DEFORMING CONVEX PROJECTIVE MANIFOLDS

DARYL COOPER, DARREN LONG, AND STEPHAN TILLMANN

ABSTRACT. We study a properly convex real projective manifold with (possibly empty) compact, strictly convex boundary, and which consists of a compact part plus finitely many convex ends. We extend a theorem of Koszul which asserts that for a compact manifold without boundary the holonomies of properly convex structures form an open subset of the representation variety. We also give a relative version for non-compact (G, X)-manifolds of the openess of their holonomies.

Given a subset $\Omega \subset \mathbb{R}P^n$ the *frontier* is $\mathrm{Fr}(\Omega) = \mathrm{cl}(\Omega) \setminus \mathrm{int}(\Omega)$ and the *boundary* is $\partial\Omega = \Omega \cap \mathrm{Fr}(\Omega)$. A *properly convex projective manifold* is $M = \Omega/\Gamma$ where $\Omega \subset \mathbb{R}P^n$ is a convex set with non-empty interior and $\mathrm{cl}(\Omega)$ is contained in the complement of some hyperplane H, and $\Gamma \subset \mathrm{PGL}(n+1, \mathbb{R})$ acts freely and properly discontinuously on Ω. If, in addition, $\mathrm{Fr}(\Omega)$ contains no line segment then M and Ω are *strictly convex*. The *boundary of M is strictly-convex* if $\partial\Omega$ contains no line segment.

If M is a *compact* (G, X)-manifold then a sufficiently small deformation of the holonomy gives another (G, X)-structure on M. In [20, 21] Koszul proved a similar result holds for closed, properly convex, projective manifolds. In particular, nearby holonomies continue to be discrete and faithful representations of the fundamental group.

Koszul's theorem cannot be generalised to the case of non-compact manifolds without some qualification—for example, a sequence of hyperbolic surfaces whose completions have cone singularities can converge to a hyperbolic surface with a cusp. The holonomy of a cone surface in general is neither discrete nor faithful. Therefore we must impose conditions on the holonomy of each end.

A group $\Gamma \subset \mathrm{PGL}(n+1, \mathbb{R})$ is a *virtual flag group* if it contains a subgroup of finite index that is conjugate into the upper-triangular group. The set of virtual flag groups is written VFG. A *generalized cusp* is a properly convex manifold C homeomorphic to $\partial C \times [0, \infty)$ with compact, strictly-convex boundary and with $\pi_1 C$ virtually nilpotent.

For an n–manifold M, possibly with boundary, define $\mathrm{Rep}(\pi_1 M) = \mathrm{Hom}(\pi_1 M, \mathrm{PGL}(n+1, \mathbb{R}))$ and $\mathrm{Rep}_{\mathrm{ce}}(M)$ to be the subset of $\mathrm{Rep}(\pi_1 M)$ consisting of holonomies of properly convex structures on M with ∂M strictly convex and such that each end is a generalized cusp. For instance, all ends of a properly convex surface with negative Euler characteristic and strictly convex boundary are generalized cusps.

Theorem 0.1. *Suppose N is a compact connected n–manifold and $\mathcal{V}$ is the union of some of the boundary components $V_1, \cdots, V_k \subset \partial N$. Let $M = N \setminus \mathcal{V}$. Assume $\pi_1 V_i$ is virtually nilpotent for each i. Then $\mathrm{Rep}_{\mathrm{ce}}(M)$ is an open subset of $\{\rho \in \mathrm{Rep}(\pi_1 N) : \forall i \ \rho(\pi_1 V_i) \in \mathrm{VFG}\}$.*

A similar statement holds for orbifolds since a properly convex orbifold has a finite cover which is a manifold, and the property of being properly convex is unchanged by coverings. This theorem is a consequence of our main theorem (6.27) that a certain map is open. By (6.9) $\rho(\pi_1 V_i) \in \mathrm{VFG}$ iff there is a finite index subgroup $\Gamma < \rho(\pi_1 V_i)$ such that every eigenvalue of Γ is real.

The author(s) acknowledge(s) support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 "RNMS: GEometric structures And Representation varieties" (the GEAR Network).
Cooper was partially supported by NSF grants DMS 1065939, 1207068 and 1045292 and thanks IAS and the Ellentuck Fund for partial support. Long was partially supported by grants from the NSF.
Tillmann was partially supported by Australian Research Council grant DP140100158
MSC 57N16, 57M50 .

Every end of a geometrically finite hyperbolic manifold M with a convex core that has compact boundary is topologically a product and is foliated by strictly convex hypersurfaces. These surfaces are either convex *towards* M so that cutting along one gives a manifold with convex boundary and the holonomy contains only hyperbolics, or else convex *away* from M in which case the end is a cusp and the holonomy contains only parabolics.

For strictly convex geometrically finite projective manifolds this dichotomy holds, but for properly convex manifolds there are ends that contain both hyperbolic and parabolic elements. We have chosen to study manifolds whose ends are convex outwards or convex inwards. Generalized cusps are those that are convex outwards with virtually nilpotent fundamental group.

There is a Margulis lemma for *properly* convex manifolds that says the local fundamental group is virtually nilpotent (0.1) in [10], see also [11]. There is a *thick-thin* decomposition for *strictly* convex manifolds (0.2) in [10], but not for *properly* convex manifolds. Each component of the thin part of a *strictly* convex manifold is a Margulis tube or a *cusp* and has virtually nilpotent fundamental group consisting of parabolics. This motivates the definition of *generalized cusp* above. There is a discussion of cusps in properly convex manifolds in §5 of [10].

Section 1 describes the (G,X)-*Extension Theorem* (1.7). This generalizes a well-known result for compact manifolds (the holonomies of (G,X)-structures form an open subset of the representation variety) by providing a *relative* version. Section 2 recalls the definition and properties of the tautological bundle. Section 3 reviews Hessian metrics and gives a characterization of properly convex manifolds in terms of the existence of a certain kind of Hessian metric on the tautological line bundle. This material is due to Koszul. Section 4 shows that various functions on properly convex projective manifolds are uniformly bounded, including a proof of the folklore result that they admit Riemannian metrics with all sectional curvatures bounded in terms of dimension.

The *Convex Extension Theorem* (5.7) is a version of (1.7) for properly convex manifolds with strictly convex boundary. A consequence is (0.2) below. Roughly this says that if you can convexly deform the ends of a properly convex manifold then you can convexly deform the manifold.

Theorem 0.2. *Suppose $M = A \cup \mathcal{B}$ is a properly convex manifold with (possibly empty) compact strictly convex boundary and A is a compact submanifold of M with $\partial A = \partial M \sqcup \partial\mathcal{B}$ and $\mathcal{B} = B_1 \sqcup \cdots \sqcup B_k$ has $k \geq 0$ connected components $B_i \cong \partial B_i \times [0,\infty)$.*

Suppose $\rho : (-1,1) \to \mathrm{Rep}(\pi_1 M)$ is continuous and $\rho_t := \rho(t)$ and ρ_0 is the holonomy of M. Let $\mathcal{C}$ denote the space of closed subset of $\mathbb{RP}^n$ with the Hausdorff topology. Suppose for all $1 \leq i \leq k$ and all $t \in (-1,1)$ that

- *there is a properly convex set $\Omega_i(t) \subset \mathbb{RP}^n$, that is preserved by $\rho_t(\pi_1 B_i)$,*
- *$P_i(t) = \Omega_i(t)/\rho_t(\pi_1 B_i)$ is a properly convex manifold and $\partial P_i(t)$ is strictly convex,*
- *there is a projective diffeomorphism from $P_i(0)$ to B_i,*
- *$P_i(t)$ is diffeomorphic to B_i,*
- *the two maps $t \mapsto \mathrm{cl}(\Omega_i(t))$ and $t \mapsto \mathrm{cl}(\Omega_i(t)) \setminus \Omega_i(t)$ into $\mathcal{C}$ are continuous.*

Then there is $\epsilon > 0$ such that for all $t \in (-\epsilon,\epsilon)$ there is a properly convex projective structure on M with holonomy $\rho(t)$ such that ∂M is strictly convex and B_i is projectively diffeomorphic to $P_i(t)$.

Section 6 proves that generalized cusps contain *homogeneous* cusps (6.5):

Theorem 0.3. *Suppose $C = \Omega/\Gamma$ is a generalized cusp. Then C contains a generalized cusp $C' = \Omega'/\Gamma$ such that $\mathrm{PGL}(\Omega')$ acts transitively on $\partial\Omega'$.*

Frequent use is made of the fact that C is *maximal* in the sense that, after taking an orientation double cover if needed, $H_{n-1}(C) \cong \mathbb{Z}$ where $n = \dim C$. An algebraic argument shows (6.13) that if $C = \Omega/\Gamma$ is a generalized cusp then Γ has a finite index subgroup that is a lattice in a connected Lie group $T = T(\Gamma)$ that is conjugate into the upper-triangular group.

Next (6.22) shows that the T-orbit of some point $p \in \Omega$ is a strictly convex hypersurface $S = T \cdot p$. The convex hull of S is a domain Ω_T and which is preserved by all of Γ and we may shrink C to be Ω_T/Γ giving (0.3).

From (0.3) it follows that generalized cusps are *stable* (6.25): if Γ is deformed to a nearby virtual flag group Γ' then $T' = T(\Gamma')$ is a nearby Lie group so $S' = T(\Gamma') \cdot p$ is a nearby strictly convex hypersurface which gives a nearby domain $\Omega_{T'}$ and a nearby generalized cusp $C' = \Omega_{T'}/\Gamma'$.

The convex extension theorem and the stability of generalized cusps implies the main theorem (0.1). In [2] generalized cusps are classified and their properties are studied. This classification for 3-manifolds is given without proof in section 7.

A function is *Hessian-convex* if it is smooth and has positive definite Hessian. This property is preserved by composition with diffeomorphisms that are close to affine. Section 8 contains a theorem about approximating strictly-convex functions on affine manifolds by Hessian-convex ones. Section 9 is a short proof of Benzécri's Theorem. We have put these results at the end of the paper with the intention of not breaking the narrative.

There is an entirely PL approach to (0.1) which, however, we do not develop in this paper. It is based on using the convex hull of the orbit of one point instead of a characteristic surface.

Theorem (0.1) does not always remain true if ∂M is convex but not strictly convex. However, in some cases, the theorem can still be applied. For instance, a hyperbolic manifold M with compact, totally geodesic boundary is a submanifold of a finite volume hyperbolic manifold with strictly convex smooth boundary obtained by fattening. In particular, any small deformation in $\mathrm{PGL}(4, \mathbb{R})$ of the holonomy in $PO(3, 1)$ of a compact Fuchsian manifold is the holonomy of a strictly convex projective structure on (surface)$\times[0, 1]$.

The reader only interested in the proof of (0.1) when M is compact need only read section 1 up to (1.2), and then sections 2 to 4 stopping before (4.3). Those interested only in the proof of (0.2) can omit section 6.

Most of sections 1-4 is not new and there is considerable overlap in the first 5 sections with the results and methods of Choi in [5]. Marquis determined the holonomies of properly convex surfaces with cusps. In [9] a method of constructing fundamental domains for some strictly convex manifolds with cusps is given. Using the main result of this paper, new properly convex structures have been found on the figure eight knot obtained by deforming the complete hyperbolic structure [1]. The type of geometry in a generalized cusp can change during a deformation. For example a generalized cusp with diagonal holonomy can *transition* to one with parabolic holonomy. This is related to the study of geometric transition in [8].

This paper has evolved over several years as the authors gradually discovered the nature of generalized cusps. The first author has lectured on earlier versions that involved the *Radial Flow Convexity Theorem* which was used to show the existence of convex structures on the ends for certain deformations. Our improved understanding allows us to avoid this by using Hessian metrics. The first author apologizes for the long delay in completing this paper.

1. (G,X) structures and Extending Deformations

The goal of this section is a relative version of the well-known fact (1.2) that for *compact* manifolds the set of holonomies of (G, X)-structures is an open subset of the representation variety. The *Extension Theorem* (1.7) implies that if $\mathcal{B}$ is a codimension-0 submanifold of M with $M \setminus \mathcal{B}$ compact then, given a (G, X)-structure on M with holonomy ρ together with a nearby representation σ, and given a nearby (G, X)-structure on $\mathcal{B}$ with holonomy the restriction of σ, there is a nearby (G, X)-structure on M with holonomy σ that extends the structure on $\mathcal{B}$.

A *geometry* is a pair (G, X) where G is a Lie group which acts transitively and real-analytically on a manifold X. A (G, X) *structure* on a manifold M (possibly with boundary) is a maximal atlas of charts which takes values in X so that transitions maps are locally the restriction of elements of

G. A map between (G,X) manifolds is a (G,X) *map* if locally it is conjugate via (G,X)-charts to an element of G.

Let $\pi : \widetilde{M} \to M$ be (a fixed choice for) the universal cover of M. We regard $\pi_1 M$ to be defined as the group of covering transformations of this covering. A local diffeomorphism $f : \widetilde{M} \to X$ determines a (G,X)-structure on $\widetilde{M}$. If covering transformations are (G,X)-maps then there is a unique (G,X)-structure on M such that the covering space projection is a (G,X)-map. In this case f is called a *developing map* for this structure and determines a homomorphism $\mathrm{hol} = \mathrm{hol}(f) : \pi_1 M \to G$ called *holonomy.*

For smooth manifolds M^m and N^n the set of smooth maps $C^\infty_w(M,N)$ has the *weak topology* [17, page 35]. The space of diffeomorphisms $\mathrm{Diff}(M)$ is a subspace of $C^\infty_w(M,M)$. If $N = \mathbb{R}$ then $C^\infty_w(M) := C^\infty_w(M,\mathbb{R})$.

The space of all developing maps is denoted $\mathcal{D}ev(M,(G,X))$ or just $\mathcal{D}ev(M)$. The (G,X)-structure on M given by $\mathrm{dev} \in \mathcal{D}ev(M)$ is written (M, dev). There is a natural embedding of $\mathcal{D}ev(M)$ into $C^\infty_w(\mathrm{int}\, \widetilde{M}, X)$ given by restricting the developing map to $\mathrm{int}\, \widetilde{M}$.

Definition 1.1. *The* geometric topology *on $\mathcal{D}ev(M)$ is the subspace topology from $C^\infty_w(\mathrm{int}\, \widetilde{M}, X)$.*

Thus two developing maps are close if they are close on a large compact set in the universal cover that is disjoint from the boundary. The following is due to Thurston [29], see also Goldman [14] and Choi [4]. The topology on $\mathrm{Hom}(\pi_1 M, G)$ is the compact-open topology.

Proposition 1.2 (holonomy is open)**.** *Suppose M is a compact connected smooth manifold possibly with boundary. Then $\mathcal{H}ol : \mathcal{D}ev(M,(G,X)) \to \mathrm{Hom}(\pi_1 M, G)$ is continuous and open.*

Given $\mathrm{dev}_M \in \mathcal{D}ev(M)$ and $\mathrm{dev}_N \in \mathcal{D}ev(N)$ a smooth map $f : M \to N$ is *close* to a (G,X) map if it is covered by $F : \widetilde{M} \to \widetilde{N}$ and there is $g \in G$ such that $g \circ \mathrm{dev}_N \circ F$ is close to dev_M in $C^\infty_w(\widetilde{M}, X)$. This means there is a large compact set $K \subset \mathrm{int}\, \widetilde{M}$ and some $g \in G$ such that for each $x \in K$ there is an open neighborhood $U \subset \widetilde{M}$ with $V = \mathrm{dev}_M(U \cap K)$ and the map $g \circ \mathrm{dev}_N \circ F \circ (\mathrm{dev}_M|_{U\cap K})^{-1}$ is very close to the inclusion map in $C^\infty(V,X)$. This notion of *close* depends on dev_M but not on the choice of developing map dev_N for a given (G,X)-structure on N.

There is a nice description of what it means for developing maps in $\mathcal{D}ev(M)$ to be close when one of them is *injective*. Suppose $\mathrm{dev} \in \mathcal{D}ev(M)$ is injective and $\Omega = \mathrm{dev}(\widetilde{M})$ and $\rho = \mathcal{H}ol(\mathrm{dev})$ and $\Gamma = \rho(\pi_1 M)$. Then $N = \Omega/\Gamma$ is a (G,X) manifold that is (G,X)-diffeomorphic to M. We choose the universal cover $\widetilde{N} = \Omega$ then $\pi_1 N = \Gamma$ by our definition as the group of covering transformations. There is a homeomorphism from $\mathcal{D}ev(M)$ to $\mathcal{D}ev(N)$.

Definition 1.3. *Replacing $\mathcal{D}ev(M)$ by $\mathcal{D}ev(N)$ is called* choosing dev as the basepoint for the space of developing maps.

The developing map $\mathrm{dev}_* \in \mathcal{D}ev(N)$ for N is the inclusion map $i : \widetilde{N} \hookrightarrow X$ and $\mathcal{H}ol(\mathrm{dev}_*) : \Gamma \hookrightarrow G$ is also the inclusion map. If N has no boundary then $\mathcal{D}ev(N)$ is a subspace of $C^\infty_w(\widetilde{N}, X)$ so $\mathrm{dev}' \in \mathcal{D}ev(N)$ is close to dev_* if dev' is close to i in $C^\infty_w(\widetilde{N}, X)$.

Lemma 1.4 (lifting developing maps)**.** *In this statement all manifolds and maps are (G,X). Suppose N and P are connected manifolds and $\theta : \pi_1 N \to \pi_1 P$ is a homomorphism such that $\mathrm{hol}_N = \mathrm{hol}_P \circ \theta$. Suppose $\pi_P : \widetilde{P} \to P$ and $\pi_N : \widetilde{N} \to N$ are universal covers and $i : Q \hookrightarrow \widetilde{N}$ is the inclusion map of a connected set Q with $\pi_N(Q) = N$. Suppose $\mathrm{dev}_N \circ i : Q \to X$ lifts to a map $j : Q \to \widetilde{P}$ such that $\mathrm{dev}_P \circ j = \mathrm{dev}_N \circ i$. Then there is $k : N \to P$ covered by $\tilde{k} : \widetilde{N} \to \widetilde{P}$ that extends j.*

$$\begin{tikzcd}
& & \widetilde{P} \arrow[d, "\mathrm{dev}_P"] \\
Q \arrow[urr, "j"] \arrow[r, "i"] & \widetilde{N} \arrow[ur, dashed, "\exists \tilde{k}"'] \arrow[r, "\mathrm{dev}_N"] & X
\end{tikzcd}$$

Proof. Because the covering translates of Q cover $\tilde{N}$ and the manifolds N and P have (via θ) the same holonomy, j can be extended by analytic continuation to an equivariant (G,X)-map $\tilde{k}:\widetilde{N}\to\widetilde{P}$. Equivariance implies $\tilde{k}$ covers a (G,X)-map $k:N\to P$. $\square$

If P is a smooth manifold then $\mathrm{Diff}(\widetilde{P},\pi_1 P)\subset\mathrm{Diff}(\widetilde{P})$ is the subgroup of diffeomorphisms that cover an element of $\mathrm{Diff}(P)$. The next result says that if two developing maps are close and have the same holonomy then, after changing one by a small isotopy, the developing maps are equal on a compact submanifold in the interior.

