theorem 3 holds, as does the comparison Theorem 4, once we finish with the equivalences.

PROOF OF: The Elemental Comparsion Theorem.

Arguing by contradiction, we suppose that

$$M := \sup_U (u - v) > \sup_{U \setminus U_\varepsilon} (u - v). \tag{29}$$

If $x_j \in U$ and $u(x_j) - v(x_j) \uparrow M$, then $x_j \in U_\varepsilon$ for large j by (29). Picking a convergent subsequence, we may assume that $x_j \to z$ for some $z \in \bar{U}_\varepsilon$, so $z \in U$, and $u(z) - v(z) = M$; therefore $z \in U_\varepsilon$ by (29). Thus $E = \{x \in U : u(x) - v(x) = M\}$ is closed, nonempty, and contained in U_ε. Define $F = \{x \in E : u(x) = \max_E u\}$ and choose a point $x_0 \in \partial F$. From

$x_0 \in F \subset E$, we have $u(x) - v(x) \leq u(x_0) - v(x_0)$ for $x \in U$, and therefore

$$u_\varepsilon(x) - u(x_0) \leq v_\varepsilon(x) - v(x_0) \tag{30}$$

for $x \in U_\varepsilon$.

We consider two cases. First, if $u^\varepsilon(x_0) = u(x_0)$, then (24) (i) (which said $u - u_\varepsilon \leq u^\varepsilon - u$) and (30) with $x = x_0$ imply

$$0 = u(x_0) - u_\varepsilon(x_0) = v(x_0) - v_\varepsilon(x_0).$$

Finally, (24) (ii) (which said $v^\varepsilon - v \leq v - v_\varepsilon$) implies $v^\varepsilon(x_0) = v(x_0)$. Thus u and v are constant in $B(x_0, \varepsilon)$, contradicting the choice of $x_0 \in \partial F$.

Second, if $u^\varepsilon(x_0) > u(x_0)$, choose $z \in \overline{B}(x_0, \varepsilon)$ such that $u(z) = u^\varepsilon(x_0)$. Since $u(z) > u(x_0)$, we have $z \notin E$ by the definitions of E and x_0, and then

$$u^\varepsilon(x_0) - u(x_0) = u(z) - u(x_0) < v(z) - v(x_0) \leq v^\varepsilon(x_0) - v(x_0),$$

that is

$$u^{\varepsilon}(x_0) - u(x_0) < v^{\varepsilon}(x_0) - v(x_0).$$

Now use the hypotheses (24) to conclude that

$$u(x_0) - u_{\varepsilon}(x_0) < v(x_0) - v_{\varepsilon}(x_0).$$

However, this contradicts (30) (which said $u_{\varepsilon}(x) - u(x_0) \leq v_{\varepsilon}(x) - v(x_0)$) at $x = x_0$.

The final assertion follows from the assumed validity of (24) for any $\varepsilon > 0$; this entails (25) for every $\varepsilon > 0$, and (26) follows in the limit $\varepsilon \downarrow 0$, owing to the continuity of u, v on $\bar{U}$. $\square$

PROOF: ComCo $\Longrightarrow$ IH

We have to explain the meaning of "solution of $\Delta_\infty u \geq 0$," etc., and we will do so in the course of proof. The subtleties involved are illustrated by Gunnar's example (15): $u(x, y) = |x|^{4/3} - |y|^{4/3}$. It is not smooth enough to calculate $\Delta_\infty u$ pointwise at points where x or y vanish.

By ComCo:

$$u(x) \leq u(y) + \left(\frac{\max_{\{w:|w-y|=r\}} u(w) - u(y)}{r} \right) |x - y| \tag{31}$$

for $x \in B(y, r) \ll U$. Rewrite (31) as

$$u(x) - u(y) \leq \max_{\{w:|w-y|=r\}} (u(w) - u(x)) \frac{|x - y|}{r - |x - y|} \tag{32}$$

If x is a local maximum point of $u - \varphi$ for some smooth φ, then

$$\varphi(x) - \varphi(y) \leq u(x) - u(y) \text{ and } u(w) - u(x) \leq \varphi(w) - \varphi(x)$$

for y, w near x. Thus we may replace u by φ in (32) to find

$$\phi(x) - \phi(y) \le \max_{\{w:|w-y|=r\}} (\phi(w) - \phi(x)) \frac{|x-y|}{r - |x-y|}$$

We have the Taylor's expansion

$$\phi(z) = \phi(x) + \langle p, z - x\rangle + \frac{1}{2}\langle X(z-x), z-x\rangle + \mathrm{o}\left(|z-x|^2\right),$$

where $p := D\phi(x),\ X := D^2\phi(x)$.

We will show that

$$\Delta_\infty \phi(x) = \langle D^2\phi(x) D\phi(x), D\phi(x)\rangle = \langle Xp, p\rangle \ge 0. \tag{33}$$

That is, ComCo from above implies that if $u - \phi$ has a local max at x, then

$$\Delta_\infty \phi(x) \ge 0.$$

This is the meaning of

u is a viscosity subsolution of $\Delta_\infty u = 0$ or, for short, $\Delta_\infty u \ge 0$.

