Appendix to A rigidity theorem for the solvable Baumslag-Soleter groups.

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Introduction

The study of properties of a metric space which are preserved by bilipschitz homeomorphism occurs in the study of groups via a word metric. It was also studied in [1] for a certain type of Cantor set embedded in the real line. The Cantor sets concerned are mild generalizations of the original middle-third Cantor set. This Cantor set has the basic property that it is the union of two exact copies of itself each scaled down in size by a factor of 1/3. The generalization allows finitely many linear scale factors. It is easy to see that the Hausdorff dimension of this type of Cantor set depends only on these scale factors. Now a bilipschitz homeomorphism preserves Hausdorff dimension, and so a natural question is what, if any, further invariants other than Hausdorff dimension are there. An almost complete answer was given in [1], using invariants derived from the Hausdorff measure. This work was generalized to analogous Cantor sets in Euclidean space of dimension n by H. Vuong in his thesis, [2], [3]. In this section, we generalize in a different direction to abstract metric Cantor sets which possess a certain linear self-similarity structure. There is a further generalization to a much wider class of Cantor set, where the self-similarity structure is smooth rather than linear. This will not be dealt with here.

The main result is Theorem (0.6) which states that every bilipschitz homeomorphism between two Cantor sets of the type we consider has constant Radon-Nikodym derivative on some clopen. The reason is, roughly speaking, that a bilipschitz map only permits a limited amount of metric distortion. The self-similar bumpiness of a Cantor set forces non-linearity to come in discrete amounts. Thus a finite number of discrete distortions exhausts the amount of metric distortion possible. Actually, this argument should be made at the level of Hausdorff measure rather than at the level of the metric.

The main ideas are already in [1], except that there the topological properties of the real line are heavily used. This obstacle was overcome in [2] with the introduction of *separation* defined below.

Cantor Sets

A map $f : X \longrightarrow Y$ between metric spaces is called a *similarity map* if there is a constant $K > 0$ such that $d_Y(fx_1, fx_2) = K\ d_X(x_1, x_2)$ for all x_1, x_2 in X. Consider a Cantor set C equipped with a metric d_C then a *clone structure* on C is the following data. Let $A_1, A_2, \cdots, A_n$ be a partition of C into clopens and for each A_i let τ_i be a similarity map of A_i onto C. We will call such a Cantor set a *clone Cantor set.*

Any composition of the maps τ_i is called a *clone map,* (we include the identity map as a clone map) thus the set of clone maps forms a semi-group. A *clone* of C is the pre-image of C under a clone map. Thus C and each of the A_i are clones. More generally any pre-image of a clone under a clone map is also a clone. For every clone there is a clone map sending the clone onto C. The clone

structure is not unique, since one may replace any clone by finitely many clones which partition it. The *level* of a clone is defined inductively. The clone C is level 0, and a clone has level n if the smallest clone properly containing it has level $n-1$.

Two clones are either disjoint or else one is a subset of the other. Thus one may construct a tree where the vertices are clones and there is an edge between a clone of level n and any clone of level $n+1$ that it meets (hence contains.)

Lemma 0.1 *Every clopen is a finite union of clones.*

Given a subset A of a Cantor set C define the *separation* of A to be the minimum distance between A and its complement,

$$sep(A) = \inf \{ d_C(x,y) \ : \ x \in A, \ \ y \in C - A \ \}.$$

By convention, $sep(C) = \infty$. Recall that the *diameter* of a metric space is the supremum of the distance between points in the metric space. Now define the *relative separation* of A to be $rel(A) = sep(A)/diam(A)$. We will make frequent use of the minimum separation between level-1 clones:

$$\alpha(C) = \min\{ \ sep(A) \ : A \text{ is a level-1 clone of } C \ \}.$$

An ϵ-*cover* of a metric space, X, is a cover by sets each of which has diameter at most ϵ. Given a cover $\mathcal{U} = \{ \ U_i \ \}$ of X and $\delta > 0$ define

$$\mathcal{H}_\delta(\mathcal{U}) = \sum_i \ [diam(U_i)]^\delta.$$

The *Hausdorff-δ outer measure* of X is the limit as $\epsilon \to 0$ of the $inf \ \mathcal{H}_\delta(\mathcal{U}_\epsilon)$ where the infimum is taken over all ϵ-covers, $\mathcal{U}_\epsilon$, of X. We will assume that the Hausdorff measure μ_C of C in its Hausdorff dimension is finite and not zero.

