Hand in Problems Due February 13

Prologue

The web assignments on solving first-order linear ordinary differential equations are carefully rigged so that the integrations can be done in closed form. This creates the very wrong impression that this is the usual situation. It is important to have some problems which can be solved explicitly, they help us understand things. However, it is very important to understand the general, usual, case when the integrations cannot be done explicitly.

THERE WILL NOT BE A PROBLEM LIKE THIS ON THE MIDTERM, BUT THE CHANCES OF SOME VARIANT ON THE FINAL ARE OVER 50%.

We recall that the solution of the problem

$$Ly = y' + p(t)y = f(t), \quad y(t_0) = y_0 \tag{1}$$

is given by the "variation of parameters formula"

$$y(t) = e^{-\int_{t_0}^{t} p(r)\,dr} y_0 + \int_{t_0}^{t} e^{-\int_{s}^{t} p(r)\,dr} f(s)\,ds. \tag{2}$$

Part 1

Derive (2) by any method. Justify all your steps with appropriate verbal explanations, invoking the Chain Rule, the Fundamental Theorem of Calculus, etc., explicitly as necessary. Use your own words, not mine.

Part II

In this part, you are asked to check that (2) works, not to find it. First show, by direct computation, that

$$u_h(t) = e^{-\int_{t_0}^{t} p(s)\,ds} y_0$$

solves

$$Lu_h = 0 \quad \text{and} \quad u_h(t_0) = y_0.$$

Then show, by direct computation, that

$$u_p(t) = \int_{t_0}^{t} e^{-\int_{s}^{t} p(r)\,dr} f(s)\,ds$$

solves

$$Lu_p = f(t) \quad \text{and} \quad u_p(t_0) = 0.$$

Explain why it then follows that $L(u_h + u_p) = f$ and $u_h(t_0) + u_p(t_0) = y_0$.

This part might well confuse you, so ask for help if it does. You may want to use that

$$\frac{d}{dt} \int_{t_0}^{t} h(t,s)\,ds = h(t,t) + \int_{t_0}^{t} h_t(t,s)\,ds \tag{3}$$

for any reasonable function $h(t,s)$. If you use this, say "by equation (3) of the handout" at that point.

Part 3

We shouldn't get completely hooked on using y's and t's, so use (2) to write down the solution of

$$\frac{d}{dx} w(x) + e^{x} w(x) = \cos(x), \quad w(5) = \pi.$$