Recalling

$$\phi(x) - \phi(y) \le \max_{\{w:|w-y|=r\}} (\phi(w) - \phi(x)) \frac{|x-y|}{r - |x-y|} \tag{34}$$

and

$$\phi(z) = \phi(x) + \langle p, z-x \rangle + \frac{1}{2}\langle X(z-x), z-x \rangle + \mathrm{o}\left(|z-x|^2\right). \tag{35}$$

Do:

Step 1: Put $y = x - \lambda p$ where p is from (35), then put $z = y$ in (35), and use the result on the left of (34)

Step 2. Let $w_{r,\lambda}$ be a value of w for which the maximum on the right of (34) is attained. Put $z = w_{r,\lambda}$ in (35) and use the result on the right of (34).

Step 3. Study the iterated limits $\lambda \downarrow 0$, then $r \downarrow 0$.

DETAILS OF STEP 3**.(starred material will not be presented in detail in lecture, but is written out for you here)

Performing Steps 1 and 2 and dividing by $\lambda > 0$ yields

$$\begin{aligned} &|p|^2 + \lambda \frac{1}{2}\langle Xp, p\rangle + \mathrm{o}(\lambda) \leq \\ &\left(\langle p, w_{r,\lambda} - x\rangle + \frac{1}{2}\langle X(w_{r,\lambda} - x), w_{r,\lambda} - x\rangle + \mathrm{o}((r+\lambda)^2)\right)\frac{|p|}{r - \lambda|p|} \end{aligned} \tag{36}$$

Sending $\lambda \downarrow 0$ yields

$$\begin{aligned} |p|^2 &\leq \left(\langle p, \frac{w_r - x}{r}\rangle + \frac{1}{2}\langle X\left(\frac{w_r - x}{r}\right), w_r - x\rangle\right)|p| + |p|\mathrm{o}(r) \\ &\leq |p|^2 + \frac{1}{2}\langle X\left(\frac{w_r - x}{r}\right), w_r - x\rangle|p| + |p|\mathrm{o}(r), \end{aligned} \tag{37}$$

where w_r is a any limit point of the $w_{r,\lambda}$ as $\lambda \downarrow 0$ and therefore $w_r \in \partial B(x, r)$ - so $(w_r - x)/r$ is a unit vector. Since the second term inside the parentheses on the right of the first inequality above has size r and $(w_r - x)/r$ is a unit vector, it follows from the first inequality that $(w_r - x)/r \longrightarrow p/|p|$ as $r \downarrow 0$. (We are assuming that $p \neq 0$, as we may.) Then the inequality of the extremes in (37), after dividing by r and letting $r \downarrow 0$, yields $0 \leq \langle Xp, p\rangle$, as desired.

The above proof contains a bit more information than $0 \le \langle Xp, p\rangle$ if $p = D\phi(x) = 0$. In this case, choosing y so that $|x-y| = r/2$, we have

$$\phi(x) - \phi(y) = \mathrm{O}(r^2)$$

and then (32) yields

$$\mathrm{O}(r^2) \le \frac{1}{2}\langle X\left(\frac{w_{r,y}-x}{r}\right), w_{r,y}-x\rangle + \mathrm{o}(r)$$

where $(w_{r,y}-y)/r$ is a unit vector. Dividing by r, sending $r \downarrow 0$ and using compactness, any limit point of $(w_{r,y}-x)/r$ as $r \downarrow 0$ is a unit vector q for which $0 \le \langle Xq, q\rangle$. In particular, if $D\phi(x) = 0$, then

$$D^2\phi(x) \text{ has a nonnegative eigenvalue.} \tag{38}$$

For insiders, this means that u is also a viscosity subsolution of the normalized infinity Laplace equation:

$$\frac{\Delta_\infty u}{|Du|^2} = 0.$$

The other half is this: if u satisfies ComCo from below, then

$$\varphi \in C^2, u - \varphi \text{ has a local min at } x \implies \Delta_\infty \varphi(x) \le 0. \tag{39}$$

That is, u is also a viscosity subsolution of $\Delta_\infty u = 0$. These results follow directly from what was already shown because $-u$ satisfies ComCo from above.

The meaning of "u is a viscosity solution of $\Delta_\infty u = 0$ is exactly that it is both a subsolution and a supersolution. $\square$

REMARK: The same manipulations may be used to derive the pde associated with AML when the Euclidean norm is replaced by a general norm, but the result is in general an equation with *discontinuous* ingredients.

PROOF: IH $\implies$ ComCo

Suppose that $\Delta_\infty u \geq 0$ i.e., u is a viscosity subsolution of $\Delta_\infty u = 0$, on the bounded set U. Computing the ∞-Laplacian on a radial function $x \mapsto G(|x|)$ yields

$$\Delta_\infty G(|x|) = G''(|x|)G'(|x|)^2$$

if $x \neq 0$ and from this we find that

$$\Delta_\infty (a|x-z| - \lambda|x-z|^2) = -2\lambda(a - 2\lambda|x-z|)^2 < 0$$

for all $x \in U$, $x \neq z$, if $\lambda > 0$ is small enough. But then if $\Delta_\infty u \geq 0$, $u(x) - (a|x-z| - \lambda|x-z|^2)$ cannot have a local maximum in $V \ll U$ different from z, by the very definition of a viscosity solution of $\Delta_\infty u \geq 0$. Thus if $z \notin V \ll U$ and $x \in V$, we have

$$u(x) - (a|x-z| - \lambda|x-z|^2) \leq \max_{w \in \partial V}(u(w) - (a|w-z| - \lambda|w-z|^2)).$$