A map $f : (X, \mu_X) \longrightarrow (Y, \mu_Y)$ between measure spaces is called *measure linear* if there is a constant $K > 0$ such that for every measurable set A in X we have $\mu_Y(fA) = K\mu_X(A)$. In other words, the Radon-Nikodym derivative of f is K everywhere. We also describe this by saying that the map is linear at the level of measure theory. A similarity map between metric spaces is measure linear for Hausdorff measures, thus in particular clone maps are measure linear. Similarly the notion of bilipschitz at the level of measure theory may be defined. Let f be a bilipschitz homeomorphism between two such Cantor sets with bilipschitz constant K. Then f is K^δ-bilipschitz for δ-dimensional Hausdorff measure. It follows that the two Cantor sets have the same Hausdorff dimension.

We will use the term *approximates* to mean that the ratio of two numbers is bounded away from 0 and ∞ by numbers depending only on certain constants.

The following says that the separation of a clone approximates its diameter.

Lemma 0.2 *Given a clone Cantor set C there is a constant $\xi(C) > 1$ such that for every clone A of C with $A \neq C$ we have*

$$\xi(C)^{-1} \ \leq \ rel(A) \ \leq \ \xi(C).$$

Proof Let x, y be points in A and $C - A$ of minimal distance apart and let B be the smallest clone containing containing both of them. Let τ be the clone map taking B onto C, thus $rel(\tau A) = rel(A)$. After replacing A by τA we may assume that $B = C$ and τ is the identity.

Now x, y are in different level-1 clones by choice of B so $d_C(x,y) \geq \alpha(C)$. Let D be the smallest clone properly containing A. Suppose that $diam(D) < \alpha$. Then there every point in $D - A$ is closer to A than x is to y, a contradiction. Thus $diam(D) \geq \alpha$ and there are only finitely many clones this

large, so that there is a bound on $level(D)$. By choice of D we have $level(A) = level(D) + 1$ and so there are only finitely many possibilities for A. Hence $\xi(C)$ is the maximum of a finite set. ■

The next result says that the smallest clone containing a given set has approximately the same diameter as that set.

Lemma 0.3 *Given a clone Cantor set C there is a constant $\mu(C)$ with the following property. Let X be any subset of C of positive diameter and let A be the smallest clone of C containing X. Then*

$$diam(X) \leq diam(A) \leq diam(X)\mu(C).$$

Proof Let τ be the clone map sending A onto C. Then $\tau(X)$ is not contained in any level-1 clone of C by definition of A. Thus $\tau(X)$ contains points in two different level-1 clones of C, and so $diam(\tau X) \geq \alpha(C)$. Hence

$$\frac{diam(A)}{diam(X)} = \frac{diam(\tau A)}{diam(\tau X)} \leq \frac{diam(C)}{\alpha(C)}.$$

Taking $\mu(C) = diam(C)\alpha(C)^{-1}$ gives the result. ■

The clone structure of a clone Cantor set provides a natural way to magnify phenomena, in particular a way to magnify subsets. Given a subset, A, of a clone Cantor set C there is some smallest clone, B, which contains A. Now B is a copy of C but linearly scaled down. The clone structure provides an identification, $\tau : B \longrightarrow C$, of B with C. Then τA is a magnified copy of A in C, in particular the diameter of τA is at least α. This technique is used to enlarge phenomena which happen on a very small scale.