Corollary 1.5. *Suppose P is a smooth manifold. Let $\rho\in\mathrm{Hom}(\pi_1 M,G)$ be the holonomy of* $\mathrm{dev}\in\mathcal{D}ev(P,(G,X))$ *and $\mathcal{D}ev_\rho(P)\subset\mathcal{D}ev(P)$ the subspace of developing maps with holonomy ρ. Then the map* $\mathrm{Diff}(\widetilde{P},\pi_1 P)\to\mathcal{D}ev_\rho(P)$ *given by* $f\mapsto\mathrm{dev}\circ f$ *is an open map.*

It follows that if N is a compact codimension-0 manifold in the interior of P and $\mathrm{dev}'\in\mathcal{D}ev_\rho(P)$ *is close enough to* dev *then there is $k\in\mathrm{Diff}(P)$ covered by $\tilde{k}\in\mathrm{Diff}(\widetilde{P},\pi_1 P)$ such that* $\mathrm{dev}=\mathrm{dev}'\circ\tilde{k}$ *on N and k is isotopic to the identity by a small isotopy supported in a small neighborhood of N.*

Proof. Let $\pi_P:\widetilde{P}\to P$ and $\pi_N:\widetilde{N}\to N$ be universal covers. Let $Q\subset\widetilde{N}$ be a compact connected manifold such that $\pi_N(Q)=N$. Since $\mathrm{dev}|_Q:Q\to X$ factors through the inclusion $j:Q\hookrightarrow\widetilde{P}$ and $\pi(Q)\subset\mathrm{int}(P)$ it follows that if dev' is close enough to dev then $\mathrm{dev}'|Q:Q\to X$ has a nearby lift $j':Q\to\widetilde{P}$. By (1.4) there is a (G,X)-map $k:(N,\mathrm{dev}'|N)\to(P,\mathrm{dev}|N)$ that lifts to a map that extends j'. If dev' is sufficiently close to dev the result now follows from the fact that a diffeomorphism close to an inclusion is ambient isotopic to the inclusion by a small ambient isotopy [23]. $\square$

Suppose M is a smooth manifold with (possibly empty) boundary and $\mathcal{B}\subset M$ is a codimension-0 submanifold that is a closed subset such that $A=\mathrm{cl}(M\setminus\mathcal{B})$ is compact manifold. Suppose $\mathcal{B}=B_1\sqcup\cdots B_k$ has $k<\infty$ connected components. Define the *relative-holonomy space*

$$\mathcal{R}el\mathcal{H}ol(M,\mathcal{B},(G,X))\subset\mathrm{Hom}(\pi_1 M,G)\times\prod_{i=1}^{k}\mathcal{D}ev(B_i,(G,X))$$

to be the subset of all $(\rho,\mathrm{dev}_1,\cdots,\mathrm{dev}_k)$ such that $\mathcal{H}ol(\mathrm{dev}_i)=\rho|\pi_1 B_i$. This space has the subspace topology of the product topology.

Given a connected submanifold $B\subset M$ we fix a choice of some component $\widetilde{B}\subset\widetilde{M}$ of the preimage B in the universal cover of M and define $\pi_1 B_i$ to be those covering transformations that preserve $\widetilde{B}$. If dev_M is a developing map for a (G,X)-structure on M the *restriction to B* is $\mathrm{dev}_{M|B}:=\mathrm{dev}_M|\widetilde{B}$.

Definition 1.6. *A developing map for M restricts to give developing maps on each component of $\mathcal{B}$ and this defines the* relative holonomy map $\mathcal{E}:\mathcal{D}ev(M,(G,X))\longrightarrow\mathcal{R}el\mathcal{H}ol(M,\mathcal{B},(G,X))$

$$\mathcal{E}(\mathrm{dev}_M)=(\mathcal{H}ol(\mathrm{dev}_M),\mathrm{dev}_{M|B_1},\cdots,\mathrm{dev}_{M|B_k})$$

This map depends on a fixed choice of one component $\widetilde{B}_i\subset\widetilde{M}$ for each i. In the special case that $\mathcal{B}$ is empty then $\mathcal{E}=\mathcal{H}ol$. We will apply this when $\mathcal{B}$ consists of the ends of M which is why the symbol $\mathcal{E}$ is used. However the result is of interest even when everything is compact.

Theorem 1.7 (Extension theorem)**.** *Suppose M is a smooth manifold with (possibly empty) boundary and $\mathcal{B} \subset M$ is a codimension-0 submanifold that is a closed subset such that $A = \mathrm{cl}(M \setminus \mathcal{B})$ is a compact manifold. Then $\mathcal{E} : \mathcal{D}ev(M, (G, X)) \longrightarrow \mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, (G, X))$ is continuous and open.*

Proof. Continuity is easy. We prove openess. For simplicity we will assume that $\mathcal{B} = B$ is connected; the multi-end case merely requires more notation. Suppose $\mathcal{E}(\mathrm{dev}_{\rho,M}) = (\rho, \mathrm{dev}_{\rho,M|B})$ and $(\sigma, \mathrm{dev}_{\sigma,B})$ is nearby in $\mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, (G, X))$.

Let C be a compact neighborhood of A in M so that $E = C \cap B \cong \partial B \times [0, 2]$. By (1.2) there is $\mathrm{dev}_{\sigma,C} : \widetilde{C} \to X$ close to $\mathrm{dev}_{\rho,M|C}$ with holonomy (the restriction of) σ. Using (1.5) after changing $\mathrm{dev}_{\sigma,C}$ by a small isotopy we may assume $\mathrm{dev}_{\sigma,C}$ and $\mathrm{dev}_{\sigma,B}$ are equal on $\partial B \times [0, 1]$. This gives a developing map $\mathrm{dev}_{\sigma,M} : \widetilde{M} \to X$ close to $\mathrm{dev}_{\rho,M}$ that is given by $\mathrm{dev}_{\sigma,C}$ on $\widetilde{A}$ and $\mathrm{dev}_{\sigma,B}$ on $\widetilde{B}$. □

2. Tautological Bundles

There is a bundle $\xi M \to M$ over a real projective manifold M called the tautological line bundle. In the next section we show that M is properly convex iff ξM admits a certain kind of metric.

Radiant affine geometry is $\mathbb{L} = (\mathrm{GL}(n + 1, \mathbb{R}), \mathbb{R}^{n+1} \setminus 0)$. A manifold with this structure is called a *radiant affine manifold*. It *ought* to be called a *linear manifold* since transition functions are linear maps.

Projective geometry over a real vector space V is $\mathbb{P} = (\mathrm{PGL}(V), \mathbb{P}(V))$. *Positive projective space* is $\mathbb{P}_+(V) = (V - 0)/\mathbb{R}^+$ and the action of $\mathrm{GL}(V)$ on V induces an effective action of $\mathrm{P}_+\mathrm{GL}(V) = \mathrm{GL}(V)/\mathbb{R}^+$ on $\mathbb{P}_+(V)$ which gives *positive projective geometry* $\mathbb{P}_+ = (\mathrm{P}_+\mathrm{GL}(V), \mathbb{P}_+(V))$. If $X \subset V$ we write $\mathbb{P}(X)$ for its image in $\mathbb{P}(V)$ and similarly $\mathbb{P}_+(X) \subset \mathbb{P}_+(V)$.

We identify $\mathbb{P}_+(\mathbb{R}^{n+1})$ with the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ and *radial projection* $\pi_\xi : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$ is $\pi_\xi(x) = x/\|x\|$. An action of $A \in \mathrm{SL}_\pm(n + 1, \mathbb{R})$ on $\mathbb{S}^n$ is given by $A(\pi x) = \pi(Ax)$. Clearly $\mathbb{P}_+ \cong \mathbb{S} := (\mathrm{SL}_\pm(n + 1, \mathbb{R}), \mathbb{S}^n)$.

For each of the geometries $\mathbb{G}$ above there is a space of developing maps $\mathcal{D}ev(M, \mathbb{G})$ with the geometric topology. By lifting developing maps one obtains:

Proposition 2.1. *The natural map $\mathcal{D}ev(M, \mathbb{S}) \to \mathcal{D}ev(M, \mathbb{P})$ is $2 : 1$. In other words: Every projective structure on M lifts to a positive projective structure. Thus if M is a real projective n–manifold, then the holonomy $\rho : \pi_1 M \longrightarrow \mathrm{PGL}(n + 1, \mathbb{R})$ lifts to $\tilde{\rho} : \pi_1 M \longrightarrow \mathrm{SL}_\pm(n + 1, \mathbb{R})$ and $\mathrm{dev} : \widetilde{N} \to \mathbb{RP}^n$ lifts to $\widetilde{\mathrm{dev}} : \widetilde{M} \to \mathbb{S}^n$.*

We will pass back and forth between projective geometry and positive projective geometry without mention. The *tautological bundle* over $\mathbb{S}^n$ is $\pi_\xi : \mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{S}^n$. The total space is a radiant affine manifold. There is an action of $(\mathbb{R}, +)$ on the total space called the *radial flow* given by $\Phi_t(x) = \exp(-t)x$. This group acts simply transitively on the fibers so the bundle is a principal $(\mathbb{R}, +)$-bundle. All this structure is preserved by the action of $\mathrm{GL}(n + 1, \mathbb{R})$ on $\mathbb{R}^{n+1}$ covering the action of $\mathrm{SL}_\pm(n + 1, \mathbb{R})$ on $\mathbb{S}^n$

Suppose M is a projective n–manifold defined by a developing map $\mathrm{dev}_{_M} : \widetilde{M} \to \mathbb{S}^n$ with holonomy $\rho : \pi_1 M \to \mathrm{SL}_\pm(n + 1, \mathbb{R})$ and with universal cover $\pi_{_M} : \widetilde{M} \to M$. Then pullback gives a bundle $\pi_\xi : \xi\widetilde{M} \to \widetilde{M}$ where

$$\xi\widetilde{M} = \{(\widetilde{m}, x) \in \widetilde{M} \times (\mathbb{R}^{n+1} \setminus 0) : \mathrm{dev}(\widetilde{m}) = \pi_\xi(x)\}$$

Recall that we *defined* $\pi_1 M$ as the group of covering transformations of $\widetilde{M}$. There is an action of $\pi_1 M$ on $\xi\widetilde{M}$ given by $\tau \cdot (\widetilde{m}, x) = (\tau(\widetilde{m}), (\rho(\tau))(x))$. The quotient is called the *tautological bundle* ξM. There is a natural bundle map $\xi_{_M} : \xi M \to M$ given by $\xi_{_M}[\widetilde{m}, x] = \pi_{_M}(\widetilde{m})$. There is also a natural radiant affine manifold structure on ξM with developing map $\mathrm{dev}_\xi : \xi\widetilde{M} \to \mathbb{R}^{n+1} \setminus 0$ given by $\mathrm{dev}_\xi(\widetilde{m}, x) = x$.

There is a radial flow on ξM given by $\Phi_t[m,x] = [m, \exp(-t)\cdot x]$ so ξM is a principal $(\mathbb{R},+)$ bundle. Orbits are called *flowlines.* The *tautological circle bundle* is $\xi_1 M = \xi M/\Phi_1$. It is sometimes called an *affine suspension.* Observe that the developing maps of ξM and $\xi_1 M$ are the same.

Definition 2.2. *A* flow function *is a function $c : \xi M \to \mathbb{R}$ that is* flow equivariant, *which means that $c(\Phi_t(p)) = t + c(p)$ for all p, t.*

A flow function determines a section $\sigma : M \to \xi M$ of the bundle $\xi_{_M} : \xi M \to M$ defined by $c(\sigma(x)) = 0$. Conversely a section σ determines a flow function c via $c(x) = t$ if $\Phi_t(x) = \sigma(\pi x)$. So a flow function gives the amount of time it takes for a point to flow to the zero-section.

We will mostly be concerned with the situation where $\mathrm{dev}_{_M} : \widetilde{M} \to \Omega \subset \mathbb{RP}^n$ is injective. In this case dev_ξ is a diffeomorphism onto the cone $\mathcal{C}\Omega \subset \mathbb{R}^{n+1} \setminus 0$. This identifies ξM with $\mathcal{C}\Omega/\Gamma$, where $\Gamma = \mathrm{hol}(\pi_1 M)$. Moroever $\mathrm{dev}_{_M}$ identifies $\widetilde{M}$ with a subset of $\mathbb{S}^n$. Using these identifications $\xi_{_M} : \xi M \to M$ is covered by π_ξ.

3. Hessian Metrics and Convexity

The ideas in this section go back to Koszul [20, 21], and we have followed the exposition in [26]. However our notation and terminology are somewhat different.

Suppose M is a simply connected affine manifold and $\mathrm{dev} : M \to \mathbb{R}^n$ is some developing map. Given $a, b \in M$ a *segment in M from a to b* is a map $\gamma : [u,v] \to M$ such that $\gamma(u) = a$ and $\gamma(v) = b$ and $\mathrm{dev}\circ\gamma$ is affine. We often denote such a map by $[a,b]$. It is a *unit segment* if $[u,v]$ is the unit interval $I := [0,1]$. A *ray* in M is a non-constant affine map $\gamma : [0,s) \to M$ with $s \in (0,\infty]$ which does not extend to a segment. A *unit triangle* in M is a map $\tau : \Delta \to M$ such that $\mathrm{dev}\circ\tau$ is affine where $\Delta \subset \mathbb{R}^2$ is the triangle with vertices $0, e_1, e_2$. The sides of a triangle are segments.

A C^2 function $c : M \to \mathbb{R}$ is *strictly convex* if for every (non-degenerate) segment $\gamma : [-1,1] \to M$ the function $F = c\circ\gamma$ satisfies $F'' > 0$. Then c defines a Riemannian metric on M via $\|\gamma'(0)\|^2 = F''(0)$ called a *Hessian metric.* See [25] for a discussion.

An affine manifold M has *convex boundary* if for each $p \in \partial M$ there is an affine coordinate chart (U,ϕ) with $p \in U$ and a closed halfspace $H \subset \mathbb{R}^n$ such that $\phi(U) \subset H$ and $\phi(p) \in \partial H$.

Theorem 3.1. *Suppose M is a simply-connected affine n–manifold with convex boundary and M has a Hessian metric that makes M into a complete metric space. Then M is affinely isomorphic to a convex subset of $\mathbb{R}^n$.*

Proof. It suffices to show that for every pair of segments $[p,a]$ and $[p,b]$ in M there is a segment $[a,b]$ in M. This is because every pair of points in M can be connected by a path composed of finitely many segments, and it then follows these points are contained in a single segment. This implies the developing map $\mathrm{dev} : M \to \mathbb{R}^n$ is injective and the image is convex.

Given unit segments $\alpha : I \to [p,a]$ and $\beta : I \to [p,b]$ let $\mathcal{I} \subset I$ be the set of $t \in I$ such there is a unit triangle τ in M with vertices $p = \tau(0)$ and $\alpha(t) = \tau(e_1)$ and $\beta(t) = \tau(e_2)$. Then $\mathcal{I}$ is connected and contains 0. It suffices to show $\mathcal{I} = I$ since then $\gamma(t) = \tau(te_1 + (1-t)e_2)$ is a segment containing a and b.

Since ∂M is convex it easily follows from the standard argument about sets with convex boundary that $\mathcal{I}$ is open. To show $\mathcal{I}$ is closed we may assume $\mathcal{I} = [0,1)$ by reparametrizing.

The Hessian metric is given by some function $c : M \to \mathbb{R}$. Given any segment γ define $\ell(\gamma)$ to be its length. If γ is a unit segment and $F = c\circ\gamma$ then

$$\ell(\gamma) = \int_0^1 \sqrt{F''(t)}dt$$

By the Cauchy-Schwartz inequality

$$\ell(\gamma) \leq \left(\int_0^1 F''(t)dt \right)^{1/2} \left(\int_0^1 dt \right)^{1/2} \leq \sqrt{|F'(1)| + |F'(0)|}$$

For $s \in [0,1)$ there is a unit segment γ_s given by $\gamma_s(t) = \tau(s(te_1 + (1-t)e_2))$ with endpoints $\alpha(s)$ and $\beta(s)$. By the triangle inequality

$$d(p, \gamma_s(t)) \leq d(p, \gamma_s(0)) + d(\gamma_s(0), \gamma_s(t)) \leq \ell(\alpha) + \ell(\gamma_s)$$

The function $F(s,t) = c(\gamma_s(t))$ is smooth. By compactness there is $K > 0$ such that $|\partial F/\partial t| \leq K$ for all $s \in [0,1]$ and $t \in \{0,1\}$. It follows that for all $s \in [0,1)$ and $t \in [0,1]$ we have

$$d(p, \gamma_s(t)) \leq \ell(\alpha) + \sqrt{2K} =: R.$$

Since the metric on M is complete the ball $P \subset M$ with center p and radius R is compact and contains all the segments γ_s. It follows that γ_s converges to a segment $\gamma_1 \subset P$ as $s \to 1$ so $1 \in \mathcal{I}$. □

Definition 3.2. *If M is a projective n–manifold a* convexity function *for M is a Hessian-convex flow function $c : \xi M \to \mathbb{R}$. It is* complete *if the Hessian metric given by c is complete.*

The flow-equivariance of c implies the radial flow acts by isometries of the Hessian metric on ξM given by c. The 1-form dc is preserved by the flow and therefore is the pullback of a 1-form α on $\xi_1 M$. Koszul works with α but we work with c.

Lemma 3.3 (backwards convex implies convex)**.** *Suppose M is properly convex and $N = \xi M$. Suppose $c : N \to \mathbb{R}$ is a flow function and $S = c^{-1}(0)$. Then at $x \in N$ there is a splitting $T_x N = V \oplus E$ which is orthogonal with respect to $Q := D^2_x c$ where $V = \ker d_x c \subset T_x N$ is the tangent hyperplane to the hypersurface S and $E = \langle e \rangle$ where $e = \Phi'_0(x)$ is a tangent vector to the flow.*

Moreover $Q(e,e) = \|e\|^2 = 1$ so if $\kappa \in [0,1]$ then $Q \geq \kappa \| \cdot \|^2$ iff $Q|V \geq \kappa(\| \cdot \| \ |V)^2$. Here $\| \cdot \|$ is the Hilbert-Finsler norm on $T_x N$.