Now let $\lambda \downarrow 0$ to find that u satisfies ComCo from above. $\square$

Corollary 16. *If* $u, v \in C(\bar{U})$ *and*

$$\Delta_\infty u \geq 0,\ \ \Delta_\infty v \leq 0 \ \textit{in}\ U,$$

then

$$u(x) - v(x) \leq \max_{\partial U}(u - v).$$

In particular, solutions of the Dirichlet problem $\Delta_\infty u = 0,\ \ u = b$ *on* ∂U *are unique.*

Proof. We now know, from IH $\Longrightarrow$ ComCo $\Longrightarrow$ ConvCri that u, v satisfy the assumptions of Corollary 13. □

PROOF: ConvCri $\Longrightarrow$ GradE.

Lemma 17. *Let* u *satisfy* ConvCri *(i) in* U. *Then*

$$|Du|(x) = \lim_{r \downarrow 0} \frac{u^r(x) - u(x)}{r} = \inf_{0<r<\operatorname{dist}(x,\partial U)} \frac{u^r(x) - u(x)}{r} \tag{40}$$

for $x \in U$. *In particular, if* u *is differentiable at* x, *then* $|Du|(x) = |Du(x)|$.

Proof. From Corollary 13 with $v = 0$ and $U = B(x,r)$, we have $u^r(x) = u(y)$ for some $y \in \partial B(x,r)$. It is then clear that $(u^r(x) - u(x))/r \le \mathrm{Lip}(u, B(x,r))$. Then (40) with " $\ge$ " in place of $=$ follows upon sending $r \downarrow 0$. To obtain the opposite inequality, we invoke (7) with $z = x$ to assert that

$$|Du|(x) = \lim_{r\downarrow 0} \mathrm{Lip}(u, B(x,r)) \le \lim_{r\downarrow 0} \frac{u^R(x) - u(x)}{R - 2r} = \frac{u^R(x) - u(x)}{R}.$$

Now we may let $R \downarrow 0$. The final assertion of the lemma is obtained by noting that if u is differentiable at x, then the right hand side of (40) is $|Du(x)|$, as is easily seen. $\square$

Remark 18. Lemma 17 establishes that ConvCri implies GradE.

PROOF OF: Lemma 7. Let us recall that the lemma asserts that if $r \longrightarrow u^r$ is convex and $u^r(x_0) = u(x_r)$, $|x_0 - x_r| = r$, then $|Du|(x_r) \ge |Du|(x_0)$. Set

$$x_t = x_0 + \frac{t}{r}(x_r - x_0)$$

for $0 \le t \le r$. Then

$$\|x_t - x_r\| = (r-t), \|x_t - x_0\| = t,$$

and so $u^{r-t}(x_t) = u(x_r) = u^r(x_0)$, $u^t(x_0) \ge u(x_t)$, and then, for $s \ge r-t > 0$,

$$\begin{aligned} \frac{u^s(x_t) - u(x_t)}{s} &\ge \frac{u^{r-t}(x_t) - u(x_t)}{r-t} = \frac{u^r(x_0) - u(x_t)}{r-t} \\ &\ge \frac{u^r(x_0) - u^t(x_0)}{r-t} \ge \frac{u^r(x_0) - u(x_0)}{r}. \end{aligned}$$

Letting $t \uparrow r$ we conclude that

$$\frac{u^s(x_r) - u(x_r)}{s} \ge \frac{u^r(x_0) - u(x_0)}{r}$$

for $0 < s < \operatorname{dist}(x_r, \partial U)$. Lemma 7 now follows from Lemma 17. $\square$

PROOF** : GradE $\implies$ ConvCri. (starred material will not be presented in detail in lecture, but is written out for you here)

Assume now that u satisfies GradE (i). Let $y \in U$, $0 < r < \operatorname{dist}(y, \partial U)$, $|x-y| \le r$, $\xi(t) = y + t(x-y)$. Using Lemma 9 and then GradE, we have

$$\frac{d}{dt}u(\xi(t)) \le |Du|(\xi(t))|x-y| \le \left(\frac{u^s(\xi(t)) - u(\xi(t))}{s}\right)|x-y|.$$

for $0 < s < \operatorname{dist}(y + t(x-y), \partial U)$. It is convenient to rewrite this as

$$\frac{d}{dt}u(\xi(t)) + \frac{|x-y|}{s}u(\xi(t)) \le u^s(\xi(t))\frac{|x-y|}{s}. \tag{41}$$

If $x \in B(r, y) \ll U$ and $\xi(t) = y + t(x-y)$, then $\operatorname{dist}(\xi(t), \partial U) > r - t|x-y|$. Moreover, $B(r, y) \supset B(r - t|x-y|, \xi(t))$. Thus we may take $s = r - t|x-y|$ in (41) to assert

$$\frac{d}{dt}u(\xi(t)) + \frac{|x-y|}{r - t|x-y|}u(\xi(t)) \le u^r(y)\frac{|x-y|}{r - t|x-y|}. \tag{42}$$

This simple differential inequality integrated over $0 \le t \le 1$ yields

$$u(x) \le u(y) + \frac{u^r(y) - u(y)}{r}|x-y|$$

Finally, this is exactly the consequence of ComCo from which we already derived ConvCri (see (22) and the arguments following).

