# Essential closed surfaces in bounded 3-manifolds.

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## 1 Introduction.

A question dating back to Waldhausen [10] and discussed in various contexts by Thurston (see [9]) is the problem of the extent to which irreducible 3−manifolds with infinite fundamental group must contain surface groups. To state our results precisely, it is convenient to make the definition that a map $i : S \looparrowright M$ of a closed, orientable connected surface $S$ is *essential* if it is injective at the level of fundamental groups and the group $i_*\pi_1(S)$ cannot be conjugated into a subgroup $\pi_1(\partial_0 M)$ of $\pi_1(M)$, where $\partial_0 M$ is a component of $\partial M$. This latter condition is equivalent to the statement that the image of the surface $S$ cannot be freely homotoped into $\partial M$.

One of the main results of this paper is the following:

**Theorem 1.1** *Let $M$ be a compact 3-manifold with nonempty incompressible boundary. Suppose that the interior of $M$ has a complete hyperbolic structure.*

*Then either $M$ is covered by a product $F \times I$ where $F$ is a closed orientable surface or $M$ contains an essential surface $S$ of genus at least 2.*

*Furthermore $S$ can lifted to an embedded nonseparating surface in a finite cover of $M$.*

In particular this gives a complete resolution to Waldhausen and Thurston's questions in the context of manifolds with non-empty incompressible boundary:

**Corollary 1.2** *Let $M$ be a compact irreducible manifold with nonempty incompressible boundary.*

*Then either $M$ is covered by a product $F \times I$ or $M$ contains an essential closed surface.*

In particular, the case that all the boundary components of $M$ are tori gives a generalisation of Corollaries 1 & 2 of [2].

Indeed our methods show much more is true. In the course of the proof of Theorem 1.1, in the case when $M$ is hyperbolic, we exhibit a finite cover of $M$ containing a pair of disjoint non-separating embedded essential closed surfaces. Combining this with some 3-manifold topology shows:

**Theorem 1.3** *Let $M$ be a compact irreducible 3-manifold with nonempty incompressible boundary.*

*Then either $\partial M$ consists only of tori and is covered by a product $T^2 \times I$ or $\pi_1(M)$ virtually maps onto a free group of rank 2.*

The method of proof used in the last clause of Theorem 1.1 has a further application. Defining the *peripheral homology* to be the image of the map $incl_* : H_2(\partial M; Z) \longrightarrow H_2(M; Z))$, a stronger version of Waldhausen's question is to ask if one can create nonperipheral homology by passing to a finite sheeted covering. This question was also raised by Gabai in [3]. The proof of Theorem 1.3 creates this homology:

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**Corollary 1.4** *Let $M$ be a 3-manifold with nonempty incompressible boundary. Suppose that the interior of $M$ has a complete hyperbolic structure. Then given any integer $K$, there is a finite sheeted covering $\tilde{M} \longrightarrow M$ so that $H_2(\tilde{M})/incl_*(H_2(\partial\tilde{M}))$ has rank at least $K$.*

This paper is organised as follows. Since it contains the main ideas and is perhaps of most interest, §2 is devoted to the proofs in the case that all the boundary components are incompressible tori. In §3 we prove a technical lemma which enables the proof in the case of higher genus boundary components.

The results of this paper grew out of attempts to generalise Corollaries 1 & 2 of [2]; the authors gratefully acknowledge the inspiration provided by this paper.

## 2 Proof of the case all boundary components are tori.

Throughout the following we assume that all manifolds are orientable and compact and that all homology groups have $Z$-coefficients.

Recall that if $M$ is a complete orientable hyperbolic 3-manifold of finite volume, and noncompact, then the ends of the manifold are all of the form $T^2 \times [0, \infty)$.

**Lemma 2.1** *Let $M$ be a 3-manifold with nonempty incompressible boundary, all of whose components are tori. Suppose that the interior of $M$ has a complete hyperbolic structure with finite volume.*

*Then there is a finite cover of $M$ which has at least 3 torus boundary components.*

**Proof.** Fix some torus $T$ in $\partial M$. Choose a non-trivial element $x$ not in $\pi_1 T$, and set $y = [x^2, m]$ where $m$ is a non-trivial element of $\pi_1 T$. Since $M$ is hyperbolic $y$ is non-trivial.

Moreover, since $\pi_1 M$ is residually finite, there is a finite group $G$ and an epimorphism $\theta$ of $\pi_1 M$ onto $G$ with $\theta(y)$ non-trivial. Then the finite covering $M_\theta$ which corresponds to the kernel of the homomorphism has at least 3 torus components covering $T$. To see this, note that the index of $H = \theta(\pi_1 T)$ in $G$ equals the number of pre-images of $T$ in $M_\theta$.