In particular c is Hessian-convex iff S is a Hessian-convex hypersurface that is convex in the backwards *direction of the radial flow*

Proof. This is a local question so it suffices to assume ξM is a properly convex cone $\mathcal{C} \subset \mathbb{R}^{n+1} \setminus 0$ and S is a hypersurface and the radial flow is $\Phi_t(x) = \exp(-t) \cdot x$. Since c is a flow function $c(\Phi_t(x)) = c(x) + t$. This implies $c(s \cdot x) = c(x) - \log s$. From this it follows that $D^2_x c(e,v) = 0$ for all $v \in V$ which proves the Q-orthogonallity of the direct sum.

The Hilbert-Finsler norm on $(0,\infty)$ is ds/s. The radial flow on $(0,\infty)$ is $\Phi_t(s) = \exp(-t)s$ so $e = \Phi'_0(s) = s \cdot \partial/\partial s$ and $\|e\| = 1$. Moreover

$$Q(e,e) = s^2 Q(\partial/\partial s, \partial/\partial s) = s^2 d^2(-\log s)/ds^2 = 1.$$

□

Theorem 3.4. *Suppose M is a projective manifold with (possibly empty) convex boundary and $c : \xi M \to \mathbb{R}$ is a complete convexity function. Then M is properly convex.*

Proof. By (3.1) $\text{dev} : \xi\widetilde{M} \to \mathbb{R}^{n+1} \setminus 0$ is injective and has convex image $\Omega \subset \mathbb{R}^{n+1}$. It suffices to show the image is properly convex. The function $f = c \circ \text{dev}^{-1} : \Omega \to \mathbb{R}$ is strictly convex and the hypersurfaces $S_t = f^{-1}(t)$ are connected and strictly convex and foliate Ω. The radial flow on ξM is conjugate to the radial flow $\Phi_t(x) = \exp(-t) \cdot x$ on $\mathbb{R}^{n+1}$ so $\Phi_s(S_t) = S_{t+s}$. Define $S := S_0$.

Let q be a point in the interior of S. We can choose coordinates in $\mathbb{R}^{n+1}$ so that S is tangent at $q = (1, 0, \cdots, 0) = e_1$ to the hyperplane P given by $x_1 = 1$ and S lies on the opposite side of P to 0.

The sublevel set $W = f^{-1}(-\infty, 0] = \cup_{t \leq 0} \Phi_t(S) \subset \Omega$ is obtained by flowing S backwards. Let H be the hyperplane $x_1 = 1 + \epsilon$. Refer to Figure 1. We do not know that S is *properly* embedded in $\mathbb{R}^{n+1}$. However if $\epsilon > 0$ is small enough we can work in a chart for a small neighborhood of $\text{dev}^{-1}(q)$ in $\xi\widetilde{M}$ and see that $K = H \cap W$ is a compact convex set and $\partial K = H \cap S$.

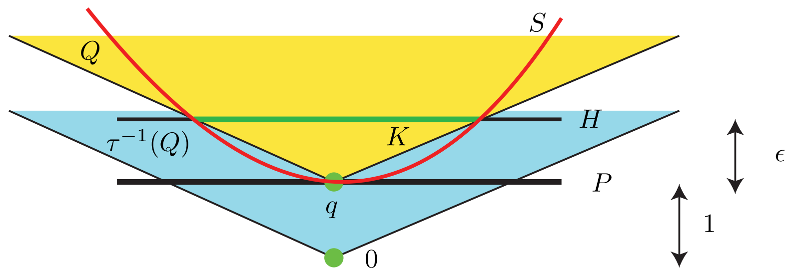

FIGURE 1. Flowing S backwards

Let Q be the convex cone consisting of the set of rays starting at q and intersecting K. Since $q \in \partial W = S$ and W is convex it follows that Q contains the subset of W above H. Unit vertical translation upwards $\tau : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by $\tau(x) = x + e_1$. Note that $\tau(Q) \subset Q$. Since $\epsilon < 1$ it follows that $\tau(S)$ is above H, therefore Q contains $\tau(S)$. Hence $\tau^{-1}(Q)$ contains S. Since $\tau^{-1}(Q)$ is the cone from 0 of $\tau^{-1}(K)$, it is preserved by Φ so it contains the entire orbit $\Phi \cdot S = \Omega$. It follows that $\pi_\xi(\Omega) \subset \mathbb{RP}^n$ is contained in $\pi_\xi(\tau^{-1}(K))$. Since $\tau^{-1}(K)$ is a compact convex set in $x_n = \epsilon$ it follows that $\pi_\xi(\Omega)$ is properly convex. □

4. THE CHARACTERISTIC CONVEXITY FUNCTION

In this section $V = \mathbb{R}^{n+1}$ and $\Omega \subset \mathbb{S}(V) = \mathbb{S}^n$ is an open properly convex set. The open convex cone $\mathcal{C}\Omega \subset V$ consists of all $t \cdot v$ with $v \in \Omega$ and $t > 0$. The *dual cone* $\mathcal{C}\Omega^* \subset V^*$ is the set of all $\phi \in V^*$ with $\phi(x) > 0$ for all $x \in \mathcal{C}\overline{\Omega}$. The *dual domain* is $\Omega^* = \mathbb{P}(\mathcal{C}\Omega^*) \subset \mathbb{P}(V^*)$. The *characteristic function* $\chi = \chi_\Omega : \mathcal{C}\Omega \longrightarrow \mathbb{R}^+$ of Koecher [19] and Vinberg [30] is defined by

$$\chi(x) = \int_{\mathcal{C}\Omega^*} e^{-\psi(x)} d\psi$$

where $d\psi$ is a Euclidean volume form on V^*. This function is real analytic, non-negative, and $\chi(tx) = t^{-(n+1)}\chi(x)$ for $t > 0$. More generally, if A is in the subgroup $\mathrm{GL}(\mathcal{C}\Omega) \subset \mathrm{GL}(V)$ that preserves $\mathcal{C}\Omega$, then $\chi(Ax) = (\det A)^{-1}\chi(x)$. The level sets of χ, called *characteristic hypersurfaces*, are smooth, convex, and meet each ray in $\mathcal{C}\Omega$ once transversely. The *characteristic section* is the map $\sigma_\Omega : \Omega \longrightarrow \mathcal{C}\Omega$ given by

$$\sigma_\Omega(x) = x \cdot (\chi(x))^{1/(n+1)}$$

It has image *the characteristic hypersurface* $S_\Omega = \chi^{-1}(1)$.

The radial flow $\Phi_t(x) = e^{-t} \cdot x$ on V preserves $\mathcal{C}\Omega$ and $\mathrm{c} = \mathrm{c}_\Omega = (n+1)^{-1} \log \chi$ is a flow function on $\mathcal{C}\Omega$. The Hessian $D^2 c$ is a positive definite quadratic form at each point of $\mathcal{C}\Omega$ and gives a complete metric on $\mathcal{C}\Omega$. Thus $\mathrm{c}_\Omega : \mathcal{C}\Omega \to \mathbb{R}$ is a complete convexity function called the *characteristic convexity function.* A reference for the above is [13], page 53 (C.8) in the 1988 version and page 68 in the 2009 version.

If $\Gamma \subset \mathrm{SL}_\pm(\mathcal{C}\Omega)$ is the holonomy of a properly convex manifold $M = \Omega/\Gamma$, then ξM is identified with $\mathcal{C}\Omega/\Gamma$. Since c_Ω is preserved by Γ it covers a map $c_M : \xi M \to \mathbb{R}$. It is a convexity function for M called the *characteristic convexity function* for M.

Definition 4.1. *The subspace $\mathcal{D}ev_c(M) \equiv \mathcal{D}ev_c(M, \mathbb{P}_+) \subset \mathcal{D}ev(M, \mathbb{P}_+)$ consists of the developing maps of properly convex structures for which ∂M is strictly convex.*

Proof of (0.1) when M is closed. If M is properly convex there is a characteristic convexity function $c_M : \xi M \to \mathbb{R}$. If the holonomy of M is changed slightly then, by (1.2), there is a radiant affine manifold N_1 and a diffeomorphism $f : \xi_1 M \to N_1$ that is everywhere close to an affine map. Taking infinite cyclic covers gives a map $F : \xi M \to N$ that is everywhere close to affine. The compact Hessian-convex hypersurface $S = c^{-1}(0) \subset \xi M$ maps to a hypersurface in N that is convex if F is close enough in $C^2(\xi M, N)$ to affine. It is also transverse to the radial flow Φ_N on N for the same reason. This section of the radial flow defines a convexity function on N by (3.3). This convexity function is complete because N_1 is compact and every Riemannian metric on a compact manifold is complete. It follows from (3.4) that N/Φ_N is properly convex. $\square$

Let C be the set of closed subsets of $\mathbb{S}^n$ equipped with the Hausdorff topology. Let $\mathcal{C}$ be the set of properly convex n–manifolds in $\mathbb{S}^n$ with (possibly empty) strictly convex boundary. There is an injective map $\iota : \mathcal{C} \to C \times C$ defined by $f(\Omega) = (\overline{\Omega}, \overline{\Omega} \setminus \Omega)$. The *Hausdorff boundary topology* on $\mathcal{C}$ is the subspace topology given by this embedding. Thus a neighborhood of Ω consists of domains Ω' which are close to Ω and $\partial\Omega'$ is close to $\partial\Omega$. This topology is given by a metric.

Definition 4.2. *The* strong geometric topology *on $\mathcal{D}ev_c(M)$ is the refinement of the geometric topology obtained by requiring the map $\mathcal{D}ev_c(M) \to \mathcal{C}$ given by* $\mathrm{dev} \mapsto \mathrm{Im}(\mathrm{dev})$ *is continuous.*

If M is closed the strong geometric topology equals the geometric topology. Two developing maps are close in this topology if they are close in C^∞ on a large compact set in the universal cover of the interior and in addition their images are close in the above sense. This can be expressed more simply using basepoints in the space of developing maps as in (1.3):

Suppose M has no boundary and $\mathrm{dev}_\rho \in \mathcal{D}ev_c(M)$ and $\rho = \mathcal{H}ol(\mathrm{dev}_\rho)$ and $\Gamma = \rho(\pi_1 M) \subset \mathrm{SL}_\pm(n+1, \mathbb{R})$ and $\Omega_\rho = \mathrm{Im}(\mathrm{dev}_\rho) \subset \mathbb{S}^n$. *Choosing* dev_ρ *as a basepoint* means: replace M by Ω_ρ/Γ. Thus $\mathrm{dev}_\rho = i : \widetilde{M} \hookrightarrow \mathbb{S}^n$ is now the inclusion. Then $\mathrm{dev}_\sigma \in \mathcal{D}ev_c(M)$ *is close to* dev_ρ *in the strong geometric topology* means: dev_σ is close to i in $C^\infty_w(\widetilde{M}, \mathbb{S}^n)$ and $\Omega_\sigma = \mathrm{Im}(\mathrm{dev}_\sigma)$ is close to Ω_ρ in $\mathcal{C}$.

There is a radiant affine manifold ξM_ρ and we have the same notions for $\mathcal{D}ev(\xi M_\rho, \mathbb{L})$. The radiant affine manifold $N = \mathcal{C}\Omega_\rho/\Gamma$ is (G, X) equivalent to ξM_ρ. The developing map for N, $\mathrm{dev}^\xi_\rho \in \mathcal{D}ev(N, \mathbb{L})$, is the inclusion $\mathrm{dev}^\xi_\rho : \mathcal{C}\Omega \hookrightarrow \mathbb{R}^{n+1}$. *A nearby developing map* $\mathrm{dev}^\xi_\sigma \in \mathcal{D}ev(N, \mathbb{L})$ *in the strong geometric topology* means: dev^ξ_σ is close to the inclusion in $C^\infty_w(\mathcal{C}\Omega_\rho, \mathbb{R}^{n+1})$ and in addition $\mathcal{C}\Omega_\sigma$ is close to $\mathcal{C}\Omega_\rho$ in the Hausdorff boundary topology on subsets of $\mathbb{S}^{n+1}$.

Let $\mathcal{C}' \subset \mathcal{C}$ be the subspace of open properly convex sets. For $K \subset V$ define $\Omega(K) \subset \mathcal{C}'$ to be those properly convex domains Ω that contain K. The map $\mathcal{C}' \to \mathcal{C}'$ given by $\Omega \mapsto \Omega^*$ is continuous.

Lemma 4.3. *If $K \subset \mathbb{R}^{n+1}$ is compact, then the function $\overline{\chi} : \Omega(K) \to C^\infty(K)$ defined by $\overline{\chi}(\Omega) = \chi_\Omega | K$ is continuous.*

Proof. Since both topologies are metrizable it suffices to show that the image of a convergent sequence converges. Suppose the sequence $\Omega_k \in \Omega(K)$ converges to $\Omega_\infty \in \Omega(K)$, and denote the respective characteristic functions by χ_k and χ^∞. Define the smooth function $h : V \times V^* \longrightarrow \mathbb{R}$ by $h(x, \phi) = \exp(-\phi x)$. Then for $x \in K$, if ∂^α is an n'th order mixed partial derivative on V, then $\partial^\alpha h(x, \phi) = p(\phi) h(x, \phi)$ where $p(\phi)$ is a monomial of degree n in the coordinates of ϕ. Let $U = \Omega^*_\infty \,\Delta\, \Omega^*_k$ be the symmetric difference then

$$|\partial^\alpha \chi_\infty(x) - \partial^\alpha \chi_k(x)| \leq \int_{\mathcal{C}U} |p(\phi) h(x, \phi)| d\phi.$$

Since $K \subset \mathcal{C}(\Omega_k \cap \Omega_\infty)$ it follows that $\phi(x) > 0$ for all $x \in K$ and $\phi \in \mathcal{C}U$. Now $p(\phi)$ is polynomial in ϕ, and $h(x, \phi)$ is exponential in ϕ, so $p(\phi) h(x, \phi) \to 0$ exponentially fast as $\phi \to \infty$ in $\mathcal{C}U$. It follows that if U is small enough, then $|\partial^\alpha \chi_\infty - \partial^\alpha \chi_k| < \epsilon$ on K. See (I.3.1) of [12] for more details. $\square$

The restriction of the Hessian metric $D^2 c$ to S_χ is Riemannian metric that is preserved by $\mathrm{SL}_\pm(\mathcal{C}\Omega)$. If $M = \Omega/\Gamma$ is a properly convex manifold, then radial projection gives a natural identification $M \equiv S$ and this puts a Riemannian metric on M called the *induced metric.* The following seems to be folklore:

Corollary 4.4 (bounded curvature)**.** *For each dimension $n > 0$ there is $k_n > 0$ such that if M is a properly convex projective manifold of dimension n, then all sectional curvatures κ of the induced metric on M satisfy $|\kappa| < k_n$. Moreover the induced metric is k_n-bi-Lipschitz equivalent to the Hilbert metric, and is therefore complete.*

Proof. If the result is false there is a sequence $M_k = \Omega_k/\Gamma_k$ and a point $x_k \in M_k$ and a sectional curvature $\kappa > k$ at x_k. By Benzécri compactness (9.2) we may assume these domains are in Benzecri position (9.1) with $x_k = 0$ and $\Omega_k \to \Omega_\infty$. The sectional curvature is given by formula involving various partial derivatives of c. By (4.3) these formulae converge to some (finite) sectional curvature for M_∞, a contradiction. This also proves the bi-Lipschitz result. $\square$

We wish to give universal bounds on the derivatives of certain real-valued functions defined on radiant affine manifolds of the form $N = \mathcal{C}\Omega/\Gamma$. If M is a smooth manifold and $f \in C^\infty(M)$ is a smooth function, then the k-th derivative $D^k f_x$ at $x \in M$ is a symmetric k-linear map on the vector space $V = T_xM$. Given a norm on V we get an operator norm $\|D^k f_x\|$ defined as the infimum of K for which $|D^k f_x(v_1, \cdots, v_k)| \le K\|v_1\| \cdots \|v_k\|$. In our case $M = \mathcal{C}\Omega$ is properly convex, and hence a Finsler manifold using the Hilbert metric on $\mathcal{C}\Omega$, and this gives a norm $\|\cdot\|_{\mathcal{C}\Omega}$ called the *Hilbert-Finsler norm* on the tangent space to $\mathcal{C}\Omega$ and corresponding operator norm. The group $\mathrm{GL}(\mathcal{C}\Omega)$ acts by isometries for this norm and so pushes down to a norm on the tangent space $N = \mathcal{C}\Omega/\Gamma$.

Given a point $x \in \mathcal{C}\Omega$ there is a *Benzécri chart* τ for $\mathcal{C}\Omega$ (see 9.1) centered on x. This chart determines a Euclidean metric d_E on $\mathcal{C}\Omega$, and there is also the Hilbert metric $d_H = d_{\mathcal{C}\Omega}$. There is a constant $K > 0$ depending only on dimension such that in the ball of d_H-radius 1 around x we have $K^{-1} \cdot d_E \le d_H \le K \cdot d_E$.

It follows that universal bounds on operator norms using the Hilbert metric give bounds in the Euclidean metric for Benzécri coordinates, and vice-versa. Thus we may regard these universal bounds as bounds on ordinary partial derivatives of functions defined in a small neighborhood of the origin in $\mathbb{R}^n$ by means of Benzécri coordinates. We now use Benzécri's compactness theorem (9.2) to provide uniform bounds on various properties of characteristic functions.

Suppose B is a properly convex submanifold of a properly convex manifold M, both without boundary so that $\xi B \subset \xi M$. The next result says that far inside B (as measured in M) the characteristic convexity functions for B and M are almost equal.