PROOF OF: Theorem 4.

Now that the equivalences are all proven, see Remark 15.

PROOF OF: Theorem 3. All that remains to explain is the existence assertion. There are various ways to treat existence. The Perron method was used in the origins of the subject by Gunnar, then the limit $p \to \infty$ by Jensen, and discrete approximations also work, Le Gruyer [25], Oberman [28], and it is immediate from viscosity solution theory, given the comparison results now in hand and the existence of subsolutions and supersolutions provided by $\mathcal{M}^{\pm}$, but applying the Perron method is now simplified by the equivalences and comparison results now in hand to nearly a triviality, and the way to go here, imo.

In our case, we can let b be Lipschitz continuous, as the comparison theorem allows us then to take limits to treat general continuous b. The McShane-Whitney extension $\mathcal{M}^{+}(b)$ is the inf of functions satisfying $r \mapsto u_r$ is concave, so it also satisfies this condition; similarly, $r \mapsto \mathcal{M}^-(b)^r$ is convex. Define

$$\mathcal{S} := \left\{ w : w \in C(\bar{U}), r \mapsto w^r \text{ is convex}, w = b \text{ on } \ \partial U \right\}$$

and

$$u(x) = \sup \{w(x) : w \in \mathcal{S}\} = \sup \left\{ w(x) : w \in \mathcal{S}, \mathcal{M}^-(b) \le w \right\}.$$

The second equality is due to $\mathcal{M}^-(b) \vee w \in \mathcal{S}$ if $w \in \mathcal{S}$. Every element of the set on the right above satisfies $w \le \mathcal{M}^+(b)$ by comparison (Corollary 13), and $\mathcal{M}^-(b)$ lies in the set, so it is not empty. Now $\mathcal{M}^-(b) \le u \le \mathcal{M}^+(b)$ and Proposition 2 imply $u \in C(\bar{U}), u = b$ on ∂U. By its definition, $r \mapsto u^r$ is convex, so u is a maximal element of the set on the right above. The punch line is to show that if $r \mapsto u_r$ is not also concave, then this contradicts this maximality.

We switch to ComCo for this last point. If u does not satisfy ConvCri (ii), then there exists a $V \ll U$ and a cone function C such that $\mathrm{ver}(C) \notin V$, and

$$\min_{\bar{V}}(u - C) < \min_{\partial V}(u - C),$$

and then, as in the proof AML $\implies$ ComCo, there exists a nonempty $W \ll V$ such that

$$u < d + C \text{ in } W, \;\; u = d + C \text{ on } \partial W, \;\; \text{where } d = \min_{\partial V}(u - C).$$

Now define

$$\tilde{u}(x) = \begin{cases} u(x) \text{ if } x \in U \setminus W, \\ d + C(x) \text{ if } x \in W. \end{cases}$$

We reach a contradiction if $\tilde{u}$ satifies ComCo from above, for then $\tilde{u} \in \mathcal{S}$ and $u < \tilde{u}$ on W contradicts the definition of u. If $\tilde{u}$ does not satisfy ComCo from above, there is a nonempty $\tilde{W} \ll U$ and a cone function $\tilde{C}$, with $\mathrm{ver}(\tilde{C}) \notin \tilde{W}$, such that

$$\tilde{u} > \tilde{d} + \tilde{C} \text{ in } \tilde{W}, \ \tilde{u} = \tilde{d} + \tilde{C} \text{ on } \partial\tilde{W} \text{ where } \tilde{d} := \max_{\partial\tilde{W}}(\tilde{u} - \tilde{C}). \tag{43}$$

But then $u \le \tilde{u} = \tilde{d} + \tilde{C}$ on $\partial\tilde{W}$ implies $u \le \tilde{d} + \tilde{C}$ in $\tilde{W}$. It follows that $\tilde{u} > u$ on $\tilde{W}$ and then, necessarily, $\tilde{W} \subset W$, and then $\tilde{u} = d + C(x)$ on $\tilde{W}$. However, cone functions satisfy ComCo on sets not containing their vertices, and comparison then contradicts (43).

Some Generalizations

Well, let us remark to begin that none of the "generalizations" below contain all of the results we have already presented. The primary reason is that they almost all place smoothness restrictions on the H, soon to appear, while in our theory, if $|\cdot|$ is a general norm, then $H(p) = |p|^*$, with $|\cdot|^*$ the norm dual to $|\cdot|$. But $p \mapsto |p|^*$ is not C^1 in general.

In the spirit of the problem MinG_∞, consider $H : \mathbb{R}^n \times \mathbb{R} \times U \longrightarrow \mathbb{R}$ and define the absolutely minimizing property

$$\text{AMH} : \operatorname{ess\,sup}_{x\in V} H(Du(x), u(x), x) \le \operatorname{ess\,sup}_{x\in V} H(Dv(x), v(x), x)$$
$$\forall\, V \ll U, v \in C(\bar{V}) \ni u = v \text{ on } \partial V.$$

We will write generic arguments of H as (p, z, x), and H_p below denotes the gradient of $H(p, z, x)$ in p, etc.

What is the PDE in general?