Lemma 0.4 *Given $K > 1$, $\epsilon > 0$ and clone Cantor sets C, C' there is a constant M with the following property. Let f be any K-bilipschitz map of C onto a clopen in C', and suppose that $rel(fC) \geq \epsilon$. Let A be any clone of C. If P' is the smallest clone of C' containing fA and τ' is the clone map taking P' onto C' then there are at most $M = M(C, C', K, \epsilon)$ possibilities for the image $\tau'(fA)$ as a subset of C'.*

Proof Since f is K-bilipschitz $diam(fA) \leq K\ diam(A)$. Choose x in fA and y in $C' - fA$ minimizing the distance between x and y. If y is in $C' - fC$ then $sep(fA) \geq sep(fC)$ and $diam(fA) \leq diam(fC)$ so $rel(fA) \geq \epsilon$. Otherwise if y is in fC then,

$$sep(fA) = d_{C'}(x, y) \geq K^{-1} d_C(f^{-1}x, f^{-1}y) \geq K^{-1} sep(A).$$

Using lemma (0.2) we get that $sep(fA) \geq K^{-1} diam(A)\xi(C)^{-1}$. Hence in either case

$$rel(fA) \geq min\left(\frac{K^{-1} diam(A)\xi(C)^{-1}}{K\ diam(A)}, \epsilon\right) \equiv \gamma.$$

Since P' is the smallest clone containing fA it follows that C' is the smallest clone containing $\tau' fA$ thus $diam(\tau'(fA)) \geq \alpha(C')$. Since τ' is a similarity, $rel(\tau'(fA)) \geq rel(fA) \geq \gamma$, thus

$$sep(\tau'(fA)) = rel(\tau'(fA)) diam(\tau'(fA)) \geq \gamma\alpha(C') \equiv \beta.$$

Consider a clone B of C' contained in $\tau'(fA)$ and not a proper subset of any other clone contained in $\tau'(fA)$. We will call such a clone a *maximal clone* in $\tau'(fA)$. Let D be a clone containing B with $level(D) = level(B) - 1$. There is a point $y \in D$ which is not in $\tau'(fA)$ for otherwise B would not be maximal. Hence $sep(\tau'(fA)) \leq diam(D)$. Combining this with the previous inequality gives $diam(D) \geq \beta$. The argument of the last paragraph of lemma (0.2) again shows that $level(D)$ is bounded and so there only a bounded number of possibilities for B. This bound depends only on β and C'.

Thus there is a bounded number of choices for the collection of maximal clones. By lemma (0.1) the clopen $\tau'(fA)$ is a finite union of (maximal) clones, hence there are a bounded number of possibilities for $\tau'(fA)$. ■

Corollary 0.5 *Given $K > 1$, $\epsilon > 0$ and clone Cantor sets C, C' there is a finite set S of positive real numbers with the following propery. Suppose that f is any K-bilipschitz map of C onto a clopen in C', and suppose that $rel(fC) \geq \epsilon$. If B is any clone of level n containing a clone D of level $n+1$ in C then $\mu_{C'}(fD)/\mu_{C'}(fB)$ is in S.*

Proof Let P' be the smallest clone of C' containing fB and τ' the clone map of P' onto C'. Since τ' is a similarity

$$\frac{\mu_{C'}(fD)}{\mu_{C'}(fB)} = \frac{\mu_{C'}(\tau'(fD))}{\mu_{C'}(\tau'(fB))}.$$

Then there are finitely many possible images $\tau'(fB)$ by lemma (0.4). We will show that there are only finitely many possibilities for the set $\tau'(fD)$, then the result follows.

The ratio $diam(D)/diam(B)$ is one of the finitely many numbers $diam(A)/diam(C)$ for A a level-1 clone of C, and is thus bounded away from 0. By lemma (0.3) $diam(P')$ approximates $diam(fB)$ which approximates $diam(B)$ since f is bilipschitz. Let Q' be the smallest clone of C' containing fD. Then $diam(Q')$ approximates $diam(D)$ and so the ratio $diam(Q')/diam(P')$ approximates $diam(D)/diam(B)$ and so is bounded away from 0. Now

$$\frac{diam(Q')}{diam(P')} = \frac{diam(\tau'(Q'))}{diam(\tau'(P'))}$$

and $\tau'(P') = C'$ hence $diam(\tau' Q')$ is bounded away from 0.