Lemma 4.5 (convexity functions on submanifolds)**.** *Given $\epsilon > 0$ and a dimension n, there is $R = R(\epsilon, n) > 0$ with the following property. Suppose $B \subset M$ are properly convex n–manifolds with characteristic convexity functions c_B and c_M. Let $U \subset B$ be the subset of all x with $d_M(x, M\setminus B) > R$ and define $g = c_M - c_B : \xi U \to \mathbb{R}$. Then $\|D^k g\| < \epsilon$ for $0 \le k \le 2$.*

Proof. Let $\Omega_U \subset \Omega_B \subset \Omega_M \subset \mathbb{S}^n$ be images of the developing maps of $U \subset B \subset M$ respectively. Since h is constant along rays from the origin in $\mathcal{C}\Omega_U$ it suffices to show the bounds hold for $x \in \Omega_U := \mathbb{S}^n \cap \mathcal{C}\Omega_U$. Choose a Benzécri chart for Ω_M centered on x. In this chart the Euclidean distance between $\partial\Omega_M$ and $\partial\Omega_B$ is bounded above by a function $f(R)$ independent of Ω_M and Ω_B and $f(R) \to 0$ as $R \to \infty$. The result now follows from (4.3). $\square$

It also follows from (4.3) that nearby properly convex manifolds have nearby characteristic convexity functions; we state this as:

Lemma 4.6. *The map $\mathcal{D}ev_c(\xi M_\rho) \to C^\infty_w(\xi M_\rho)$ given by $\mathrm{dev}_\sigma \mapsto c_\Omega \circ \mathrm{dev}_\sigma$ is continuous, where $\Omega \subset \mathbb{RP}^{n+1}$ is the image of dev_σ. Here, the strong geometric topology is used on $\mathcal{D}ev_c(\xi M_\rho)$.*

Lemma 4.7 (uniform Hessian-convexity)**.** *For each dimension n there is $0 \le \kappa = \kappa(n) \le 1$ with the following property. Suppose $\Omega \subset \mathbb{R}P^n$ is open and properly convex and $c : \mathcal{C}\Omega \longrightarrow \mathbb{R}$ is the characteristic convexity function. Then $D^2\,\mathrm{c} \ge \kappa\|\cdot\|^2_{\mathcal{C}\Omega}$ everywhere.*

Proof. Since c is preserved by each element of $\mathrm{GL}(\mathcal{C}\Omega)$ up to adding a constant, it suffices to show there is κ such that the result holds at the center of every Benzécri domain $\Omega = \mathbb{S}^n \cap \mathcal{C}\Omega$. Since the set of all such domains is compact, and by (4.3) the characteristic function varies smoothly with the Benzécri domain, the result follows. □

If $f : (-\epsilon, \epsilon) \to \mathcal{C}\Omega$ is an arc parameterized by arc length then $(c \circ f)''$ is a *second directional derivative*. The conclusion can be rephrased as $(c \circ f)'' \geq \kappa$ for every second directional derivative. We will abuse notation and write this as $c'' \geq \kappa$.

5. Deforming Properly Convex Manifolds rel ends

In this section we prove a version of (1.7) for convex manifolds. We show that the only obstruction to deforming a properly convex manifold is whether the ends have such a deformation. Suppose M_ρ is a properly convex manifold with holonomy ρ. The main result of this section (5.7) is that for representations σ sufficiently close to ρ, if the ends of M_ρ can be deformed to properly convex manifolds with holonomy the restriction of σ, then these deformations can be extended to all of M_ρ to give a properly convex structure M_σ.

Definition 5.1. *A Finsler manifold $M = \Omega/\Gamma$ has* controlled ends *if there is a smooth proper function, called an* exhaustion function*, $f : M \to [0, \infty)$ and $K > 0$ such that $\|Df\|, \|D^2 f\| < K$ in the Finsler norm.*

For example every finite volume complete hyperbolic manifold has controlled ends. If $C \cong \partial C \times [0, \infty)$ is a horocusp in a hyperbolic manifold M then the horofunction $f(x) = d_M(x, \partial C)$ is an exhaustion function. A similar construction works on a *generalized cusp* (6.26). There are complete Riemannian manifolds with no exhaustion function. However:

Proposition 5.2. *Every properly convex manifold has controlled ends.*

Proof. By (4.4) every properly convex manifold admits a complete Riemannian metric that is bi-Lipschitz equivalent to the Hilbert metric and which has bounded sectional curvature. It is a result of Schoen and Yau [24] (see also [28] and Proposition 26.49 in [7]) that a complete Riemannian manifold of bounded sectional curvature has a proper function with bounded gradient and Hessian. □

Definition 5.3. *A* localization function *on a Finsler manifold M is a smooth function $\lambda : M \to [0, 1]$ with compact support and $\|D\lambda\|, \|D^2\lambda\| \leq 1$.*

Corollary 5.4. *If M is a properly convex manifold and $K \subset M$ is compact, then there is a localization function λ on M with $\lambda(K) = 1$.*

Suppose $M = A \cup \mathcal{B}$ is a connected n–manifold and A is a compact submanifold with $\partial A = \partial M \sqcup \partial \mathcal{B}$ and $\mathcal{B}$ has k components B_i with $1 \leq i \leq k$ such that $B_i = \partial B_i \times [0, \infty)$. By (1.6) there is an relative holonomy map

$$\mathcal{E}_{\mathbb{P}} : \mathcal{D}ev(M, \mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, \mathbb{P})$$

The subspace $\mathcal{D}ev_c(M, \mathbb{P}) \subset \mathcal{D}ev(M, \mathbb{P})$ consists of the developing maps of properly convex structures for which ∂M is strictly convex. The subspace $\mathcal{R}el\mathcal{H}ol_e(M, \mathcal{B}, \mathbb{P}) \subset \mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, \mathbb{P})$ consists of the data for which each B_i is properly convex with strictly convex boundary. Then $\mathcal{D}ev_e(M, \mathbb{P}) = \mathcal{E}_{\mathbb{P}}^{-1}\, \mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, \mathbb{P})$ consists of developing maps for which these ends are properly convex with strictly convex boundary. Finally $\mathcal{D}ev_{ce}(M, \mathbb{P}) = \mathcal{D}ev_c(M, \mathbb{P}) \cap \mathcal{D}ev_e(M, \mathbb{P})$ is the subspace of developing maps for properly convex structures on M with ∂M strictly convex and for which these ends are properly convex and have strictly convex boundary.

Theorem 5.5. *$\mathcal{E}_{\mathbb{P}} : \mathcal{D}ev_{ce}(M, \mathbb{P}) \to \mathcal{R}el\mathcal{H}ol_e(M, \mathcal{B}, \mathbb{P})$ is open using the geometric topology on the domain and the strong geometric topology on the codomain.*

Proof. First assume M has no boundary. By (1.7) $\mathcal{E}_{\mathbb{P}} : \mathcal{D}ev(M, \mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M, \mathcal{B}, \mathbb{P})$ is open using the geometric topologies in domain and codomain. Hence the restriction $\mathcal{E}_{\mathbb{P}} : \mathcal{D}ev_e(M, \mathbb{P}) \to \mathcal{R}el\mathcal{H}ol_e(M, \mathbb{P})$ is also open with these topologies. Thus it is open using the strong geometric topology (which is finer than the geometric topology) on the codomain and the geometric topology on the domain. The *end geometric topology* on $\mathcal{D}ev_e(M, \mathbb{P})$ is the smallest refinement of the geometric topology such that $\mathcal{E}_{\mathbb{P}}$ is continuous. Then $\mathcal{E}_{\mathbb{P}}$ is open and continuous with the end geometric topology on the domain and the strong geometric topology on the codomain.

As usual we will assume that $\mathcal{B} = B$ is connected. It suffices to show that $\mathcal{D}ev_{ce}(M, \mathbb{P})$ is open in $\mathcal{D}ev_e(M, \mathbb{P})$ with respect to the end geometric topology. A neighborhood $\mathcal{U} \subset \mathcal{D}ev_e(M, \mathbb{P})$ of dev_ρ in this topology consists of all developing maps dev_σ that are nearby in $C^\infty_w(\widetilde{M}, \mathbb{RP}^n)$ and in addition have the property that $\mathrm{dev}_\sigma(\widetilde{B})$ is close in $\mathcal{C}$ to $\mathrm{dev}_\rho(\widetilde{B})$.

Suppose $\mathrm{dev}_\rho \in \mathcal{D}ev_{ce}(M, \mathbb{P})$ has holonomy ρ and $\mathrm{dev}_\sigma \in \mathcal{U}$ has holonomy σ. The corresponding projective structures (charts) on M are denoted by M_ρ and M_σ. We must show $\mathrm{dev}_\sigma \in \mathcal{D}ev_c(M, \mathbb{P})$. To do this we construct a complete convexity function on the tautological bundle ξM_σ. It then follows that M_σ is properly convex by (3.4).

We will use M_ρ as a basepoint for $\mathcal{D}ev_c(M)$ as in (1.3), see also (4.2). Thus we replace M by M_ρ and will usually omit the subscript ρ. Then $\widetilde{M} = \Omega_\rho \subset \mathbb{S}^n$ and $\mathrm{dev}_\sigma : \Omega_\rho \to \mathbb{S}^n$. Similarly we use $\xi M := \mathcal{C}\Omega_\rho/\Gamma$ as a basepoint for $\mathcal{D}ev(\xi M_\rho, \mathbb{L})$ and write this as $\mathcal{D}ev(\xi M, \mathbb{L})$.

We use the Hilbert-Finsler metric on ξM to calculate operator norms. Recall $\xi_1 M = \xi M/\Phi_1$ is the affine suspension and has an infinite cyclic cover ξM. Let $\kappa = \kappa(\dim(M)) > 0$ be the lower bound on the Hessian of characteristic functions given by (4.7) and $\epsilon = \kappa/10$. Let $R = R(\epsilon, \dim(\xi M))$ be the constant given by (4.5) and $K \subset M$ a compact connected submanifold such that $\xi_1 K$ contains the R-neighborhood of $\xi_1 A$ in $\xi_1 M$. Then $M \setminus K \subset B$ and for $x \in \xi(M \setminus K)$ the characterisic functions $c_{\rho,B}$ and $c_{\rho,M}$ are ϵ–close in $C^2(\xi B)$.

By (5.4) there is a localization function $\lambda : \xi_1 M \to [0, 1]$ with $\lambda(\xi_1 K) = 1$ that has support inside a compact connected submanifold $\xi_1 L$. Define $J = \mathrm{cl}(L \setminus K)$. Then every point in $\xi_1 J$ is distance at least R from $\partial(\xi_1 B)$. All these submanifolds depend on the choice of ϵ. Let $\widetilde{\lambda} : \xi M \to [0, 1]$ be the function that covers λ. We will recklessly abuse notation by writing $\widetilde{\lambda}$ as λ. Observe that $\lambda^{-1}(0, 1) \subset \xi J$.

Claim 1. *There is a convexity function $c : \xi M \to \mathbb{R}$ which equals $c_{\rho,M}$ on ξK and equals $c_{\rho,B}$ on $\xi(M \setminus L)$ and $D^2 c \geq (\kappa/2)\| \cdot \|^2$.*

Proof of Claim 1. First blend $c_{\rho,M}$ and $c_{\rho,B}$ inside ξJ using λ to get $f : \xi M \to \mathbb{R}$ given by

$$f = \lambda \cdot c_{\rho,M} + (1 - \lambda) \cdot c_{\rho,B} = c_{\rho,M} + (1 - \lambda) \cdot g$$

where $g = c_{\rho,B} - c_{\rho,M}$. The map f is well defined even though $c_{\rho,B}$ is only defined on ξB because $(1 - \lambda) = 0$ outside ξB.

Subclaim: $D^2 f \geq (\kappa/2)\| \cdot \|^2$. Outside ξJ this follows from (4.7) since f is $c_{\rho,M}$ on ξK and $c_{\rho,B}$ on $\xi(M \setminus L)$. On ξJ we show this using directional derivatives. By the product rule

$$f'' = c''_{\rho,M} + g'' - (\lambda'' g + 2\lambda' g' + \lambda g'').$$

Since M_ρ is properly convex $c''_{\rho,M} \geq \kappa$ by (4.7). Also $|\lambda|, |\lambda'|, |\lambda''| \leq 1$ because λ is a localization function and $|g|, |g'|, |g''| < \epsilon = \kappa/10$ on ξJ by definition of R and K so

$$|\ g'' - (\lambda'' g + 2\lambda' g' + \lambda g'')\ | \leq 5\epsilon = \kappa/2.$$

Thus $f'' \geq \kappa/2$ which proves the subclaim. The level set $S = f^{-1}(0)$ is Hessian-convex in the backwards direction of the flow and is the 0-set of a unique flow function c which coincides with $c_{\rho,B}$ outside ξL. It follows from (3.3) that $c'' \geq \kappa/2$ also. This proves claim 1. ∎

To avoid a proliferation of notation, and because what we are about to do is similar to what we just did, we reuse notation as follows. We define the new K to be the old L and the new λ is a localization function on $\xi_1 M$ with $\lambda(\xi_1 K) = 1$ and the new $L \subset M$ is a compact connected manifold so that $\xi_1 L$ contains the support of λ. Then redefine $J = \mathrm{cl}(L \setminus K)$. Let $E := \mathrm{cl}(M \setminus K) \subset B$. Again we write the lift as $\lambda : \xi M \to \mathbb{R}$. There are characteristic convexity functions $c_{\rho,B} : \xi B_\rho \to \mathbb{R}$ and $c_{\sigma,B} : \xi B_\sigma \to \mathbb{R}$.

Since $\xi_1 L_\rho$ is compact, if $\mathcal{U}$ is small there is a diffeomorphism $H : \xi_1 M_\rho \to \xi_1 M_\sigma$ such that $H|\xi_1 L_\rho$ is very close in C^∞ to the identity in the following sense. The map H is covered by $\widetilde{H} : \xi\widetilde{M}_\rho \to \xi\widetilde{M}_\sigma$, and the restriction of $\widetilde{H}$ is very close to the inclusion $\xi\widetilde{L}_\rho \hookrightarrow \mathbb{R}^{n+1}$ in $C^\infty_w(\xi\widetilde{L}_\rho, \mathbb{R}^{n+1})$. The map H also covers $h : \xi M_\rho \to \xi M_\sigma$.

Set $g = (c_{\sigma,B}) \circ h - c_{\rho,B} : \xi E_\rho \to \mathbb{R}$ then by (4.6) $\|D^k g\| < \epsilon$ for $k \in \{0, 1, 2\}$ everywhere on $\xi_1 J_\rho$. Define $f : \xi M_\rho \to \mathbb{R}$ by

$$f = \lambda \cdot c + (1 - \lambda) \cdot (c_{\sigma,B}) \circ h.$$

As before this is well defined.

Claim 2. *$f'' \geq \kappa/2$ on ξL_ρ.*

Proof of Claim 2. When $\lambda = 1$ then $f'' = c'' \geq \kappa/2$ by claim 1. The set where $\lambda < 1$ is contained in ξJ_ρ. On ξJ_ρ and $c = c_{\rho,B}$ so

$$f = \lambda \cdot c_{\rho,B} + (1 - \lambda) \cdot (c_{\sigma,B}) \circ h = c_{\rho,B} + (1 - \lambda) \cdot g$$

and

$$f'' = c''_{\rho,B} + g'' - (\lambda'' \cdot g + 2\lambda' g' + g'').$$

Then $c''_{\rho,B} \geq \kappa/2$ by (4.7). As before $|\lambda|, |\lambda'|, |\lambda''| \leq 1$ and by the above $|g|, |g'|, |g''| < \epsilon$. Since $\epsilon < \kappa/10$ this proves claim 2. □

Since H is very close to the inclusion in $C^\infty_w(\xi\widetilde{L}_\rho, \mathbb{R}^{n+1})$ it follows that $f \circ h^{-1}$ is Hessian-convex on ξL_σ. Outside this set $f \circ h^{-1} = c_{\sigma,B}$ which is Hessian-convex. This proves $f : \xi M_\sigma \to \mathbb{R}$ is Hessian-convex everywhere.

Again it follows from (3.3) that there is a Hessian-convex flow function $c_\sigma : \xi M_\sigma \to \mathbb{R}$ defined by $f \circ h^{-1}$. The corresponding Hessian metric on $\xi_1 M_\sigma$ is complete because $\xi_1 L_\sigma$ is compact so the metric is complete on $\xi_1 L_\sigma$, and outside $\xi_1 L_\sigma$ it is the complete metric given by the properly convex end $\xi_1 B_\sigma$. It follows that the Hessian metric on ξM_σ is also complete. This completes the proof when M has no boundary.

Now suppose M has (compact) boundary and set $P = \mathrm{int}(M)$. Then P is properly convex with a characteristic convexity function $c : \xi P_\rho \to \mathbb{R}$. By (8.3) there is a submanifold $N \subset M$ with Hessian-convex compact boundary such that $\mathrm{cl}(M \setminus N)$ is a collar of ∂M. The restriction of c to ξN is a complete convexity function. There is a diffeomorphism $F : \xi M \to \xi N$ close to the identity in C^2 that is the identity outside a small collar of ∂M. Then $c_{\rho,M} := (c|_{\xi N}) \circ F : \xi M \to \mathbb{R}$ is a complete convexity function. The pullback of the restriction to ξN of the Hilbert metric on ξP is a complete metric on ξM. The proof now proceeds as above to construct a complete convexity function on ξM_σ. □

To apply (5.5) involves finding deformations of the cusps that are nearby in the strong geometric topology. This involves finding a diffeomorphism from the original cusp to the deformed cusp that is close to projective. To make this task easier we show such a map exists for a small deformation of the holonomy if the deformed domain is close to the original domain.

The *projective Kleinian group space* for a smooth manifold M is

$$\mathcal{K}(M) = \{(\Omega, \rho) \ \in \mathcal{C} \times \mathrm{Rep}(M) : \ M \text{ diffeomorphic to } \Omega/\rho(\pi_1 M)\}$$

with topology given by the subspace topology of the product topology on $\mathcal{C}\times\mathrm{Rep}(M)$. This topology is given by a metric. There is a natural map

$$\mathcal{K} : \mathcal{D}ev_c(M,\mathbb{P}) \to \mathcal{K}(M)$$

given by $\mathcal{K}(\mathrm{dev}) = (\mathrm{dev}(\widetilde{M}), \mathrm{hol}(\mathrm{dev}))$. With the *strong* geometric topology on the domain it is obvious that this map is continuous.

Proposition 5.6. *Suppose $M \cong \partial M \times [0,\infty)$ is a connected smooth manifold and ∂M is compact. Then $\mathcal{K}$ is an open map.*

Proof. Suppose $\mathrm{dev}_\rho \in \mathcal{D}ev_c(M,\mathbb{P})$ and $\mathcal{K}(\mathrm{dev}_\rho) = (\Omega_\rho, \rho)$ and that $(\Omega_\sigma,\sigma) \in \mathcal{K}(M)$ is very close. Then $Q = \Omega_\sigma/\sigma(\pi_1 M)$ is a properly convex manifold. We identify $M \equiv \Omega_\rho/\rho(\pi_1 M)$. It suffices to show there is a diffeomorphism $M \to Q$ which is almost a projective map between large compact sets in the interiors.

Using (8.3) there is a diffeomorphism $M \cong \partial M \times [0,\infty)$ so that $\partial M \times t$ is $\mathrm{dev}_{\rho,M}$-Hessian-convex for all $t \le 1$. For $k > 1$ define $N = \partial M \times [1/k, k]$ and $W = \partial M \times [0, k+1]$ and $B = \partial M \times 1/k$. By (1.2) there is a $\mathrm{dev}_{\sigma,W} \in \mathcal{D}ev(W,\mathbb{P})$ with holonomy σ that is very close to $\mathrm{dev}_{\rho,M|W}$ over a compact set in $\widetilde{W}$ that covers N.

By (1.5) we may change $\mathrm{dev}_{\sigma,W}$ by a small isotopy so that there is a projective embedding $f : N \to Q$. If σ is close enough to ρ then, since B is Hessian-convex for $\mathrm{dev}_{\rho,M}$ it follows that B is also Hessian-convex for $\mathrm{dev}_{\sigma,W}$. Hence $f(B)$ is Hessian-convex in Q.