What is the equation corresponding to AMH as $\Delta_\infty u = 0$ corresponds to our old AMG? It is appropriately called the Aronsson Equation, and it is formally given by

$$\mathcal{A}_H[u] = \langle H_p(Du, u, x), H(Du, u, x)_x\rangle = \langle H_p(Du, u, x), D^2uH_p(Du, u, x)\rangle$$
$$+ H_z(Du, u, x)\langle H_p(Du, u, x), Du\rangle + \langle H_p(Du, u, x), H_x(Du, u, x)\rangle = 0.$$

This equation has been shown to be satisfied by AMH functions when $\{p : H(p, z, x) \le \lambda\}$ is convex for all λ, i.e., H is quasiconvex in p, in a variety of circumstances. See [7] for a first proof in this setting, and [17] for greater generality - $H \in C^1$ and either $H = H(p, x)$ or H is convex in p and $H = H(p, z, x)$. The paper [14] is intermediate in generality, $H \in C^2$, but perhaps "easier on the eyes."

DOES $\mathcal{A}_H[u] = 0$ IMPLY AMH?

The answer ([19]) is YES if $H = H(p)$ is C^2, quasiconvex and $H(p) \to \infty$ as $|p| \to \infty$. The answer is YES ([33]) if $H = H(p,x)$ is convex in p and $H(p,x) \to \infty$ as $|p| \to \infty$ uniformly in x. The answer is NOT ALWAYS if $H = H(p,x)$ if H is merely quasiconvex in p. In [33] a counterexample of the form $H(p,x) = (p^2 - 3p)^3 + V(x)$ is given where U is the interval [0,1]. The answer is NOT ALWAYS for $H = H(p,z)$ even if H is convex in p. In [33] it is shown that $H(p,z) = p^2 - z$, $U = [0,2]$ provides a counterexample.

IS THERE UNIQUENESS FOR THE DIRICHLET PROBLEM FOR $\mathcal{A}_H[u] = 0$?

The answer is NOT ALWAYS. In [33] (again! congrats, Yifeng) one finds the example $H(p,x) = p^2 + \sin^2 x$ with $U = [0, 2\pi]$; both $u = \sin x$, and $u \equiv 0$ satisfy $\mathcal{A}_H[u] = 0$, $u(0) = u(2\pi) = 0$. The answer is YES ([21]) if $H = H(p)$ is convex, C^2, tends to infinity at infinity, and the set $\left\{p : H(p) = \min_{\mathbb{R}^n} H\right\}$

has no interior points. In the same paper, sophisticated conditions guaranteeing nonuniqueness in the case $H = H(p, x)$ are also presented.

What are the cones?

We note the nice quote from [24]: "The characterization of solutions in terms of the cone functions $x \mapsto a|x - x_0| + b$, discovered by Crandall, Evans and Gariepy, is arguably the most important tool in the theory of the infinity Laplace equation." We add to this sentiment the remark that the idea has permeated much of the research already mentioned concerning variations and generalizations.

In [13], the case $H = H(p, x)$ is considered, with H quasiconvex in p. In addition, $H(p, x) \geq H(0, x) = 0$, $H(p, x) \to \infty$ as $|p| \to \infty$ uniformly in x, and $(p, x) \to H(p, x)$ is lower semicontinuous. There are no further regularity assumptions. For open $V \subset U$, the "distances"

$$d_\lambda^V(x,y) = \inf\left\{\int_0^1 L(\dot{\xi}(t), \xi(t), \lambda)\, dt : \xi \in \mathrm{path}_V(x,y)\right\}$$

are defined, where

$$\begin{cases} L(p,x,\lambda) = \max_{H(q,x)\le\lambda}\langle q,p\rangle, \\ \mathrm{path}_V(x,y) = \{\text{absolutely continuous } \xi : [0,1] \to V; \xi(0) = x, \xi(1) = y\}. \end{cases}$$

It is shown that u satisfies AMH in U iff $\forall\ V \ll U$, $0 \le \lambda$, $x_0 \in \bar{V}$,

$$\begin{cases} \max_{x\in\bar{V}}(u(x) - d_\lambda^V(x_0,x)) = \max_{x\in\partial V\cup\{x_0\}}(u(x) - d_\lambda^V(x_0,x)), \\ \min_{x\in\bar{V}}(u(x) + d_\lambda^V(x,x_0)) = \min_{x\in\partial V\cup\{x_0\}}(u(x) + d_\lambda^V(x,x_0)). \end{cases} \tag{44}$$

The quantity $d_\lambda^V(x,y)$ has to be defined carefully at boundary points of V. Nice technical tools, applications to Γ-limits, and extensions to metric spaces are also given in [13].

If $H = H(p)$ is convex, $H(p) > H(0) = 0$ for $p \neq 0$, then $p \mapsto L(p, \lambda)$ is basically a norm (up to the possible failure of $L(p, \lambda) = L(-p, \lambda)$).