There are only finitely many clones $\tau' Q'$ in C' which are this large. If we map Q' onto C' by a clone map ϕ, then $\phi\tau'(fD)$ is one of finitely many possibilities by lemma (0.4), and since there are only finitely many possible choices for ϕ there are only finitely many possible sets $\tau'(fD)$. ■

The following is the basic result in this subject.

Theorem 0.6 *Suppose that C, C' are clone Cantor sets and that f is a bilipschitz map of C onto a clopen subset of C'. Then there is a clopen A in C such that the restriction $f|A$ of f to A is measure linear.*

Proof Given a measurable subset A of C of positive measure define the *mass ratio* of A to be

$$MR(A) = \frac{\mu_{C'}(fA)}{\mu_C(A)}.$$

This is the multiplicative factor by which f changes the mass of A. Since f is bilipschitz the mass ratio function is bounded above, let L be the supremum. We remark that

$$MR(A \cup B) \leq \max\{MR(A), MR(B)\}$$

and the inequality is strict unless the mass ratios are equal. Thus given $\epsilon > 0$ there is a clone A of C with $MR(A) > L - \epsilon$. We will show that for ϵ small enough, that $f|A$ is measure linear.

Assertion. There is a finite set S of real numbers such that for every pair of clones $E \subset D$ of C with $level(E) = level(D) + 1$ then $MR(E)/MR(D)$ is in S.

Assuming this, suppose that $f|A$ is not measure linear. Then there is some clone E in A such that $MR(A) < MR(E)$. Otherwise for every clone E in A we would have $MR(E) \leq MR(A)$ but this would imply, by the remark, there is always equality and this implies that $f|A$ is measure linear.

Thus if $f|A$ is not measure linear we may choose a clone E in A with $MR(A) < MR(E)$ and with $n = level(E)$ minimal. Note that if D is any clone in A of level less than n then $MR(D) = MR(A)$. Let D be a clone in A containing E with $level(D) = n - 1$. Then $MR(E)/MR(D)$ is in S. Define λ

to be the minimum of the elements of S which are larger than 1. Thus $MR(E)/MR(D) \geq \lambda$. If we choose ϵ such that $(L-\epsilon)\lambda > L$ then

$$MR(E) > \lambda MR(D) = \lambda MR(A) > \lambda(L-\epsilon) > L$$

which contradicts the definition of L, proving that $f|A$ is measure linear.

To prove the assertion observe that

$$\frac{MR(E)}{MR(D)} = \frac{\mu_{C'}(fE)/\mu_{C'}(fD)}{\mu_C(E)/\mu_C(D)}.$$

Since $level(E) = level(D)+1$, as discussed in the proof of (0.5), there are only finitely many values for the ratio $\mu_C(E)/\mu_C(D)$. By (0.5), applied with $\epsilon = rel(fC)$, there are only finitely many values for $\mu_{C'}(fE)/\mu_{C'}(fD)$, and this proves the assertion. ■

One may now define various invariants of the bilipschitz type of clone Cantor sets. Let C be a clone Cantor set with Hausdorff measure μ_C in its Hausdorff dimension. We will call two sets of real numbers A and B *similar* if there are non-zero scalars α, β such that

$$\alpha A \subset B \qquad and \qquad \beta B \subset A.$$

The *clopen invariant* of C is the similarity class of the countable set of real numbers $\{\mu_C(A) : A \text{ a clopen in } C\ \}$.

Corollary 0.7 *If C, C' are clone Cantor sets and there is a bilipschitz map of C onto a clopen in C', then C and C' have the same clopen invariant.*

Proof By the theorem, there is a clopen, hence a clone, A in C and a measure linear map of A onto a clopen in C'. The set A is a clone Cantor set and the clone map of A onto C shows that it has the same clopen invariant as C. Let α be the Radon-Nikodym derivative of the measure linear map of A into C'. The image of a clopen B in A is a clopen in C' with α times the measure of B. Using that the inverse of a bilipschitz map is bilipschitz, one obtains the reverse relationship. ■

There is another invariant which measures the expansions of the similarity self maps of a clone Cantor set. A clone map is an example of such a map. In order to obtain a notion which is an invariant of bilipschitz homeomorphism, one must allow for conjugating a map which is a metric similarity by a bilipschitz map which is measure preserving, and this leads to the following.