However, if $H$ has index at most 2, then since $H$ is normal in $G$, $\theta(x)^2$ would lie in $H$, an abelian group, so that $[\theta(x^2), \theta(m)] = \theta([x^2, m]) = \theta(y)$ would be trivial, a contradiction. □

**Definition** An oriented surface $F$ which is properly embedded in a 3-manifold $M$ is said to *meet the boundary efficiently* if for each component $A$ of $\partial M$ the following condition is satisfied. Either $F \cap A = \emptyset$ or $F \cap A \neq \emptyset$, and giving the boundary components of $F$ the induced orientation, every non-empty subset of components of $F \cap A$ is non-zero in $H_1(A)$.

In this section, we are primarily interested in the case that $\partial M$ consists of tori, in which case the condition may be stated as: all the components of $\partial F$ are parallel circles, oriented the same way, and not bounding a disc in $A$.

**Lemma 2.2** *Suppose $M$ is a compact orientable 3-manifold with toriodal boundary having at least 3 torus boundary components. Then there is a connected, non-separating properly embedded incompressible surface $F$ in $M$ which is disjoint from at least one torus boundary component. Furthermore $[\partial F]$ is not zero in $H_1(\partial M)$ and $F$ meets the boundary of $M$ efficiently.*

**Proof.** Let $T$ be a torus boundary component of $M$ and $\partial_- M = \partial M - T$. Then from the long exact sequence of the pair $(M, \partial M)$ and Poincaré-Lefschetz duality, the rank of

$$K \equiv ker[incl_* : \ H_1(\partial_- M) \longrightarrow H_1(M)]$$

is at least

$$\frac{\beta_1(\partial M)}{2} - 2 > 0.$$

Let $\alpha$ be a primitive class in $K$ then there is a non-separating properly embedded surface $F$ in $M$ with

$$[\partial F] = \alpha.$$

We now arrange that $F$ meets the boundary of $M$ efficiently. First, any components of $\partial F$ which bound discs in $\partial M$ can be capped off. After this, whenever $F \cap \partial M$ contains a pair of curves which are adjacent and parallel but are oppositely oriented on some component of $\partial M$, we can cap them off with an annulus. This surface is therefore disjoint from $T$, and meets the boundary efficiently. Now compress $F$ to obtain an incompressible surface. The resulting surface represents the same relative homology class and so there is a component with boundary non-zero in $H_1(\partial M)$. $\square$

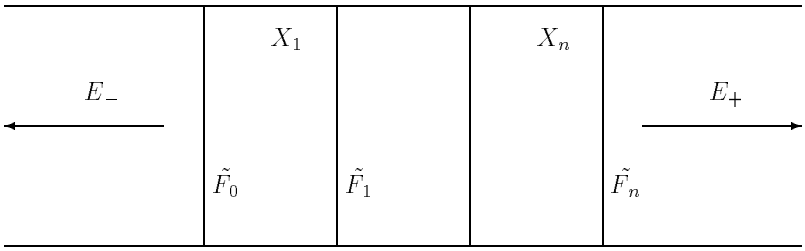


**Figure 1.** The covering $M_F$

**Theorem 2.3** *Let $M$ be a 3-manifold with nonempty incompressible boundary, all of whose components are tori. Suppose that the interior of $M$ has a complete hyperbolic structure with finite volume.*

*Then $M$ contains an essential surface $S$ of genus at least 2.*

*Furthermore $S$ can lifted to an embedded nonseparating surface in a finite cover of $M$.*

**Proof.** By Lemma 2.1, after passing to a finite cover, we may suppose that the boundary of $M$ contains at least 3 tori.

Now it follows from Lemma 2.2, that we can find $F$, a 2-sided non-separating incompressible surface properly embedded in $M$, with $[\partial F]$ non-zero in $H_1(\partial M)$. We may also suppose that $F$ meets the boundary of $M$ efficiently, and is disjoint from some torus boundary component of $M$.

Set $X \cong M - intN(F)$ and let $X_i = X \times i$ be a copy of $X$. Then the infinite cyclic cover

$$p : M_F \longrightarrow M$$

dual to $F$ is $M_F = \bigcup_i X_i$. Define a submanifold $Y_n = \bigcup_{i=0}^{n} X_i$. Let $\tilde{F}_i$ be the components of $p^{-1}F$ labelled so that $\tilde{F}_i = X_i \cap X_{i+1}$. Since $F$ is incompressible in $M$ it follows that $\tilde{F}_i$ is incompressible in $M_F$. Then

$$M_F = E_- \cup_{\tilde{F}_0} Y_n \cup_{\tilde{F}_n} E_+$$

where $E_- = \bigcup_{i \leq 0} X_i$ and $E_+ = \bigcup_{i > n} X_i$. Two applications of Van Kampen's theorem then implies that

$$incl_* : \pi_1 Y_n \longrightarrow \pi_1 M_F$$

is injective and composing with the covering space projection one sees that

$$(p|Y_n)_* : \pi_1 Y_n \longrightarrow \pi_1 M$$

is injective.