Writing $C_\lambda(p)$ instead of $L(p, \lambda)$, our simple proof of AMG $\implies$ ComCo adapts (I checked it at the "back of an envelope level") to prove that AMH in U implies

$$\text{(45)} \qquad \begin{cases} \text{(i)} \ \max_{x \in \bar{V}}(u(x) - C_\lambda(x - z)) = \max_{x \in \partial V}(u(x) - C_\lambda(x - z), \\ \text{(ii)} \ \min_{x \in \bar{V}}(u(x) + C_\lambda(z - x) = \min_{x \in \partial V}(u(x) + C_\lambda(z - x)), \end{cases}$$

provided that $V \ll U$, $z \notin V$. Note that (ii) follows from (i) because $-u$ is absolutely minimizing for $H(-p)$. With these "cones," [19] extends much of the program we have outlined for $H(p) = |p|$ to $H \in C^2$, quasiconvex, nonnegative, $H(0) = 0$, $H(p) \to \infty$ as $|p| \to \infty$, and the level sets of H have no interior points. In particular, if (45) (formulated somewhat differently, as in our special case (31) of ComCo) holds, there is an upper-semicontinuous representative of $H(Du)$ (our old $|Du|(x)$), and (45) is equivalent to $\mathcal{A}_H[u] = 0$. A tricky point is that the equation implies (45), the simple proof we gave not being available here.

What becomes of the convexity criteria?

In [24], it is shown that the convexity condition in the case $H = H(p)$, $H \in C^2$, H locally uniformly convex and superlinear, $H(p) \geq H(0) = 0$, $u \in$ AMH in $\mathbb{R}^n$ is equivalent to the convexity of $t \mapsto w(x,t)$ and the concavity of $t \mapsto v(x,t)$, where w, v are the unique (viscosity) solutions of

$$\begin{cases} w_t - H(Dw) = 0 \text{ in } \mathbb{R}^n \times (0,\infty), \\ w(x,0) = u(x), \end{cases} \tag{46}$$

$$\begin{cases} v_t + H(Dv) = 0 \text{ in } \mathbb{R}^n \times (0,\infty), \\ v(x,0) = u(x). \end{cases} \tag{47}$$

This answers a conjecture in [6]. See, e.g., [24] for some discussion of the interaction between Hamilton-Jacobi flows and degenerate elliptic pde's. While this statement is for functions defined in $\mathbb{R}^n$, it is "localized" in [24].

Who Else Cares?

Weak KAM Theory

In [34], applications of the theory of AMH functions to Weak KAM Theory are given. We'll say no more about that.

Image Processing

The sources [1], [12], [11], [27] are examples of papers which use "absolutely minimizing" in one way or another in image processing. The following pic was lifted from [27].