Definition. Given $K > 1$ we will call a map $\psi : (X, d_X) \longrightarrow (Y, d_Y)$ between metric spaces *K-quasi-conformal* if

$$K^{-1} \leq \frac{d(\psi x, \psi y)/d(x,y)}{d(\psi x, \psi z)/d(x,z)} \leq K$$

for all distinct points x, y, z. Thus a similarity is 1-quasi-conformal. A K-bilipschitz map is K^2-quasi-conformal. The composition of K and K' quasi-conformal maps is (KK')-quasi-conformal. Suppose that x, y, w are distinct then

$$K^{-1} \leq \frac{d(\psi y, \psi w)/d(y,w)}{d(\psi y, \psi x)/d(y,x)} \leq K$$

thus

$$K^{-2} \leq \frac{d(\psi y, \psi w)/d(y,w)}{d(\psi x, \psi z)/d(x,z)} \leq K^2.$$

Regard x and z as fixed, then we see that ψ is M-bilipschitz with $M = K^2 \max(L, 1/L)$ where $L = d(\psi x, \psi z)/d(x,z)$.

Lemma 0.8 *Suppose that $\psi : X \longrightarrow Y$ is a K-quasi-conformal bijection. Then ψ is M-bilipschitz with $M = K^4 \max(N, 1/N)$ where $N = diam(Y)/diam(X)$.*

Proof Choose x, y, z, w in X with $d_X(y, w) = diam(X)$ and $d_Y(\psi x, \psi z) = diam(Y)$. Then

$$K^{-2} \leq \frac{d_Y(\psi y, \psi w)/d_X(y, w)}{d_Y(\psi x, \psi z)/d_X(x, z)} = \frac{d_Y(\psi y, \psi w)}{d_X(y, w)} \frac{d_X(x, z)}{diam(Y)}.$$

Now $d_X(x, z) \leq diam(X)$ thus

$$(K^2 N^{-1})^{-1} = K^{-2} \frac{diam(Y)}{diam(X)} \leq \frac{d_Y(\psi y, \psi w)}{d_X(y, w)} \leq \frac{diam(Y)}{diam(X)} = N \leq K^2 N.$$

It follows that ψ is M-bilipschitz with $M = K^2 \max(K^2 N, K^2 N^{-1})$. ■

Definition. A map $\sigma : (X, d_X) \longrightarrow (X, d_X)$ of a metric space (X, d_X) to itself has *bounded distortion* or is *BD* if there is a constant $K > 1$ such that for every integer $n > 0$ the map σ^n is K-quasi-conformal. Informally, the iterates of σ are almost similarities. For example the conjugate of a contracting similarity by a bilipschitz map is BD. This definition is not quite equivalent to the one in [1], but is better for present purposes.

Definition. A *local similarity* of a clone Cantor set C is a continuous map $\sigma : U \longrightarrow U$ defined on a non-empty clopen U of C which is a homeomorphism onto $V = \sigma U$. We require that V is a clopen contained in U, and the following properties:
(1) σ has a unique fixed point x called a *similarity point* of C.
(2) σ is measure linear with respect to Hausdorff measure. The Radon-Nikodym derivative of σ is called the *scale factor* of σ.
(3) σ is BD.

If x is a similarity point of C we define the *similarity group* $S_x(C)$ of C at x to be the subgroup of the multiplicative group $\mathbb{R}^+$ generated by the scale factors of the local similarities of C fixing x.