Let P be the closure of the component of $Q \setminus f(N)$ that contains ∂Q. Since ∂M is compact, for homology reasons $f(B)$ separates ∂Q from the end of Q, thus P is compact and $\partial P = \partial Q \sqcup f(B)$. We claim P is diffeomorphic to $B \times I$.

Suppose N is a smooth manifold that is homeomorphic to $\partial N \times I$. By [31] smooth manifolds are PL. The $M \times I$ *theorem* [18] says that if M is a PL manifold, then every smoothing of $M \times I$ is diffeomorphic to a product. Thus N is diffeomorphic to $\partial N \times I$. Hence it suffices to show P is homeomorphic to $B\times I$. Since B is $\mathrm{dev}_{\sigma,N}$-Hessian-convex there is a nearest point retraction (using the Hilbert metric on Q) $r : P \to B$ with fibers that are lines and this gives a homeomorphism $P \to B \times I$ which proves the claim.

It follows that P is a collar of ∂Q so $R = P \cup f(N) \cong B \times [0,k]$ is also a collar of ∂Q. Thus $Q' = \mathrm{cl}(Q \setminus R)$ is diffeomorphic to $B \times [k,\infty)$. Clearly P lies in a small neighborhood of ∂Q. We can now extend f to a diffeomorphism $f : M \to Q$ by sending $\partial M \times [0, 1/k]$ to P and $\partial M \times [k,\infty)$ to Q'. This is close to a projective map on N. Define $\mathrm{dev}_{\sigma,M} : \widetilde{M} \to \mathbb{RP}^n$ by $\mathrm{dev}_{\sigma,M} = \mathrm{dev}_{\sigma,Q} \circ \widetilde{f}$. Since f is close to projective over N it follows that $\mathrm{dev}_{\sigma,M}$ is close to $\mathrm{dev}_{\rho,M}$. □

Suppose $M = A \cup \mathcal{B}$ is a smooth manifold with (possibly empty) boundary and A is a compact submanifold of M with $\partial A = \partial M \sqcup \partial\mathcal{B}$ and $\mathcal{B} = B_1 \sqcup \cdots B_k$ has $k < \infty$ connected components, and $B_i \cong \partial B_i \times [0,\infty)$. Define the *Kleinian relative-holonomy space*

$$\mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K}) \subset \mathrm{Rep}(\pi_1 M) \times \prod_{i=1}^{k} \mathcal{K}(B_i)$$

to be the subset of all $(\rho, (\Omega_1,\rho_1), \cdots, (\Omega_k,\rho_k))$ such that $\rho_i = \rho|\pi_1 B_i$. This space has the subspace topology of the product topology.

For each $B_i \subset M$ we fix a choice of some component $\widetilde{B}_i \subset \widetilde{M}$ of the preimage B_i in the universal cover of M. Then $\Omega_i = \mathrm{dev}(\widetilde{B}_i)$ and $\Gamma_i = \mathrm{hol}(\pi_1 B_i)$ gives a point in $\mathcal{K}(B_i)$. This defines the *Kleinian relative holonomy map*

$$\mathcal{E}_{\mathcal{K}} : \mathcal{D}ev_{ce}(M,\mathbb{P}) \longrightarrow \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K})$$

Theorem 5.7 (Convex Extension Theorem)**.** *$\mathcal{E}_{\mathcal{K}} : \mathcal{D}ev_{ce}(M,\mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K})$ is open using the geometric topology on the domain.*

Proof. This follows immediately from (5.5) and (5.6). □

Decoding what this says proves:

Proof of (0.2). There is a continuous map $\gamma : (-1,1) \to \mathcal{R}el\mathcal{H}ol(M,\mathcal{B},\mathcal{K})$ defined by

$$\gamma(t) = (\rho_t, (\Omega_1(t), \rho_t|_{\pi_1 B_1}), \cdots, (\Omega_k(t), \rho_t|_{\pi_1 B_k})$$

Since $\mathcal{E}_{\mathcal{K}}$ is open and $\gamma(0) \in \mathrm{Im}(\mathcal{E}_{\mathcal{K}})$ it follows that for some $\epsilon > 0$ that $g(-\epsilon,\epsilon) \subset \mathrm{Im}(\mathcal{E}_{\mathcal{K}})$. So for $|t| < \epsilon$ there is $\mathrm{dev}_t \in \mathcal{D}ev_{ce}(M,\mathbb{P})$ with $\mathcal{E}_{\mathcal{K}}(\mathrm{dev}_t) = \gamma(t)$. Define M_t to be the projective structure on M defined by dev_t. Then M_t is properly convex with strictly convex boundary because $\mathrm{dev}_t \in \mathcal{D}ev_{ce}(M)$. Moreove restricting this structure to B_i gives $P_i(t)$ by definition of $\mathcal{E}_{\mathcal{K}}$. □

6. Generalized Cusps

A generalized cusp is a certain kind of properly convex projective manifold. The main result of this section is that holonomies of generalized cusps with fixed topology form an open subset in a certain semi-algebraic set (6.25). This follows from the fact that *a generalized cusp contains a homogeneous cusp* (6.5). We then prove the main theorem (6.27). To keep this paper from becoming too long we only consider *maximal* cusps, i.e. those with boundary a closed manifold.

A cusp in a *hyperbolic manifold* viewed as a projective manifold is characterized by being projectively equivalent to an affine manifold that has a foliation by strictly convex hypersurfaces that are images of horospheres, together with a transverse foliation by parallel lines. This characterization does not work in general. Consider the affine manifold $M = U/\Gamma \cong T^2 \times [0,\infty)$, where

$$U = \{(x_1,x_2,x_3) : x_3 \geq x_1^2 + x_2^2 > 0\ \}$$

and Γ is the cyclic group generated by $(x_1,x_2,x_3) \mapsto (2x_1,2x_2,4x_3)$. It has a foliation by tori that are the images of the strictly convex hypersurfaces $z = K(x^2+y^2)$ for $K \geq 1$, and it has a transverse foliation by vertical lines. However M is *not convex.*

Definition 6.1. *A* generalized cusp *is a properly convex manifold $C = \Omega/\Gamma$ homeomorphic to $\partial C \times [0,\infty)$ with ∂C a closed manifold and $\pi_1 C$ virtually nilpotent such that $\partial\Omega$ contains no line segment, i.e. ∂C is strictly convex. The group Γ is called a* generalized cusp group.

A quasi-cusp *is a properly convex manifold homeomorphic to $Q \times \mathbb{R}$ with Q compact and $\pi_1 Q$ virtually nilpotent.*

If Γ contains no hyperbolics, then C is called a *cusp* and Γ is conjugate to a subgroup of $PO(n,1)$ by Theorem (0.5) in [10]. An example of a quasi-cusp is Δ/Γ for any discrete subgroup $\Gamma \cong \mathbb{Z}^{n-1}$ of the diagonal group in $SL(n+1,\mathbb{R})$, where $\Delta \subset \mathbb{RP}^n$ is the interior of an n-simplex that is preserved by Γ.

Definition 6.2. *A generalized cusp Ω/Γ is* homogeneous *if $\mathrm{PGL}(\Omega)$ acts transitively on $\partial\Omega$. The group $\mathrm{PGL}(\Omega)$ is called a* (generalized) cusp Lie group.

For example a cusp in a hyperbolic manifold is homogeneous if and only if it is the quotient of a horoball $\Omega \subset \mathbb{H}^n$. In this case $\mathrm{PGL}(\Omega)$ is conjugate to the subgroup of $PO(n,1) \cong Isom(\mathbb{H}^n)$ that fixes one point at infinity. Cusp Lie groups for 3–manifolds are listed in section 7.

Every finite volume cusp in a complete hyperbolic manifold contains a homogeneous cusp. There is an equivalence relation on generalized cusps generated by the property that one cusp can be projectively embedded in another. Equivalent cusps have conjugate holonomy. One can always shrink a cusp by removing a collar from the boundary. However sometimes one can remove a submanifold at the other end. For example there might by a totally geodesic codimension–1 compact submanifold in the interior of the cusp, which one could cut along. It simplifies matters to do this ahead of time:

Definition 6.3. *A generalized (maximal) cusp C is* minimal *if, for every cusp $C' \subset C$, $\partial C' = \partial C$ implies $C = C'$.*

Lemma 6.4. *Every generalized cusp contains a unique minimal cusp. A finite cover of a minimal cusp is minimal.*

Proof. Suppose $C = \Omega/\Gamma$ is a generalized cusp. Let Ω' be the convex hull of $\partial\Omega$. Then $\Omega' \subset \Omega$ is properly convex and Γ-invariant and $\partial\Omega' = \partial\Omega$. The cusp $C' = \Omega'/\Gamma$ is the unique minimal cusp contained in C. If M is a finite cover of C' then $M' = \mathrm{CH}(\partial M)$ so M is also minimal. □

The subgroup $\mathrm{UT}(n) < \mathrm{GL}(n,\mathbb{R})$ consists of upper-triangular matrices with positive diagonal entries. To find a homogeneous cusp in a generalized cusp involves removing a collar so the boundary is the right shape to be homogeneous.

Theorem 6.5. *Every generalized cusp contains a homogeneous generalized cusp.*

Proof. Suppose $C = \Omega/\Gamma$ is a generalized cusp. We may assume C is minimal by (6.4). Since Γ is virtually nilpotent by (6.10) there is a finite index subgroup $\Gamma' < \Gamma$ that is conjugate into $\mathrm{UT}(n+1)$ where $n = \dim C$. We will assume this conjugacy has been done. Then $\widetilde{C} = \Omega/\Gamma'$ is a generalized cusp that is a finite cover of C and is minimal by (6.4). At this point, results proved later in this section will be evoked. It follows from (6.13) that Γ' is a lattice in a connected upper-triangular Lie group $T = T(\Gamma)$. By (6.20) it follows that $\widetilde{C}$ is a radial flow cusp for a radial flow Φ with stationary hyperplane H. Let $\mathbb{R}^n = \mathbb{RP}^n \setminus H$. By (6.21) $\Omega \subset \mathbb{R}^n$ is a closed strictly convex set bounded by the strictly convex hypersurface $\partial\Omega$. By (6.22) there is a properly convex $\Omega_T \subset \Omega$ that is T-invariant and thus Γ'-invariant. By (6.23) Ω_T is preserved by all of Γ hence Ω_T/Γ is a homogeneous cusp in C and $\Gamma < \mathrm{PGL}(\Omega_T)$. □

In view of the fact that the holonomy of a projective structure lifts (2.1) to $\mathrm{GL}(n+1,\mathbb{R})$ we will do this in what follows.

A *connected* nilpotent subgroup Γ of $\mathrm{GL}(n,\mathbb{C})$ preserves a complete flag. However if Γ is not connected this need not be true. For example the quaternionic group of order 8 in $\mathrm{GL}(2,\mathbb{C})$ does not preserve a flag. First we show (6.9) that there is a finite index subgroup of Γ that preserves a complete flag. The index of a subgroup $H < G$ is written $|G : H|$. A subgroup $H \leq G$ is *characteristic* if every automorphism of G preserves H.

Lemma 6.6. $\exists\ h(n,k)$ *such that if the group G is generated by k elements, then there is a characteristic subgroup $C \leq G$ with $|G : C| \leq h(n,k)$ such that $|G : H| \leq n$ implies $C \leq H$ for all subgroups $H \leq G$.*

Proof. We show $h(n,k) = (n!)^{(n!^k)}$. Let S be the group of permutations of n elements so $|S| = n!$. If $\theta : G \longrightarrow S$ is a homomorphism, then $|G : \ker\theta| \leq |S|$. The number of such homomorphisms is at most $p = |S|^k$. Then $C = \cap \ker\theta$ (where the intersection is over all such homomorphisms) is a characteristic subgroup of G and $|G : C| \leq |S|^p$. Suppose $H \leq G$ and $m = |G : H| \leq n$. Then G permutes the m left cosets of H. This gives a homomorphism $\theta : G \to S$ and $C \leq \ker\theta \leq H$. □

‘ Suppose V is a vector space over $\mathbb{C}$. A *weight* of a subgroup $\Gamma \subset \mathrm{GL}(V)$ is a homomorphism (*character*) $\lambda : \Gamma \to \mathbb{C}^*$ such that the *weight space* $E(\lambda)$ and *generalized weight space* $V(\lambda)$ are both non-trivial. Here,

$$E(\lambda) = \bigcap_{\gamma\in\Gamma}(\ker(\gamma - \lambda(\gamma)) \quad \text{and} \quad V(\lambda) = \bigcup_{n>0}\bigcap\nolimits_{\gamma\in\Gamma}(\ker(\gamma - \lambda(\gamma))^n.$$

A (generalized) weight space is Γ invariant. A one-dimensional weight space is the same thing as a one-dimensional Γ–invariant subspace. The vector space V has a *generalized weight decomposition* if $V = \oplus V(\lambda)$, where the sum is over all weights.

The group Γ is *polycyclic of (Hirsch) length (at most)* k if there is a subbormal series $\Gamma = \Gamma_k \rhd \Gamma_{k-1} \cdots \rhd \Gamma_1 \rhd \Gamma_0 = 1$ with Γ_{i+1}/Γ_i cyclic. A subgroup of a polycyclic group of length k is polycyclic of length k. Every finitely generated nilpotent group is polycyclic. The following implies the Lie-Kolchin theorem.

Lemma 6.7. *$\exists\ c = c(n,k)$ such that if $\Gamma < \mathrm{GL}(\mathbb{C}^n)$ is polycyclic of length at most k, then there is a characteristic subgroup $C \leq \Gamma$ with $|\Gamma : C| \leq c$ and C preserves a one-dimensional subspace of $\mathbb{C}^n$.*

Proof. We use induction on n and k. For $n = 1$ the result is obvious. For $k = 1$ the result follows from Jordan normal form with $c = 1$. Assume the result true for k. Suppose Γ is polycyclic of length $k+1$. Then Γ contains a normal polycyclic group Γ_k of length k with Γ/Γ_k cyclic. There is a characteristic subgroup $C_k \leq \Gamma_k$ of index at most $c(n,k)$ that preserves a one-dimensional subspace W.

There is some weight $\lambda : C_k \longrightarrow \mathbb{C}^*$ with W contained in the weight space $E = E(\lambda)$. There are at most n such weights. If θ is an automorphism of C_k then $\lambda \circ \theta$ is a weight for C_k. Since C_k is a characteristic subgroup of Γ_k, and Γ_k is normal in Γ, it follows that C_k is perserved by inner automorphisms of Γ. Thus an inner automorphism of Γ permutes these weights, so an element $\gamma \in \Gamma$ induces a permutation with order $m \leq n$ of the weights. Choose $\gamma \in \Gamma$ which generates Γ/Γ_k. Then γ^m induces the identity permutation. Hence the subgroup $\Gamma' = \langle C_k, \gamma^m \rangle$ preserves E. Applying Jordan normal form to $\gamma^m|E$ gives a one-dimensional subspace of E that is preserved by γ^m. This subspace is also preserved by C_k. Then $|\Gamma : \Gamma'| \leq m|\Gamma_k : C_k| \leq n \cdot c(n,k)$ since $m \leq n$. By (6.6) there is a characteristic subgroup $C \leq \Gamma_k \leq \Gamma$ with $|\Gamma : C| \leq c(n, k+1) = h(n \cdot c(n,k), k+1)$. □

Proposition 6.8. *$\exists\ d(n,k)$ such that for all polycyclic groups G of length at most k there is $C \leq G$ with $|G : C| \leq d(n,k)$ such that if $\rho : G \longrightarrow \mathrm{GL}(n,\mathbb{C})$, then $\rho(C)$ preserves a complete flag in $\mathbb{C}^n$.*

Proof. Below we show by induction on n that for a fixed ρ there is a subgroup of index at most $e(n,k) = \prod_{i=1}^n c(i,k)$ that preserves a complete flag. The result follows from (6.6) with $d(n,k) = h(e(n,k),k)$.

For $n = 1$ the result is clear. By (6.7) there is a subgroup $\Gamma' < \Gamma = \rho(G)$ of index at most $c(n,k)$ that preserves a one-dimensional subspace $E \subset V = \mathbb{F}^n$. Then Γ' acts on $V/E \cong \mathbb{F}^{n-1}$. By induction there is $\Gamma'' < \Gamma'$ with $|\Gamma' : \Gamma''| \leq e(n-1,k)$ that preserves a complete flag $\mathcal{F}$ in V/E. The preimage of $\mathcal{F}$ in V, together with E, forms a complete flag for V which is preserved by γ''. Moreover $|\Gamma : \Gamma''| = |\Gamma : \Gamma'| \cdot |\Gamma' : \Gamma''| \leq c(n,k)e(n-1,k) = e(n,k)$. □

A group $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$ is conjugate into $\mathrm{UT}(n)$ if and only if Γ preserves a complete flag and every weight of Γ is positive.

Corollary 6.9. *Suppose G is finitely generated and virtually nilpotent. Let $m > 0$. Then there is a finite index subgroup $H = \mathrm{core}(G,m) < G$ called the m–core of G such that for every homomorphism $\rho : G \to \mathrm{GL}(m,\mathbb{F})$:*

1. *If $\mathbb{F} = \mathbb{C}$, then $\rho(H)$ preserve a complete flag in $\mathbb{C}^m$.*
2. *If $\mathbb{F} = \mathbb{R}$ and every weight of $\rho(H)$ is real, then $\rho(H)$ is conjugate into $UT(m)$.*

Moreover the set $\rho \in \mathrm{Hom}(G, \mathrm{GL}(m,\mathbb{R}))$ for which $\rho(G)$ has a finite index subgroup that preserves a complete flag in $\mathbb{R}^m$ is a semi-algebraic set $\mathrm{VFG}(G, \mathrm{GL}(m,\mathbb{R}))$.