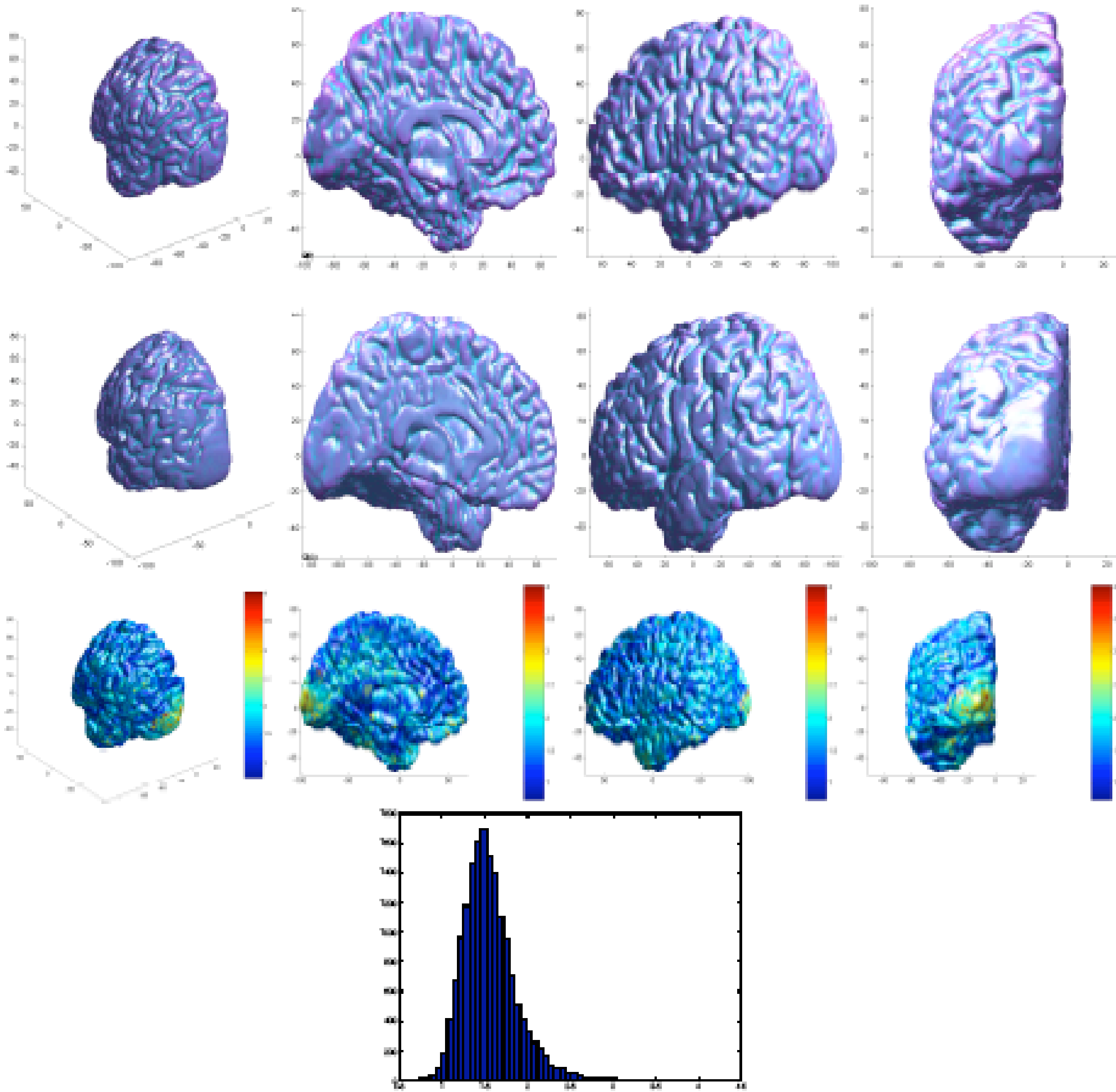


Figure 3: *Warping between the cortical surfaces of two brains. In the first row we show 4 views of $\mathbb{B}_1$: posterior, medial, lateral and directly viewing the occipital cortex. The corresponding 4 views of $\mathbb{B}_2$ are shown in the second row. In the third row, we show $\mathbb{B}_1$ with texture $I(x_i) = L_i(\Phi)$ which can interpreted as a measure of local deformation needed to match $x_i \in \mathbb{B}_1$ to $\Phi(x_i) \in \mathbb{B}_2$. Relatively little deformation (blue colors) is required to match features across subjects on the flat interhemispheric*

Game Theory

To the surprise of the community, the equation $\Delta_\infty u = 0$, or, more generally, the equation

$$\frac{\Delta_\infty u}{|Du|^2} = f(x) \tag{48}$$

arose from game theory in [29]. The associated operator

$$\Delta_\infty^N u = \frac{\Delta_\infty u}{|Du|^2}, \tag{49}$$

where N stands for "normalized," is itself often simply denoted by Δ_∞, and it seems to be the correct one to consider for "forced" equations such as (48). This theme expanded in [6]. Hereafter, in the current discussion, Δ_∞ means Δ_∞^N.

To give a feeling for the set up, we have lifted the following text from [3], modified slightly here, (and any errors are due to us).

"Let us briefly review the notion of two-player, zero-sum, random-turn tug-of-war games, which were first introduced by Peres, Schramm, Sheffield, and Wilson [29]. Fix a number $\varepsilon > 0$. The dynamics of the game are as follows. A token is placed at an initial position $x_0 \in \Omega$. At the kth stage of the game, Player I and Player II select points x_k^I and x_k^{II} , respectively, each belonging to a specified set $A(x_{k-1}, \varepsilon) \subset \bar{\Omega}$. The game token is then moved to x_k, where $x_k$is chosen randomly so that $x_k = x_k^I$ with probability $P = P(x_{k-1}^I, x_k^I, x_k^{II})$ k) and $x_k = x_k^{II}$ with probability $1 - P$, where P is a given function. After the kth stage of the game, if $x_k \in \Omega$, then the game continues to stage k+1. Otherwise, if $x_k \in \partial\Omega$, the game ends and Player II pays Player I the amount

$$\text{Payoff } = g(x_k) + \frac{\varepsilon}{2} \sum_{j=1}^{k} q(\varepsilon, x_{j-1}, x_j) f(x_{j-1}), \tag{50}$$

where q is a given function. We call g the terminal payoff function and f the running payoff function. Of course, Player I attempts to maximize the payoff, while Player II attempts to minimize it. A strategy for a Player I is a mapping

σ_I from the set of all possible partially played games $(x_0, x_1, ..., x_{k-1})$ to moves $x_k^I \in A(x_{k-1}, \varepsilon)$, and a strategy for Player II is defined in the same way. Given a strategy I for Player I and a strategy II for Player II, we denote by $F_I(\sigma_I, \sigma_{II})$ and $F_{II}(\sigma_I, \sigma_{II})$ the expected value of the expression (50) if the game terminates with probability one, and this expectation is defined in $[-\infty, \infty]$. Otherwise, we set $F_I(\sigma_I, \sigma_{II}) = -\infty$ and $F_{II}(\sigma_I, \sigma_{II}) = \infty$. (If the players decide to play in a way that makes the probability of the game terminating less than 1, then we penalize both players an infinite amount.) The value of the game for Player I is the quantity $\sup_{\sigma_I} \inf_{\sigma_{II}} F_I(\sigma_I, \sigma_{II})$ where the supremum is taken over all possible strategies for Player I and the infimum over all possible strategies for Player II. It is the minimum amount that Player I should expect to win at the conclusion of the game. Similarly, the value of the game for Player II is $\inf_{\sigma_{II}} \sup_{\sigma_I} F_{II}(\sigma_I, \sigma_{II})$, which is the maximum amount that Player II should expect to lose at the conclusion of the game. We denote the value for Player I as a function of the starting point $x \in \Omega$ by $V_I^\varepsilon(x)$, and similarly the value for Player II by $V_{II}^\varepsilon(x)$. We extend the value functions to $\partial\Omega$ by setting $V_I^\varepsilon(x) = V_I^\varepsilon(x) = g(x)$ there. It is clear that $V_I^\varepsilon \le V_{II}^\varepsilon$. The game is said to have a value if $V_I^\varepsilon = V_{II}^\varepsilon =: V^\varepsilon$.