Corollary 0.9 *Suppose that C and C' are clone Cantor sets and there is a bilipschitz homeomorphism of C onto a clopen subset of C'. If x is a similarity point of C there is a similarity point y in C' such that $S_x(C) = S_y(C')$.*

Proof By theorem (0.6) there is a clopen A of C on which the given bilipschitz homeomorphism

$$\phi : C \longrightarrow C'$$

is measure linear. The orbit of x under the semi-group of clone maps of C contains a point in every clone and hence contains a point z in A. Clearly $S_x(C) = S_z(C)$. We claim that the conjugate of a local similarity by a measure-linear bilipschitz homeomorphism is a local similarity. To verify this, let $\sigma : A \longrightarrow A$ be a local similarity defined on the clopen A of C and with fixed point z. Let $\sigma' = \phi \circ \sigma \circ \phi^{-1}$ then σ' has a unique fixed point $\phi(z)$. Since ϕ is a homeomorphism, ϕA is a clopen of C and so the domain of σ' is the clopen ϕA. Since $\phi | A$ and σ are measure linear, σ' is measure linear. Since σ is BD then the iterates σ^n are all K-quasi-conformal. Also ϕ is bilipschitz and thus M-quasi-conformal for some M. Thus

$$\sigma'^n = \phi \circ \sigma^n \circ \phi^{-1}$$

is $(M^2 K)$-quasi-conformal. Hence σ' is BD. ■

An Application to n-adic Cantor sets

Given $n > 1$ let C_n, or just C, be the Cantor set of sequences $(x_0, x_1, \cdots)$ where x_i is an element of $\{0, \cdots, n-1\}$. The metric on C_n is defined by $d_n(x, y) = n^{-p}$ if p is minimal subject to $x_p \neq y_p$. Thus C has diameter 1. The shift map σ mapping C onto itself is defined by $\sigma(x_0, x_1, \cdots) = (x_1, x_2, \cdots)$. Define clones A_i for $0 \leq i \leq n-1$ to be the subsets of C with first digit i. Define a clone map $\tau_i = \sigma | A_i$ which linearly maps A_i onto C.

Proposition 0.10 *The Hausdorff dimension of the Cantor set C is 1 and it has measure 1 in this dimension.*

Proof There is a map ϕ of C onto the unit interval $[0, 1]$ given by regarding a sequence $(x_0, x_1, \cdots)$ as an expansion $0.x_0x_1x_2\cdots$ in base-n of a real number. Consider a subset A in C with diameter R. Choose p so that $n^{-p} \leq R < n^{1-p}$. Then all points in A consist of sequences which have the same digits up to and including x_{p-1} and therefore ϕA has Euclidean diameter at most n^{-p}. Thus

$$diameter_C(A) \geq Euclidean\ diameter(\phi A).$$

Thus the sum of the diameters of the sets in any cover of C is at least 1 and so the Hausdorff 1-measure of C is at least 1. Given $\epsilon > 0$ there is a cover of C by level-k clones each of which has diameter in the d_C-metric $n^{-k} < \epsilon$. There are n^k such clones, so the sum of their diameters is 1. Therefore the Hausdorff 1-measure is at most 1. ■

Since clone maps are measure linear it follows that the measure of a clone of C equals the Euclidean length of its image under ϕ. Thus ϕ is measure preserving. Hence the clopen invariant of C_n is the similarity class of $[0, 1] \cap \mathbb{Z}[1/n]$.

Corollary 0.11 *There is a bilipschitz homeomorphism of the clone Cantor set C_n onto a clopen subset of C_m if and only if there are integers $i, j > 0$ such that $n^i = m^j$.*

Proof If $n^i = m^j$ then it is clear that the Cantor sets are bilipschitz homeomorphic. For the converse, suppose that σ is a local similarity of C_n with scale factor $s < 1$ for Hausdorff measure. We first show that s is a rational power of n.

Let A be a clone in the domain of σ which contains the fixed point of σ. Then some iterate, σ^m, of σ maps A into A. We replace σ with $\sigma^m | A$. The scale factor of σ^m is s^m and it suffices to prove that s^m is a rational power of n. By this means we may suppose that the domain of σ is a clone A.