Proof. (1) follows from (6.8). (2) follows from (1) as follows. Set $U = \mathbb{R}^m$ and $V = U \otimes \mathbb{C}$ so $G \subset \mathrm{GL}(U) \subset \mathrm{GL}(V)$. By (1) $V = \oplus V(\lambda)$ where $V(\lambda) = \cap_{h \in H} \ker(\rho(h) - \lambda(h))^m$. Observe that $V(\lambda) \subset \mathbb{C}^m$ is given by linear equations that are defined over $\mathbb{R}$ because $\lambda(H) \subset \mathbb{R}$ and $\rho(H) \subset \mathrm{GL}(m,\mathbb{R})$. Thus $V(\lambda)$ is the complexification of $U(\lambda) = \cap_{h \in H} \ker(\rho(h) - \lambda(h))^m \subset \mathbb{R}^m$ so $U = \oplus U(\lambda)$. Hence $\rho(H)$ preserves a complete flag in $\mathbb{R}^m$. By replacing H by a subgroup of index 2^m we may ensure that all real weights are positive. Then $\rho(H)$ is conjugate into $\mathrm{UT}(m)$. The last assertion follows from (2) and the observation that every weight real is defined by the semi-algebraic equations that say every eigenvalue of every element of $\rho(H)$ is real. $\square$

Suppose U is a real vector space and $\Gamma < \mathrm{GL}(U)$ preserves a complete flag in $V = U \otimes \mathbb{C}$. Then combining each weight λ for V with the complex-conjugate weight $\overline{\lambda}$ gives a real invariant subspace $U(\lambda, \overline{\lambda}) = (V(\lambda) + V(\overline{\lambda})) \cap U \subset U$ and $U = \bigoplus U(\lambda, \overline{\lambda})$. We call $U(\lambda)$ a *conjugate generalized weights space*. For each $\gamma \in \Gamma$ the eigenvalues of $\gamma|_{U(\lambda,\overline{\lambda})}$ are $\lambda(\gamma)$ and $\overline{\lambda}(\gamma)$.

Proposition 6.10. *Suppose Ω/Γ is a quasi-cusp of dimension n. Then* $\mathrm{core}(\Gamma, n+1)$ *is conjugate into* $\mathrm{UT}(n+1)$*. In particular* $\Gamma \in \mathrm{VFG}$.

Proof. Write $V = \mathbb{R}^{n+1}$ so $\Gamma \subset \mathrm{PGL}(V)$. By (2.1) we may lift to get $\Gamma \subset \mathrm{GL}(V)$. By (6.9)(1) we can conjugate so that $H = H(\Gamma, n+1)$ is contained in the upper-triangular subgroup in $\mathrm{GL}(n+1, \mathbb{C})$. We replace Γ by H. Then $V = A \oplus B$ where A is the sum of the generalized weight spaces for real weights and $B = \oplus B_i$ is the sum of the remaining conjugate generalized weights spaces. It suffices to show $B = 0$, since then by (6.9)(2) Γ is conjugate into $\mathrm{UT}(n+1)$.

Each vector $x \in V$ is uniquely expressed as a linear combination $a + b_1 + \cdots + b_k$ with $a \in A$ and $b_i \in B_i$. Define $n(x)$ to be the number of distinct i with $b_i \neq 0$. Choose $x \neq 0$ with $[x] \in \Omega$ so that $n(x)$ is minimal.

Claim $n(x) = 0$.

Proof of the claim. If $n(x) \neq 0$, then some $b_j \neq 0$. There is $\gamma \in \Gamma$ which has eigenvalues $\lambda_j(\gamma), \overline{\lambda}_j(\gamma)$ that are not real. Let $\langle \gamma \rangle$ be the cyclic group generated by γ. Let $C \subset B_j$ be the convex hull of the orbit $\langle \gamma \rangle \cdot b_j$.

Suppose $0 \notin C$. Then $K = \mathrm{cl}(\mathbb{P}_+(C))$ is a closed convex cell in $\mathbb{P}_+(B_j)$ that is preserved by γ. By the Brouwer fixed point theorem, γ fixes a point $[v] \in K$, so $v \in B_j$ is an eigenvector of $\gamma|B_j$ with a positive eigenvalue. However every eigenvector for γ in B_j has eigenvalue $\lambda_j(\gamma)$ or $\overline{\lambda}_j(\gamma)$ which are both not real. This contradiction shows that $0 \in C$.

The convex cone $\mathcal{C}\Omega \subset V$ is preserved by Γ. Since $0 \in C$ there is a finite convex combination $\sum t_i \gamma^i b_j = 0$ with $t_i \geq 0$ and $\sum t_i = 1$. Since $x \in \mathcal{C}\Omega$ and this cone is Γ-invariant it follows that $\gamma^i x \in \mathcal{C}\Omega$. Since $\mathcal{C}\Omega$ is convex the convex combination $x' = \sum t_i \gamma^i x \in \mathcal{C}\Omega$. In particular $x' \neq 0$ and $[x'] \in \Omega$. The component of x' in B_j is $\sum t_i \gamma^i b_j = 0$. Since the conjugate weights spaces are Γ invariant, the property that a point has a zero component in some B_i is preserved by Γ, so $n(x') < n(x)$ contradicting minimality. Hence no such b_j exists this proves the claim. $\square$

Since $x \neq 0$ this implies $A \neq 0$ and $[x] \in W := \Omega \cap \mathbb{P}(A)$ is a nonempty properly convex set that is preserved by Γ. The manifold $P = \Omega/\Gamma$ is homeomorphic to $Q \times \mathbb{R}$ for some closed manifold Q. Then $M = W/\Gamma$ is a submanifold of P and $\pi_1 M \cong \Gamma$. Each B_i has real dimension at least 2 so $\dim A \leq \dim V - 2$ and thus $\dim M \leq n-2$. Passing to a double cover we may assume Q is orientable. Since $\Omega \cong \widetilde{Q} \times \mathbb{R}$ is follows that $\widetilde{Q}$ is contractible so Q is a $K(\Gamma, 1)$ and $H_{n-1}(\Gamma) \cong H_{n-1}(Q) \cong \mathbb{Z}$. The universal cover of M is W, which is contractible and so $\pi_1 M \cong \Gamma$. Hence $H_{n-1}(M) \cong H_{n-1}(\Gamma)$. However $\dim M \leq n-2$ so $H_{n-1}(M) = 0$ which is a contradiction. $\square$

A *virtual syndetic hull* of a discrete subgroup $\Gamma < H$ of a Lie group H is a connected Lie subgroup $G < H$ such that $|\Gamma : G \cap \Gamma| < \infty$ and $(G \cap \Gamma)\backslash G$ is compact. When syndetic hulls exist they are not always unique because the exponential map on $\mathfrak{gl}(n)$ is not injective for $n \geq 2$. It is useful to have a unique version of a syndetic hull.

Let $\mathfrak{r} \subset \mathfrak{gl}$ be the subset of all matrices M such that all the eigenvalues of M are real. The set $R = \exp(\mathfrak{r})$ consists of all matrices A such that every eigenvalue of A is positive. Then $\exp : \mathfrak{r} \longrightarrow R$ is a diffeomorphism with inverse log. An element of R is called an *e-matrix* and a group $G \subset R$ is called an *e-group.* For example $\mathrm{UT}(n)$ is an e-group. If G is a connected e-group, then $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism. If $S \subset R$ define $\langle \log S \rangle$ to be the vector subspace of $\mathfrak{gl}$ spanned by $\log S$.

Definition 6.11. *Given a discrete subgroup $\Gamma \subset \mathrm{GL}(n+1,\mathbb{R})$ a* virtual e-hull *for Γ is a connected Lie group G that is an e-group and $|\Gamma : G \cap \Gamma| < \infty$ and $(G \cap \Gamma)\backslash G$ is compact. There might not be such a group.*

Definition 6.12. *If $\Gamma \subset GL(n,\mathbb{R})$ and $\Gamma \in VFG$, then the* translation group *of Γ is $T(\Gamma) = \exp\langle \log(\mathrm{core}(\Gamma, n))\rangle$.*

Theorem 6.13. *Suppose Ω/Γ is a quasi-cusp with translation group $T = T(\Gamma)$. Then T is the unique virtual e-hull of Γ.*

Proof. By (6.10) $\mathrm{core}(\Gamma, n+1)$ is conjugate into $\mathrm{UT}(n+1)$ and is therefore an e-group. By (6.15) $\mathrm{core}(\Gamma, n+1)$ has an e-hull T that is conjugate into $\mathrm{UT}(n+1)$. Thus T is a virtual e-hull of Γ. Uniqueness of T follows from (6.16). It is clear that T is the group $T(\Gamma)$ in (6.12). □

Proposition 6.14 ((9.3) of [10])**.** *Suppose that Γ is a finitely generated, discrete nilpotent subgroup of $\mathrm{GL}(n,\mathbb{R})$. Then Γ contains a subgroup of finite index Γ_0, which has a syndetic hull $G \leq \mathrm{GL}(n,\mathbb{R})$ that is nilpotent, simply-connected and a subgroup of the Zariski closure of Γ_0.*

Lemma 6.15. *If $\Gamma \subset \mathrm{UT}(n)$ is nilpotent, then it has an e-hull $G \subset \mathrm{UT}(n)$.*

Proof. There is a finitely generated discrete subgroup $\Gamma' \subset \Gamma$ such that $\langle \log \Gamma \rangle = \langle \log \Gamma' \rangle$. By (6.14) there is a finite index subgroup $\Gamma_0 \subset \Gamma'$ which has a syndetic hull G. Since $\mathrm{UT}(n)$ is an algebraic subgroup it follows that the Zariski closure of Γ is in $\mathrm{UT}(n)$ so $G \subset \mathrm{UT}(n)$. Since $\Gamma_0 \subset G$ it follows that $\langle \log \Gamma_0 \rangle = \langle \log \Gamma' \rangle = \langle \log \Gamma \rangle \subset \mathfrak{g}$ thus $\Gamma \subset G$. Since $G \subset \mathrm{UT}(n)$ it is an e-group. Moroever $\Gamma\backslash G$ is a quotient of $\Gamma_0\backslash G$ and so is compact. Hence G is a syndetic hull of Γ. □

Lemma 6.16. *If G_0 and G_1 are virtual e-hulls of Γ, then $G_0 = G_1$.*

Proof. The group $H = G_0 \cap G_1$ is connected because if $h \in H$, then the one parameter group $\exp\langle \log h \rangle$ is contained in both G_0 and G_1. If $\gamma \in \Gamma \cap R$ (with R defined above), then $\gamma^n \in G_i$ for some $n > 0$. Thus $\log \gamma^n = n \log \gamma \in \log G_i$ so $\gamma \in G_i$. Thus $\Gamma \cap R = \Gamma \cap G_i \subset G_i$ so $\Gamma \cap R \subset H$. Also $(\Gamma \cap G_i)\backslash G_i$ is compact and H is a closed subset of G_i, so $(\Gamma \cap G_i)\backslash H$ is also compact. It follows that $H = G_i$ since otherwise $(\Gamma \cap G_i)\backslash G_i$ is not compact. □

The next thing to do is show that the orbit under $T(\Gamma)$ of a point $x \in \partial\Omega$ is a strictly convex hypersurface.

A *projective flow* Φ on $\mathbb{RP}^n$ is a continuous monomorphism $\Phi : \mathbb{R} \longrightarrow \mathrm{PGL}(n+1,\mathbb{R})$. There is an infinitesimal generator $A \in \mathfrak{gl}_{n+1}$ with $\Phi_t := \Phi(t) = \exp(tA)$. If $p \in \mathbb{RP}^n$ and $\Phi_t(p) = p$ for all t, then p is a *stationary point* of Φ. A *radial flow* is a projective flow that is stationary on a hyperplane $H \cong \mathbb{R}P^{n-1}$ and that is parameterized so that $\Phi_t(p) \to H$ as $t \to -\infty$ whenever p is not stationary. It follows that $\Phi_t = \exp(tA)$, where $A \in \mathfrak{gl}_{n+1}$ is a rank one matrix and H is the projectivization of $\ker A$. The projectivization of the image of A is a point $p \in \mathbb{R}P^n$, called the *center of the flow*, that is also fixed by Φ. Every orbit is contained in a line containing the center. This property characterizes radial flows.

A radial flow is *parabolic* if $p \in H$ and *hyperbolic* otherwise. Every radial flow is conjugate to one generated by an elementary matrix $E_{i,j}$. A parabolic flow is conjugate to $(I + t \cdot E_{1,n+1})$ and a hyperbolic flow is conjugate to the diagonal group $(\exp(t), 1, \cdots, 1)$. The *backward orbit* of $X \subset \mathbb{RP}^n$ is $\Phi_{(-\infty,0]}(X)$. A set $X \subset \mathbb{RP}^n$ is *backwards invariant* if X contains its backwards orbit, and it is *backwards vanishing* if $\cap_{t<0}\Phi_t(X) = \emptyset$.

A *displacing hyperplane* for a radial flow Φ is a hyperplane P such that P and $\Phi_t(P)$ are disjoint in $\mathbb{RP}^n \setminus H$ for all $t \neq 0$. A hyperplane P is displacing if and only if $P \neq H$ and P does not contain the center of Φ.

Proposition 6.17. *Suppose Ω/Γ is a quasi-cusp and $\Gamma \subset \mathrm{UT}(n+1)$. For each weight $\lambda : \Gamma \to \mathbb{R}^*$ with generalized weight space $V = V(\lambda)$ there is a radial flow $\Phi = \Phi^\lambda$ that is centralized by Γ, and Φ acts trivially on each generalized weight space other than V.*

If $\dim V \geq 2$, then Φ is parabolic, and if $\dim V = 1$, then Φ is hyperbolic. The center of Φ is the projectivization of some vector in the weightspace $\mathbb{P}(E(\lambda))$.

If the orbit of $x \in \mathbb{RP}^n$ under $T(\Gamma)$ is a strictly convex hypersurface, then the group $G(\Gamma) := \Gamma \oplus \Phi(\mathbb{R})$ generated by $T(\Gamma)$ and $\Phi(\mathbb{R})$ is a direct sum and $G(\Gamma) \cdot x \subset \mathbb{RP}^n$ is open.

Proof. We may assume Γ upper-triangular and block diagonal with one bock for each generalized weight space. We may assume V is the first block and set $m = \dim V$. As above, let $E_{i,j} \in \mathfrak{gl}(n)$ be the elementary matrix with 1 in row i and column j. Define $\Phi(t) = \exp(tE_{1,m})$. Then $\mathbb{R}^n = V \oplus W$ where W is the sum of the other generalized weight spaces and the action of Φ on W is trivial. If $m = 1$ then $\Phi(t) = \mathrm{diag}(\exp(t), 1, \cdots, 1)$ is a hyperbolic flow. If $m \geq 2$ then $\Phi(t)$ is a parabolic flow given by the unipotent subgroup with t in the top right corner of the block for V. The center is $\mathbb{P}(e_1)$ and the stationary hyperplane is $H = \mathbb{P}(\langle e_1, \cdots, e_{m-1}\rangle \oplus W)$. It is easy to check that Γ centralizes Φ.

Since $T(\Gamma) = \exp(\mathfrak{t})$ and $\Phi(\mathbb{R}) = \exp(\mathfrak{f})$ are e-groups, if they have a nontrivial intersection, then $\Phi(\mathbb{R}) \subset T(\Gamma)$. The orbits of Φ are lines. If $S = T(\Gamma) \cdot x$ is a strictly convex hypersurface, then it does not contain a line so $\Phi(\mathbb{R}) \cap \Gamma$ is trivial. Since $\Phi(\mathbb{R})$ and $T(\Gamma)$ commute they generate $G = \Phi(\mathbb{R}) \oplus T(\Gamma)$. It follows that $\Phi(\mathbb{R}) \cdot S \subset \mathbb{RP}^n$ is open. □

Definition 6.18. *A radial flow Φ_t is* compatible *with a properly convex manifold $M = \Omega/\Gamma$ if $\Phi(\mathbb{R})$ commutes with Γ and Ω is disjoint from the stationary hyperplane of Φ and Ω is backwards invariant and backwards vanishing.*

A radial flow end *is a properly convex manifold $M = \Omega/\Gamma$ with compact, strictly convex boundary and for which there is a compatible radial flow. A* radial flow cusp *is a radial flow end that is also a generalized cusp.*

The hypersurfaces $\widetilde{S}_t := \Phi_{-t}(\partial\Omega)$ are strictly convex and Γ-invariant and foliate Ω for $t \geq 0$. They are all disjoint from H. Their images under the projection $\pi : \Omega \to M$ give a product foliation of M by compact strictly convex hypersurfaces $S_t = \pi(\widetilde{S}_t)$. There is a transverse foliation of Ω by flowlines that limit on the center of Φ. These project to a transverse foliation of M by rays.

The *flow time function* is $\widetilde{T} : \Omega \to [0, \infty)$ defined by $\widetilde{T}(x) = t$ if $\Phi_t(x) \in \partial\Omega$. Thus $\widetilde{T}(x)$ is the amount of time for x to flow into $\partial\Omega$ and $\widetilde{T}(\tilde{S}_t) = t$. Let $\pi : \Omega \to M$ be the projection. Then there is a map $T : M \to [0, \infty)$ defined by $T(\pi x) = \widetilde{T}(x)$. The level sets of T are the hypersurfaces S_t.

Lemma 6.19. *Suppose Φ is a radial flow with center p and stationary hyperplane H. Suppose $\Omega \subset \mathbb{RP}^n \setminus H$ is a properly convex manifold. If Φ is hyperbolic and $p \notin \mathrm{cl}(\Omega)$, then Ω is backwards vanishing. If Φ is parabolic, then Ω is backward vanishing for either $\Phi(t)$ of $\Phi'(t) := \Phi(-t)$.*

Proof. If Φ is hyperbolic and $p \notin \mathrm{cl}(\Omega)$, then by the Hahn-Banach separation theorem there is P that separates Ω from p. If Φ is parabolic, then choose any hyperplane P disjoint from Ω that does not contain P. In either case P is a displacing hyperplane. After possibly reversing Φ in the parabolic case, the half space $U \subset \mathbb{R}^n = \mathbb{RP}^n \setminus H$ that contains Ω is backwards vanishing, and hence so is Ω. □

Proposition 6.20. *Every minimal generalized cusp $C = \Omega/\Gamma$ with $\Gamma \subset \mathrm{UT}(n+1)$ is a radial flow cusp.*

Proof. **Claim 1.** Ω is disjoint from every Γ-invariant hyperplane P

Proof of claim 1. If $P \cap \Omega \neq \emptyset$, then $P \cap \partial\Omega \neq \emptyset$ since C is minimal. Observe that $P \cap \Omega$ is properly convex and preserved by Γ. Thus $R = (P \cap \Omega)/\Gamma$ is a codimension-1 submanifold of C. Moreover ∂R

is a non-empty codimension-1 closed submanifold of ∂C. But ∂R and ∂C are homotopy equivalent, a contradiction. This proves the first claim. $\square$

There are now two cases:

Parabolic case. There is a generalized weight space W for Γ with $\dim W \geq 2$. Let Φ be the parabolic radial flow that centralizes Γ given by (6.17). Let H be the stationary hyperplane and $p \in H$ the center of Φ. Let P be a displacing hyperplane that is tangent to Ω at $q \in \partial\Omega$.

Hyperbolic case. Every generalized weight space has dimension 1, so Γ is diagonalizable. The weight spaces projectivize to give points $p_0, \cdots, p_n \in \mathbb{RP}^n$ that are in general position. The hyperplane P_i contains all these points except p_i. These hyperplanes divide $\mathbb{RP}^n$ into 2^{n-1} open n-simplices. These hyperplanes are Γ-invariant so Ω is contained in one of these simplices Δ. There is a vertex p of Δ with $p \notin \overline{\Omega}$ because $\partial\Omega$ is a strictly convex hypersurface in Δ. After relabelling $p = p_0$, let $H = P_0$ and let Φ be the radial flow with center p and stationary hyperplane H. Then Φ centralizes Γ and p is disjoint from $\mathrm{cl}(\Omega)$. By (6.19) there is a displacing hyperplane P that separates p from Ω.