Partition the boundary of $Y_n$ into two parts: $G$ is the component containing $\tilde{F}_0 \cup \tilde{F}_n$ and is diffeomorphic to the double of $F$; and $W$ is the rest of $\partial Y_n$.

Let $S$ be the possibly disconnected surface in $Y_n$ obtained by compressing $G$ in $Y_n$ as much as possible. Since $[G] + [W] = 0$ in $H_2(Y_n)$ it follows that $[S] = -[W]$. Suppose that some component of $S$ is a sphere, then since $M$ is hyperbolic, $S$ bounds a ball $B$ in $M_F$. Now $E_-, E_+$ are both non-compact, thus $B$ is contained in $Y_n$. Let $S'$ be the subsurface of $S$ obtained by deleting the sphere components. Then $[S'] = -[W]$ also. Suppose that every component of $S'$ is parallel to some component of $W$. The only relation between the classes in $H_2(Y_n)$ represented by the boundary components of $Y_n$ is that their sum is zero. Thus the components of $W$ (which excludes $G$) freely generate a subgroup of $H_2(Y_n)$. The equation $[S'] = -[W]$ thus implies that $S'$ has at least as many components as $[W]$.

Now suppose that we have chosen $n$ so that $n > \beta_1(F)$. We claim that this implies that some component $S_1$ of $S'$ is not parallel to any component of $W$.

Suppose to the contrary that every component of $S'$ is parallel to some boundary component of $W$ in $Y_n$. If we let $D$ be any boundary component of $\partial M$ which is disjoint from $F$, then $W$ has at least $n$ such components since each component of $D$ lifts. However, each component of $\partial M$ is a torus and so the observations of the previous paragraph show that we have the inequality: $\beta_1(S') \geq 2n$. But

$$\beta_1(S') \leq \beta_1(G) = 2\beta_1(F).$$

We see that this is a contradiction with our chosen $n$, so that we have shown that some component $S_1$ of $S'$ is an embedded incompressible surface in $Y_n$ not parallel to any component of $W$.

We must now show that the projection of the surface $S_1$ is not homotopic into the boundary of $M$. Suppose to the contrary that it were. Notice that this boundary component cannot be any of the boundary components which meet $F$. The reason is that $F$ meets the boundary of $M$ efficiently, so every such component in $\partial M$ which meets $F$ is covered by noncompact surfaces, in particular there would be free homotopy classes on $S_1$ which did not lift, a contradiction.

Then we lift this homotopy back into $M_F$ and see a homotopy between an embedded surface $S_1$ and some component $\partial_W M$ of the boundary of $M_F$ which is a preimage of a boundary component not meeting $F$. Two disjoint embedded surfaces in a 3-manifold which are homologous cobound an embedded submanifold $H$.

We have already argued that $\partial_W M$ cannot lie in $Y_n$. However, this means that $G$ separates the boundary components of $H$, so that $G$ lies in the interior of $H$. Since $G$ meets $\partial M_F$ along some annuli, the interior of $H$ meets these annuli in $\partial M_F$ which is absurd. This contradiction shows that the projection of $S_1$ cannot be homotoped into $\partial M$.

Notice that it follows from the construction that the surface $S_1$ embeds in a finite cyclic covering of a finite covering of the manifold $M$. □

**Corollary 2.4** *Let $M$ be an irreducible manifold with nonempty boundary, all of whose components are incompressible tori.*

*Then either $M$ is covered by $T^2 \times I$ or $M$ contains an essential surface.*

**Proof.** The manifold $M$ is Haken, so it has a torus decomposition; if this decomposition contains an essential torus, then we are finished.

So we may assume that the torus decomposition has all tori being boundary parallel. It follows from Thurston's geometrization of Haken manifolds that $M$ is either Seifert fibred or hyperbolic.

If the manifold is Seifert fibred we consider possibilities for the base. This must have boundary, so that it must be a flat or hyperbolic orbifold. One sees easily (for example by taking a manifold covering) that if the base is hyperbolic, one can always find a vertical immersed torus satisfying the conclusion of the corollary. Thus the only possibility for failure is if the base is a flat orbifold with boundary which is therefore covered by an annulus. Pulling back gives the case of a virtual product.