Since σ is BD there is $L > 1$ such that for all $p > 0$ we have that σ^p is L-quasi-conformal. Define $\alpha \equiv \alpha(C_n)$. The lemma (0.4) provides a constant, $M = M(C_n, C_n, K, \epsilon)$, for the case that C and C' are both C_n, and $K = L^6\alpha^{-1}$, and $\epsilon = K^{-1}\alpha$. Given an integer $a > 0$, define a new metric, d_{new} on C_n which is n^a times the n-adic metric thus $d_{new} = n^a d_n$. There is a unique clone, D, of diameter 1 in the d_{new}-metric which contains the unique fixed point of σ. We may choose a large enough that D is contained in $\sigma^{2M}A$. Then D with the metric d_{new} is a clone Cantor set isometric to C_n with the n-adic metric. Also $\sigma | D$ is a local similarity of D with scale factor s. Also for all $p > 0$ we have σ^p is L-quasi-conformal. We now replace (C_n, d_n) and σ by (D, d_{new}) and $\sigma | D$. By means of this we may assume our original local similarity, σ, admits an extension to a local similarity, $\overline{\sigma}$, defined on all of a clone Cantor set, E, of large diameter containing C_n as a clone. Furthermore we may assume that the image of $\overline{\sigma}^{2M}$ contains C_n.

Suppose that A_p is the smallest clone containing $\sigma^p C_n$. If τ_p is the clone map taking A_p onto C_n, define $\psi_p \equiv \tau_p \sigma^p$ then $\psi_p C_n$ is not contained in any level-1 clone of C_n thus $diam(\psi_p C_n) \geq \alpha$. The clone map τ_p has domain the clone A_p which it maps bijectively to C_n. Define $B_p = (\overline{\sigma})^{-p} A_p$ and define a bijection

$$\overline{\psi}_p \equiv \tau_p \overline{\sigma}^p : B_p \longrightarrow C_n.$$

Since $\overline{\sigma}^p$ is L-quasi-conformal and τ_p is linear $\overline{\psi}_p$ is also L-quasi-conformal. Thus, in particular, ψ_p is L-quasi-conformal. Apply lemma (0.8) to

$$\psi_p : X \equiv C_n \longrightarrow Y \equiv \psi_p C_n .$$

Then $diam(X) = 1$ and $diam(Y) = diam(\psi_p C_n) \geq \alpha$ so $\alpha \leq N \leq 1$ thus ψ_p is $(L^4\alpha^{-1})$-bilipschitz. Since $\overline{\psi}_p$ is L-quasi-conformal and $(L^4\alpha^{-1})$-bilipschitz on C_n it follows that $\overline{\psi}_p$ is K-bilipschitz with $K = L^6\alpha^{-1}$.

We wish to apply the lemma (0.4) to the maps $\psi_p : C_n \longrightarrow C_n$ with $1 \leq p \leq 2M$. These maps are all K-bilipshitz. We claim that $sep(\psi_p C_n) \geq \epsilon$. Since $p \leq 2M$ we have $\overline{\sigma}^p E$ contains C_n thus $\overline{\psi}_p(B_p) = C_n$. Since $\overline{\psi}_p$ is K-bilipschitz $sep(\psi_p C_n)$ is at least $K^{-1} d_E(C_n, B_p - C_n)$. Now $d_E(C_n, B_p - C_n) \geq \alpha$ because C_n is a clone in E and the separation of C_n in E is therefore bigger than the separation of any clone contained in C_n. Thus $sep(\psi_p A) \geq K^{-1}\alpha = \epsilon$, which proves the claim.

By lemma (0.4) there are $M = M(C_n, C_n, K, \epsilon)$ possibilities for $\psi_p C_n$. So for some $1 \leq k < j \leq 2M$ we have that $\psi_j C_n = \psi_k C_n$. The metric scale factor of every clone map of C_n is an integral power of n. Since the Hausdorff dimension of C_n is 1 this means that the scale factor for Hausdorff measure of a clone map is also an integral power of n. Thus τ_j, τ_k are measure linear with scale factors for Hausdorff measure which are powers of n. Thus the scale factor for Hausdorff measure for ψ_j and ψ_k is $n^a s^j = n^b s^k$ and so s is a rational power of n.

By (0.9) we have that s is also the scale factor of a local similarity of C_m. The above then implies that s is also a rational power of m. ■

References

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