In each case, Ω is disjoint from H by claim 1. Set $\mathbb{R}^n = \mathbb{RP}^n \setminus H$ so $\Omega \subset \mathbb{R}^n$. Let U be the closure of the component of $\mathbb{R}^n \setminus P$ that contains Ω. Choose linear coordinates on $\mathbb{R}^n$ such that $q = e_1 = (1, 0, \cdots, 0)$, U is $x_1 \geq 1$, and, moreover, $p = 0$ in the hyperbolic case and p is the limit of the positive x_1-axis in the parabolic case. Then $P = \partial U$ is the horizontal hyperplane $x_1 = 1$.

We may assume U is backward invariant after possibly reversing the flow in the parabolic case. We reparameterize Φ so that in these coordinates in the parabolic case $\Phi_t(x) = x - t \cdot e_1$ and in the hyperbolic case $\Phi_t(x) = \exp(-t) \cdot x$.

Let $p_1 : U \to P$ be the projection along flowlines. Thus in the parabolic case $p_1(x_1, \cdots, x_n) = (1, x_2, \cdots, x_n)$ and in the hyperbolic case $p_1(x_1, \cdots, x_n) = (1, x_2/x_1, \cdots, x_n/x_1)$.

Claim 2. Let Ω_1 be the backward orbit of $\operatorname{int}\Omega$. Then Ω_1 is open and properly convex.

Proof of claim 2. First we show that Ω_1 is convex. Suppose $a, b \in \Omega_1$. Then $a = \Phi_\alpha(a')$ and $b = \Phi_\beta(b')$ for some $\alpha, \beta \leq 0$ and $a', b' \in \operatorname{int}\Omega$. Let $\ell = [a', b']$ be the line segment with endpoints a', b'. Since Ω is convex $\ell \subset \Omega$. Then $\cup_{t \leq 0} \Phi_t(\ell)$ is a planar convex set in Ω_1 that contains a and b. Hence Ω_1 is convex.

Let C be the cone of Ω from 0. Since Ω is properly convex and $p \notin \overline{\Omega}$ it follows that C is properly convex. Moreover C is backward invariant and so contains Ω_1. Thus Ω_1 is properly convex proving claim 2. $\square$

Observe that Ω_1 is backward invariant. Define Ω_M to be the *flow closure* of Ω_1, i.e. the set of all points x such that $\Phi_t(x) \in \Omega_1$ for all $t < 0$. There is a homeomorphism $\widetilde{F} : \partial\Omega_M \times [0, \infty) \to \Omega_M$ given by $F(x, t) = \Phi_{-t}(x)$.

Claim 3. Γ acts freely and properly discontinuously on Ω_M.

Proof of claim 3. Since Γ commutes with Φ it follows that Ω_1 is preserved by Γ. Since Γ acts freely on Ω it contains no elliptics and therefore acts freely on Ω_1. Moreover Γ is discrete and therefore acts properly discontinuously on Ω_1. The map Φ_{-1} embeds Ω_M into Ω_1 and since Φ_{-1} commutes with Γ it follows that Γ acts freely and properly discontinuously on Ω_M. $\square$

Thus $M = \Omega_M/\Gamma$ is a properly convex manifold and there is a homeomorphism $F : \partial M \times [0, \infty) \to M$ covered by $\widetilde{F}$. The proof is not completed with the following:

Claim 4. $\Omega_M = \Omega$, and hence $M = C$.

By (8.3) there is a collar neighborhood $P \subset C$ of ∂C with $\partial P = \partial C \sqcup Q$ and Q is strictly convex. Let $R = \text{cl}(C \setminus P)$ so $\partial R = Q$ and R is a generalized cusp. Since C is a closed manifold and a $K(\Gamma, 1)$, it is easy to see that the inclusions $Q \subset R \subset M$ are homotopy equivalences. Thus there is a compact submanifold $X \subset M$ with $\partial X = \partial M \sqcup Q$ and $X \simeq \partial M \times [0, 1]$. In particular Q separates ∂M from the end of M. Since $X \subset M$ is compact and $P \setminus \partial C \subset X$ it follows that $\partial C \subset M$. Thus $\Omega \subset \Omega_M$.

For $t \geq 0$ there is a map $\phi_t : M \to M$ covered by Φ_{-t}. Let T be the supremum of $t > 0$ such that $\phi_t(\partial C) \subset \text{CH}(\partial C)$. Then $T > 0$ because ∂C is compact and strictly convex. It is easy to see that if $s, t < T$ then $s + t < T$ so $T = \infty$. Thus Ω is backward invariant, and so $\Omega_M \subset \Omega$. □

Lemma 6.21. *Suppose $\Omega \subset \mathbb{RP}^n$ and $M = \Omega/\Gamma$ is a radial flow end with radial flow Φ. Let $H \subset \mathbb{RP}^n$ be the stationary hyperplane for Φ. Then $\partial\overline{\Omega} = \partial\Omega \sqcup (H \cap \overline{\Omega})$. In particular, if M is a radial flow cusp, then it is minimal.*

Proof. Let $\mathbb{R}^n = \mathbb{RP}^n \setminus H$, so that $\Omega \subset \mathbb{R}^n$. Let p be the center of Φ. Choose a displacing hyperplane $P \subset \mathbb{RP}^n$ that is disjoint from Ω such that if Φ is hyperbolic P separates p from Ω in $\mathbb{R}^n$.

Let U be the closure of the component of $\mathbb{R}^n \setminus P$ that is the halfspace containing Ω. Then U is backward invariant. Thus U is the backward orbit of P. Define the function $\tau : U \to [0, \infty)$ by $\tau(x) = t$ if $\Phi_t(x) \in P$. This is the amount of time it takes x to flow into P. Observe that if $x, y \in \Omega$, then $\widetilde{T}(x) - \widetilde{T}(y) = \tau(x) - \tau(y)$.

Because each $\widetilde{S}_t$ is strictly convex it follow that the only critical points of the restriction of $\widetilde{T}$ to a segment are maxima, and therefore there is at most one critical point on a segment. Thus $T : M \to [0, \infty)$ has the same critical point behaviour along segments.

Choose a metrically-complete Riemannian metric on M and use the lifted metric on Ω. Suppose $\partial\Omega$ is not properly embedded in $\mathbb{R}^n$. Then there is a sequence $\widetilde{p}_k \in \partial\Omega$ which converges in $\mathbb{R}^n$ to a point $\widetilde{p}_\infty \notin \partial\Omega$.

Let α_k be the length of $[\widetilde{p}_0, \widetilde{p}_k] \subset \Omega$. Then $\alpha_k \to \infty$ because $\widetilde{p}_\infty \notin \Omega$ and the metric on Ω is complete. Let $\widetilde{\ell}_k : [0, 1] \to [\widetilde{p}_0, \widetilde{p}_k]$ be the unit segment. Hence $\widetilde{\ell}_k$ converges to $\widetilde{\ell}_\infty : [0, 1] \to [\widetilde{p}_0, \widetilde{p}_\infty]$. The restriction of $\widetilde{\ell}_\infty$ to $[0, 1)$ is a ray, $\widetilde{\ell} : [0, 1) \to \Omega$, of infinite length in Ω. Since $\widetilde{p}_k \to \widetilde{p}_\infty$ there is $\beta > 0$ such that $\tau \circ \widetilde{\ell}_k \leq \beta$ for all $k \in [0, \infty]$. Since $\widetilde{T}(x) = \tau(x) - \tau(\tilde{p}_0) + \widetilde{T}(\tilde{p}_0)$, replacing β by $\beta - \tau(\tilde{p}_0) + \widetilde{T}(\tilde{p}_0)$ gives $\widetilde{T} \circ \widetilde{\ell}_k \leq \beta$ everywhere.

The projection $\ell_k = \pi \circ \widetilde{\ell}_k : [0, 1] \to M$ is an immersed affine segment and $T \circ \ell_k \leq \beta$. Thus ℓ_k is contained in the compact set $M_\beta := \cup_{0 \leq t \leq \beta} S_t$. These segments converge to the ray $\ell = \pi \circ \widetilde{\ell}$ of infinite length that is also contained M_β. Now $T \circ \ell_\infty : [0, 1) \to [0, \beta]$ is eventually monotonic and so taking limits of subsegments of length 1 there is a segment of length 1 along which T is some constant α. This segment is contained in S_α. But this contradicts the fact that S_α is strictly convex. It follows that $\partial\Omega$ is properly embedded in $\mathbb{R}^n$. Hence Ω is a closed convex set in $\mathbb{R}^n$ □

Proposition 6.22. *Suppose $C = \Omega/\Gamma$ is a minimal generalized cusp and $\Gamma \subset \text{UT}(n + 1)$. Let $T = T(\Gamma)$ be the translation group. Then C contains a minimal homogeneous cusp $C_T = \Omega_T/\Gamma$ and T acts transitively on $\partial\Omega_T$.*

Proof. By (6.20) C is a radial flow cusp for some flow Φ. Let H be the stationary hyperplane of Φ and set $\mathbb{R}^n = \mathbb{RP}^n \setminus H$. Since T centralizes Φ it preserves H and acts affinely on $\mathbb{R}^n$. By (6.21) $\Omega \subset \mathbb{R}^n$ is a closed subset bounded by the properly embedded strictly convex hypersurface $\partial\Omega$.

Claim. There is $x \in \Omega$ such that $T \cdot x \subset \Omega$.

Proof of claim. Let $\pi : \Omega \longrightarrow C$ be the covering space projection. There is a continuous map $F : T \times \partial\Omega \to \mathbb{R}^n/\Gamma$ given by $F(t, x) = \pi(t \cdot x)$. Since $C = \Omega/\Gamma$ is compact there is a compact subset $D \subset \partial\Omega$ such that $\Gamma \cdot D = \partial\Omega$. So $T \cdot \partial\Omega = T \cdot D$ because $\Gamma \subset T$. There is compact $X \subset T$ such that $\Gamma X = T$. So $T \cdot D = (\Gamma X) \cdot D$. Hence $\text{Im}(F) = \pi(X \cdot D)$ because $\pi(\Gamma \cdot x) = \pi(x)$. Thus

$\mathrm{Im}(F) \subset \mathbb{R}^n/\Gamma$ is compact because it equals $F(X \times D)$ and $X \times D$ is compact. Thus $K = \mathrm{Im}(F) \cap C$ is compact. Choose $x \in \Omega$ such that $\pi(x) \notin K$, then $T \cdot x \subset \Omega$. This proves the claim. $\square$

The set $\Omega_0 = \mathrm{cl}(\mathrm{CH}(T \cdot x)) \subset \Omega$ is properly convex and T–invariant. Since Ω_0 is a closed properly convex set in $\mathbb{R}^n$ there is an extreme point $y \in \partial\Omega_0$ at which $\partial\Omega_0$ is strictly convex. Thus $\partial\Omega_0$ is strictly convex at every point in the orbit $S = T \cdot y \subset \partial\Omega_0$. Then $\Omega_T = \mathrm{CH}(S) \subset \Omega_0$ is properly convex. Since $\Gamma \subset T$ it follows that Ω_T is Γ-invariant. Thus $C_T = \Omega_T/\Gamma$ is a generalized cusp and a submanifold of C. Both C_T and C are contractible with the same fundamental group. Since $C \cong \partial C \times [0,1)$ with C closed it follows that $C_T \cong \partial C_T \times [0,1)$ with C_T closed and $\dim C_T = \dim C$. Hence $\dim \Omega_T = \dim \Omega$. Moreover $S \subset \partial\Omega_0$ so $S \subset \partial\Omega_T$. Since $\Omega_T = CH(S)$ it follows that $S = \partial\Omega_T$ and $\Omega_T = \Omega_0$. $\square$

Lemma 6.23. *Suppose $C = \Omega/\Gamma$ is a minimal generalized cusp and $T = T(\Gamma) \subset \mathrm{UT}(n+1)$ and $\Gamma_0 = T \cap \Gamma$. Suppose Ω/Γ_0 contains a homogeneous cusp Ω_T/Γ_0 and Ω_T is preserved by T. Then Γ preserves Ω_T so C contains the homogeneous generalized cusp Ω_T/Γ.*

Proof. By (6.20) $C^* = \Omega_T/\Gamma_0$ is a radial flow cusp and by (6.21) $\Omega_T \subset \mathbb{R}^n$ is bounded by the strictly convex properly embedded hypersurface $\partial\Omega_T$. By (6.13) $T = T(\Gamma)$ is the unique translation group that contains Γ.

Since Γ normalizes itself it follows that Γ normalizes T and therefore Γ permutes the decomposition of $\mathbb{RP}^n$ into T–orbits. The domain Ω_T is foliated by T–orbits and $\Omega_T/T \cong [0,1)$. Since $\Gamma \cap T$ preserves Ω_T and $|\Gamma : \Gamma \cap T| < \infty$ it follows the Γ–orbit of Ω_T is a finite number of pairwise disjoint convex sets all contained in Ω. Thus $\Gamma \cap T$ permutes these domains. There is a finite index subgroup $\Gamma_1 \subset \Gamma \cap T$ that preserves each domain. We may assume Γ_1 is normal in Γ. Thus $M = \Omega/\Gamma_1$ is a regular cover of C that contains one copy of $P = \Omega_T/\Gamma_1$ for each domain. However M and each copy of P is a generalized cusp. Each copy of ∂P separates ∂M from the end of M. Since the copies of P are disjoint there is only one copy of P and Γ preserves Ω_T. $\square$

Lemma 6.24. *Suppose G is a connected group with $\dim G = n - 1$. For $x \in \mathbb{RP}^n$ the subset of $\mathrm{Hom}(G, \mathrm{GL}(n+1, \mathbb{R}))$ consisting of all ρ with $\rho(G) \cdot x$ a strictly convex hypersurface is open.*

Proof. Suppose the map $f : G \longrightarrow \mathbb{RP}^n$ given by $f(g) = (\rho(g)) \cdot x$ has image a strictly convex hypersurface S. Because G acts transitively on S by projective maps it follows that S is strictly convex everywhere if and only if it is strictly convex at the single point x. Strict convexity of S at x is equivalent to the quadratic form $Q = \nu \cdot D_e^2 f$ being positive or negative definite where ν is a normal to S at x and $e \in G$ is the identity. This form $Q = Q(\rho)$ is a smooth function of ρ. The set of definite quadratic forms is open in the set of all quadratic forms. $\square$

Theorem 6.25 (stability of generalized cusps)**.** *Suppose M is a generalized cusp of dimension n. Then*

$$\mathrm{VFG}(M) := \{\rho : \pi_1 M \to \mathrm{GL}(n+1, \mathbb{R}) \mid \rho(\pi_1 M) \in \mathrm{VFG}\ \}$$

is semi-algebraic. Let $\mathcal{U}$ be the set of the holonomies of properly convex structures on M with ∂M strictly convex. Then $\mathcal{U}$ is open in $\mathrm{VFG}(M)$ *and consists of the holonomies of homogenous generalized cusp structures on M.*

Proof. By (6.9) $\mathrm{VFG}(M)$ is semi-algebraic. By (6.10) $\mathcal{U} \subset \mathrm{VFG}(M)$ and by (6.5) every element of $\mathcal{U}$ is the holonomy of a homogeneous generalized cusp. It remains to show $\mathcal{U}$ is open in $\mathrm{VFG}(M)$.

Set $H = \mathrm{core}(\pi_1 M, n+1)$. By (6.12) for $\rho \in \mathrm{VFG}(M)$ the translation group is

$$T(\rho) := T(\rho(\pi_1 M)) = \exp\langle \log(\rho H) \rangle.$$

This is clearly a continuous function of $\rho \in \mathrm{VFG}(M)$.

If $\rho \in \mathcal{U}$ there is a properly convex domain Ω preserved by $T(\rho)$. Choose $x \in \partial\Omega$. By (6.24) for $\sigma \in \mathrm{VFG}(M)$ close enough to ρ the hypersurface $S = T(\sigma) \cdot x$ is strictly convex. By (6.17) there is a radial flow Φ that is centralized by $T(\sigma)$ and the group $G = T(\sigma) \oplus \Phi(\mathbb{R})$ has an open orbit W in $\mathbb{RP}^n$. Moreover W is foliated by the strictly convex hypersurfaces $S_t = \Phi_t(S)$.

After replacing t by $-t$ we may assume for $t < 0$ and close to 0 that S_t is on the convex side of $S = S_0$. Let $\Omega^+ = \cup_{t\leq 0} S_t$. Then $\partial\Omega^+ = S_0$. This set is preserved by $T(\sigma)$ and therefore by $\sigma(H)$. It is contained in a properly convex cone by the argument of (3.4) using Figure 1. Hence $\Omega(\sigma) := \mathrm{CH}(\Omega^+)$ is properly convex and $T(\sigma)$–invariant. The argument of claim 3 in (6.20) shows that $\sigma(H)$ acts freely and properly discontinuously on $\Omega(\sigma)$. Since it is $T(\sigma)$–invariant, $\partial\Omega(\sigma)$ is strictly convex. Thus $\Omega(\sigma)/\sigma(H)$ is a homogeneous generalized cusp.

It only remains to show that $\Omega(\sigma)$ is preserved by all of $\sigma(\pi_1 M)$. The argument is very similar to the proof of (6.23). The $\sigma(\pi_1 M)$–orbit of $\Omega(\sigma)$ is finite because $|\pi_1 M : H| < \infty$. By (6.16) $T(\sigma)$ is the unique virtual e-hull of $\sigma(\pi_1 M)$. Thus $\sigma(\pi_1 M)$ preserves the decomposition of $\mathbb{RP}^n$ into $T(\sigma)$–orbits. Moreover $\Omega(\sigma)$ is a union of such orbits. Thus if $g \in \pi_1 M$ then $(\sigma g)(\Omega(\sigma))$ is either $\Omega(\sigma)$ or disjoint from $\Omega(\sigma)$. We need only look at finitely many such g. Observe that $\Omega(\sigma)$ is close to $\Omega(\rho)$ and $\rho(g)$ is close to $\sigma(g)$ and $\rho(\pi_1 M)$ preserves $\Omega(\rho)$. Thus $\rho(g)$ preserves $\Omega(\rho)$ so $(\sigma g)(\Omega(\sigma))$ intersects $\Omega(\sigma)$. It follows that $(\sigma g)(\Omega(\sigma)) = \Omega(\sigma)$. □

Lemma 6.26. *Every homogeneous cusp has an exhaustion function.*

Proof. Suppose $M = \Omega/\Gamma$ is a homogeneous cusp and $T = T(\Gamma)$ is the translation subgroup. Then Ω has a codimension–1 foliation by T–orbits that covers a smooth foliation of M. Pick y in the interior of Ω and define $F : \Omega \to \mathbb{R}$ by $F(x) = d_\Omega(x, T\cdot y)$ if $T \cdot y$ separates $\partial\Omega$ from x and 0 otherwise. This covers a map $f : M \to \mathbb{R}$ and $C = f^{-1}(0)$ is a compact collar neighborhood of ∂M and $f(x) = d(y, C)$ and $f^{-1}(t)$ is a leaf of the foliation of M for $t > 0$.