The tug-of-war game studied in [29], which in this paper we call standard *standard ε-step tug-of-war,* is essentially the game described above for

$$A(x,\varepsilon) = \bar{\Omega}(x,\varepsilon) = \left\{ y \in \Omega : d^{\bar{\Omega}}(y,x) \le \varepsilon \right\}, P \equiv \frac{1}{2},\ q(\varepsilon,x,y) = \varepsilon.$$

where $d^{\bar{\Omega}}(x,y)$ is the infimum of the length of Lipschitz continuous paths in $\bar{\Omega}$ from x to y. In other words, the players must choose points in the ε-balls centered at the current location of the token, a fair coin is tossed to determine where the token is placed, and Player II accumulates a debt to Player I which is increased by $\frac{1}{2}\varepsilon^2 f(x_{k-1})$ after the kth stage."

The value functions for the standard ε-step tug-of-war game satisfy the relation

$$2V(x) - \left(\sup_{\bar{\Omega}(x,\varepsilon)} V + \inf_{\bar{\Omega}(x,\varepsilon)} V \right) = \varepsilon^2 f(x), \ x \in \Omega, \tag{51}$$

which is easily seen to be an approximation, if $V \in C^2$, to

$$-\Delta_\infty V = f(x). \tag{52}$$

However, V in general is not even continuous ([3]). Above and later, Δ_∞ is the "normalized" version.

It was proved in [29] that the standard ε-step tug-of-war game has a value V^ε and $V^\varepsilon \longrightarrow$ the one and only viscosity solution of (52) satisfying $V = g$ on $\partial\Omega$, provided that $f \equiv 0$ or $\inf f > 0$ or $\sup f < 0$.

By modifying the P and q of the standard ε-step tug-of-war game to

$$P(x,y,z) = \frac{\rho_\varepsilon(x,z)}{\rho_\varepsilon(x,z) + \rho_\varepsilon(y,z)}, \ q(\varepsilon,x,y) = \rho_\varepsilon(x,y),$$

where

$$\rho_\varepsilon(x,y) = \begin{cases} \max(d^\Omega(x,y),\varepsilon) & \text{if } \ x,y\in\Omega \\ d^{\bar\Omega}(x,y) & \text{if } \ x \text{ or } y\in\partial\Omega \end{cases}$$

to obtain a game they call *the boundary-biased ε-step tug-of-war*, Armstrong and Smart [3] found a modification of (51), namely

$$-\varepsilon\Delta_\infty^\varepsilon V := \sup_{y\in\bar\Omega(x,\varepsilon)} \frac{V(x)-V(y)}{\rho_\varepsilon(x,y)} - \sup_{y\in\bar\Omega(x,\varepsilon)} \frac{V(y)-V(x)}{\rho_\varepsilon(x,y)} = \varepsilon f(x) \tag{53}$$

which has continuous solutions, and use it to prove a remarkable array of results concerning the existence of solutions of the Dirichlet problem for (52), information about the degree to which solutions are not unique, as well as a new simplified proof of the uniqueness in the cases $f\equiv 0$ (we called that case "The Elemental Comparison Theorem") and $f>0$ and $f<0$. Uniqueness fails for some f's which change sign ([29]). The first pde proof of this uniqueness was given in [26]; the proof of [3] is free of the heavy probability theory of [29] and the nontrivial machinery of viscosity solution theory used in [26].

Equations closely related to (51), (53) are already present in [28], [25].

What Has Been Left Out?

Well, a tremendous lot. The good way to get a feeling for developments not touched on here might be, repeating ourselves, to look at the articles citing [5] on Google Scholar, or, even better, the about 180 articles citing Gunnar's kick off paper [4] on Google and the about 190 articles citing Bob's original uniqueness proof [20].

We mention just two more lines. One is the "∞ -eigenvalue problem" of [23], which itself has 55 cites on Google Scholar. Another centers around the Harnack inequality - we haven't mentioned it in these notes, but, in a simple form, it comes easily from either ComCo or ConvCri; see, e.g., [5]. The "boundary" version is more subtle, see, e.g., [22] for a recent presentation and references. It is involved in the proofs of interesting special facts, such as the proof that a nonnegative IH function in an upper half space which vanishes on the boundary must be a multiple of the distance to the boundary ([9]), and the proof that IH functions are differentiable at points of $\partial B(x,r)$ at which $u^r(x)$ is assumed ([10]). However, in the latter case, the Harnack inequality can be avoided, I think. See other articles by the author of [9], [10] as well.

A Couple of Open Problems

The outstanding open problem in the theory is regularity. The Aronsson example (15) limits the possibilities to $C^{1,\alpha}$. It is proved in [30] that solutions of $\Delta_\infty u = 0$ are C^1 if $n = 2$, and this proof was extended to suitable $H(p)$ in [32]. The C^1 regularity was sharpened to $C^{1,\alpha}$ in [18]. But $n = 2$?

Think about it: if $r \mapsto u^r$ is convex and $r \mapsto u_r$ is concave, is u necessarily C^1? Anybody can think about that.

A much less important problem, but one I like, is this: in the case of a general norm $|\cdot|$, does satisfaction of the - generally discontinuous - Aronsson equation imply ComCo? An affirmative answer is given for the cases of the maximum norm and the l_1^n norm in [16], where you can learn a precise formulation of the question. My guess is that it is true in general.

Here is another, easily stated: are solutions of the Dirichlet problem for (52) unique if merely $f \geq 0$? See the end of [3] for other problems about the infinity Poisson equation.

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