It is clear that $\|df\| \leq 1$ when $f > 0$. Thus it suffices to show that $\|D^2 f\|$ is bounded. Suppose there is a sequence (M_k, f_k, x_k) such that $\|D^2 f_k\|_{x_k} > k$. Then $M_k = \Omega_k/\Gamma_k$ and Γ_k is a lattice in $G_k = \mathrm{PGL}(\Omega_k)$. We may assume all the Ω_k are in Benzecri position and 0 covers x_k. We may also assume $\Omega_k \to \Omega$ in the Hausdorff topology. Then $G_k \to G \subset \mathrm{PGL}(\Omega)$. The T orbits are a smooth foliation of Ω and we define a smooth function $F : \Omega \longrightarrow \mathbb{R}$ using $y = 0$ as above. Then F_k converges to F in C^∞ on compact sets. But $\|D^2 f_k\| = \|D^2 F_k\| \to \infty$ contradicts $\|D^2 F\| < \infty$ because F is smooth. □

Theorem 6.27 (Main Theorem)**.** *Suppose N is a compact connected n–manifold and $\mathcal{V}$ is the union of some of the boundary components $V_1, \cdots, V_k \subset \partial N$. Let $M = N \setminus \mathcal{V}$. Assume $\pi_1 V_i$ is virtually nilpotent for each i. The natural map $\mathcal{D}ev_{ce}(M) \to \{\rho \in \mathrm{Rep}(\pi_1 N) : \forall i\ \rho(\pi_1 V_i) \in \mathrm{VFG}\}$ is open.*

Proof. Let B_i be the end of M corresponding to V_i. By (5.7) the map

$$\mathcal{E}_{\mathcal{K}} : \mathcal{D}ev_{ce}(M, \mathbb{P}) \to \mathcal{R}el\mathcal{H}ol(M, \mathcal{K})$$

is open. By (6.25) for each B_i the map $\mathcal{K}(B_i) \to \mathrm{VFG}(B_i)$ is open. Composing these gives the openness of

$$\mathcal{D}ev_{ce}(M, \mathbb{P}) \to \{\rho \in \mathrm{Rep}(\pi_1 N) : \forall i\ \rho(\pi_1 V_i) \in \mathrm{VFG}\}. \qquad \square$$

By definition $\mathcal{H}ol : \mathcal{D}ev_{ce} \to \mathrm{Rep}_{\mathrm{ce}}(M)$ is onto. It is clearly continuous. So (6.27) implies (0.1).

7. Three dimensional generalized cusps

An orientable three-dimensional generalized cusp is diffeomorphic to $T^2 \times [0, \infty)$. Leitner [22] shows in this dimension that every generalized cusp Lie group is conjugate to a unique group of the form $C_n(\alpha, \beta)$ with $\beta \geq \alpha > 0$, where n is the number of non-trivial weights:

$$\begin{array}{cccc} C_0 & C_1 & C_2(\alpha) & C_3(\alpha,\beta) \\ \begin{pmatrix} 1 & s & t & \frac{s^2+t^2}{2} \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & 1 & t & \frac{1}{2}t^2 - s \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & 1 & -t-\alpha s \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} e^s & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 \\ 0 & 0 & e^{-\alpha s-\beta t} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

There is a *compact*, properly convex domain $\Omega_n = \Omega_n(\alpha,\beta)$ preserved by $C_n = C_n(\alpha,\beta)$ and $\partial\Omega_n = A \sqcup B$ where $A = C_n \cdot x$ is an orbit and B is a simplex contained in a projective hyperplane. If $\Gamma_n \subset C_n$ is a lattice then $\overline{M} = \Omega_n/\Gamma_n$ is a compactification of the generalized cusp $M = (\Omega_n \setminus B)/\Gamma_n$ obtained by adding $\partial_\infty \overline{M} = B/\Gamma$ which is a point for C_0, a circle for C_1, and a torus for C_2 or C_3. The group C_n is a *translation group* and is contained in the cusp Lie group $\mathrm{PGL}(\Omega_n)$.

The group C_0 is conjugate into $PO(3,1)$ and contains the holonomy of a cusp of a hyperbolic 3-manifold. Ballas [1] found a lattice in C_1 that is the holonomy of a generalized cusp for a properly convex structure on the figure eight knot complement. The groups $C_3(\alpha,\beta)$ are diagonal affine groups that satisfy the *uniform middle eigenvalue condition* of Choi [6]. Gye-Seon Lee found lattices in some of these groups that are holonomies of generalized cusps for a properly convex structure on the figure eight knot complement. At the time of writing it is not known if there is 3-manifold that admits a finite volume hyperbolic structure and also a properly convex structure that is a lattice in some $C_2(\alpha)$. The classification of generalized cusps in all dimensions is given in [2].

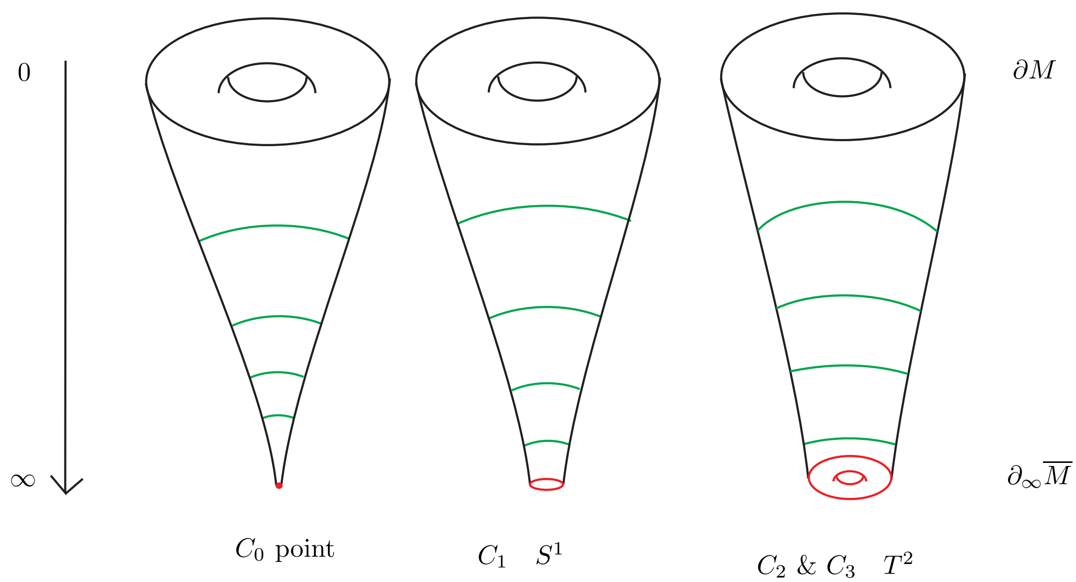

FIGURE 2. Generalized cusps in dimension 3

8. CONVEX SMOOTHING

A function $f : (a,b) \to \mathbb{R}$ is *Hessian-convex* if it is smooth and $f'' > 0$ everywhere. A smooth function on an affine manifold is *Hessian-convex* if its restriction to each line segment is. For affine manifolds, we show how to approximate a convex function *which is strictly convex somewhere* by a smooth, Hessian-convex one.

The main application is that given a projective manifold which has a convex boundary that is strictly convex at some point, we can shrink the manifold slightly to produce a submanifold with Hessian-convex boundary: locally the graph of a Hessian-convex function. One might imagine using sandpaper to smooth the boundary and produce a submanifold with smooth strictly convex boundary.

The idea is to improve a convex function which is already Hessian-convex on some open subset, by changing it in a small convex set C, and leaving it unchanged outside C. This is done so that it is Hessian-convex inside a slightly smaller convex set $C^- \subset C$, and also Hessian-convex at any point, where it was previously Hessian-convex. In this way the problem is reduced to a local one in Euclidean space.

Greene and Wu [16, Theorem 2], see also [15], showed that on a Riemannian manifold, any function f with the property that locally there is a function g with positive definite Hessian such that $f - g$ is convex along geodesics (**they** call f *strictly convex*) can be uniformly approximated by smooth, Hessian-convex functions. Smith [27] gives an example, for each $k \geq 0$, of a C^k convex function on a non-compact Euclidean surface which is not approximated by a C^{k+1} convex function.

A function f is *convex down* if $-f$ is convex. This means secant lines lie *below* the graph: $tf(a) + (1-t)f(b) \leq f(ta + (1-t)b)$ for all a, b and $0 \leq t \leq 1$. Equivalently the set of points *below* the graph of f is convex.

If f, g are smooth convex down functions, then $\min(f, g)$ is convex down, but need not be smooth at points where $f = g$. We construct a smooth approximation m^κ on $\mathbb{R}^2_+$ which agrees with min outside a certain neighborhood of the diagonal and has good convexity properties.

Lemma 8.1 (smoothing min)**.** *Given $\kappa \in (0, 1)$ there is a smooth function $m^\kappa : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$, which is convex-down and non-decreasing in each variable such that if $x \leq \kappa y$ or $y \leq \kappa x$, then $m^\kappa(x, y) = \min(x, y)$. It follows that if $f, g : C \longrightarrow \mathbb{R}_+$ are convex down, then so is $h(x) = m^\kappa(f(x), g(x))$.*

Proof. On $\mathbb{R}^2_+$

$$\min(x, y) = (x + y) \cdot k(x/(x + y)), \qquad \text{where } k(t) = \min(t, 1 - t).$$

Choose δ so that $\kappa = \delta/(1 - \delta)$. Then $\delta \in (0, 1/2)$. Let $K : [0, 1] \longrightarrow [0, 1]$ be a convex-down smooth function that agrees with k outside $(\delta, 1 - \delta)$. Define $m : \mathbb{R}^2_+ \longrightarrow \mathbb{R}$ by

$$m(x, y) = (x + y) \cdot K(x/(x + y)).$$

If $x/(x + y) \leq \delta$, then $m(x, y) = x$. This happens when $x \leq \kappa y$. Similarly $m(x, y) = y$ when $y \leq \kappa x$, thus

$$m(x, y) = \min(x, y) \quad \text{if} \quad x \leq \kappa y \text{ or } y \leq \kappa x.$$

The subset of $\mathbb{R}^2_+$ where neither $x \leq \kappa y$ nor $y \leq \kappa x$ is called the *transition region*. Outside the transition region $m = \min$.

The graph of m is a convex-down surface above $\mathbb{R}^2_+$ that is a union of rays starting at the origin. One can picture the graph of m: it is the cone from the origin of the convex-down arch that is the part of the graph lying above $x + y = 1$. This arch is given by $K(x)$. Since $K(x)$ is convex down the graph of m is convex-down; though in the radial direction it is, of course, linear.

This surface is comprised of three parts. The central part is curved down. The other two parts are sectors of flat planes, one containing the x–axis and the other containing the y–axis.

We claim $m(x, y)$ is a non-decreasing function of each variable. The is clear on the two parts of the graph of m that are flat, since they are planes containing either the x-axis or the y-axis. Now

$$m_x = \frac{\partial m}{\partial x} = K(x/(x + y)) + (x + y) \cdot y(x + y)^{-2} K'(x/(x + y)).$$

Since $m_x(tx, ty) = m_x(x, y)$ we may assume $x + y = 1$. Then

$$\frac{\partial m}{\partial x} = K(x) + (1 - x)K'(x).$$

Recalling the defining properties of K and looking at its graph one sees this is non-negative. Similar calculations work for m_y. This proves the claim.

We deduce that h is convex-down using these two properties of m, where

$$h(ta + (1-t)b) = m(f(ta + (1-t)b), g(ta + (1-t)b)).$$

Since m is non decreasing in the first variable and f is convex-down

$$m(f(ta + (1-t)b), g(ta + (1-t)b)) \geq m(tf(a) + (1-t)f(b), g(ta + (1-t)b))$$

Similarly m is non decreasing in the second variable and g is convex-down

$$\begin{aligned} m(tf(a) + (1-t)f(b), g(ta + (1-t)b)) &\geq m(tf(a) + (1-t)f(b), tg(a) + (1-t)g(b)) \\ &= m(t(f(a), g(a)) + (1-t)(f(b), g(b))). \end{aligned}$$

Finally since m is convex down

$$\begin{aligned} m(t(f(a), g(a)) + (1-t)(f(b), g(b))) &\geq t \cdot m(f(a), g(a)) + (1-t)m(f(b), g(b)) \\ &= t \cdot h(a) + (1-t)h(b). \end{aligned}$$

□

Corollary 8.2 (relative convex smoothing)**.** *Suppose $C \subset \mathbb{R}^n$ is a compact convex set with nonempty interior and C^- is a compact convex set in the interior of C. Suppose $f : C \longrightarrow \mathbb{R}$ is a non-constant, convex function, which is Hessian-convex on a (possibly empty) subset $S \subset C$. Assume $f|\partial C = 0$. Then there is a convex function $F : C \longrightarrow \mathbb{R}$ such that F is Hessian-convex on $S \cup C^-$ and $f = F$ on some neighborhood of ∂C.*

Proof. Observe $f < 0$ on the interior of C. Let g be a Hessian-convex function on $\mathbb{R}^n$ which is negative everywhere on C and $g \geq f/2$ everywhere on C^-. Since f is not identically zero this can be done with, for example, $g(x) = \alpha\|x\|^2 + \beta$ with suitable constants.

For $\kappa \in (0, 1/2)$ define $F(x) = -m^\kappa(-f(x), -g(x))$ and observe that $F(x) = \max(f(x), g(x))$ except when $f(x)$ is close enough to $g(x)$, depending on κ. Since $g < f = 0$ on ∂C it follows that $F = f$ on some neighborhood of ∂C. Moreover $F = g$ on C^- and therefore F is Hessian-convex on C^-.

By (8.1) F is convex. Since $m = m^\kappa$ and g are smooth and the composition of smooth functions is smooth, it follows F is smooth on S. It only remains to show D^2F is positive definite on $S \cup C^-$. It suffices to show for every $a \in S \cup C^-$ and every unit vector $u \in \mathbb{R}^n$ the function $p(t) = -F(a + t \cdot u)$ satisfies $p''(0) < 0$. Computing

$$p' = -m_x f_u - m_y g_u$$

where

$$m_x = \frac{\partial m}{\partial x}, \ m_y = \frac{\partial m}{\partial y}$$

and f_u, g_u are the derivatives in direction u at $a \in C$,

$$f_u = df(u), \quad g_u = dg(u).$$

Then

$$p'' = [m_{xx}(f_u)^2 + 2m_{xy}f_u g_u + m_{yy}(g_u)^2] \ - \ [m_x f_{uu} + m_y g_{uu}].$$

Since m is smooth and convex down it follows that D^2m is negative semi-definite, so the first term is non-negative. Now m_x and m_y are both non-negative. Also $g_{uu} > 0$ everywhere and $f_{uu} > 0$ on S. □

A component N of the boundary of a projective manifold M is *Hessian-convex* if N is locally the graph over the tangent hyperplane of a smooth function with positive definite Hessian in some chart.

Proposition 8.3 (smoothing convex boundary)**.** *Suppose M is a projective manifold and ∂M is everywhere locally convex, and also strictly convex at one point on each component of ∂M. Then there is a submanifold $N \subset M$ such that $M \cong N \cup ([0,1] \times \partial N)$ and ∂N is Hessian-convex.*

Proof. Suppose ∂M is strictly convex at $x \in \partial M$. Choose a (subset of a) hyperplane $H \subset M$ close to x so that the component C of $M \setminus H$ containing x is a small convex set V. Using local affine coordinates, $S = C \cap \partial M$ is the graph over H of a convex function f, which is 0 on $H \cap \partial M$. Apply (8.2) to produce a smooth function g with positive definite Hessian and satisfying $0 \leq g \leq f$. The graph of g is a smooth hypersurface between H and S. Replace S by this graph. This smoothes out part of ∂M. Repeating this procedure smoothes the entire boundary. □

In a similar way one can prove:

Corollary 8.4 (smoothing convex functions)**.** *Suppose M is a connected affine manifold and $f : M \longrightarrow \mathbb{R}$ is a convex function, which is strictly convex at some point. Given $\epsilon > 0$ there is $g : M \longrightarrow \mathbb{R}$, which is smooth, Hessian-convex and satisfies $|f - g| < \epsilon$.*

9. Benzécri's Theorem

Theorem 9.1 (Benzécri)**.** *For each $n > 1$ there is a* Benzécri constant *$R = R_{\mathcal{B}}(n) \leq 5^{n-1}$ with the following property. Suppose Ω is a properly convex open subset of $\mathbb{RP}^n$ and $p \in \Omega$. Then there is a projective transformation $\tau \in \mathrm{PGL}(n+1, \mathbb{R})$ such that $\tau(p) = 0$ and $B(1) \subset \tau(\Omega) \subset B(R)$, where $B(t)$ is the closed ball of radius t in $\mathbb{R}^n$ centered at 0.*

The projective transformation τ is called a *Benzécri chart for Ω centered at p* and the image $\tau(\Omega, p)$ is called *Benzécri position.* The following proof provides an algorithm to find one. The set of Benzécri charts for (Ω, p) is a compact subset of $\mathrm{PGL}(n+1, \mathbb{R})$.

Proof. The proof is by induction on n. If $n = 1$, then Ω is an open interval in $\mathbb{R}P^1$ with closure a closed interval. There is a projective transformation taking Ω to $(-1, 1)$ and p to 0 so $R_{\mathcal{B}}(1) = 1$.

For the inductive step, choose a projective hyperplane $H^{n-1} \subset \mathbb{RP}^n$ containing p. Then $\Omega' = \Omega \cap H$ is an open convex set in $H \cong \mathbb{R}P^{n-1}$ and $p \in \Omega'$. Since Ω is properly convex, $\overline{\Omega}$ is disjoint from some projective hyperplane K^{n-1}. Thus $\overline{\Omega}' = \overline{\Omega} \cap H$ is disjoint from $H \cap K$, which is a hyperplane in H. It follows that Ω' is properly convex in H. By induction, and after choosing appropriate coordinates on an affine patch in H (or using a fixed coordinate system and applying a Benzécri transformation to Ω'), we may assume that $\Omega' \subset \mathbb{R}^{n-1} \subset H$ with $p = 0$ and $B^{n-1}(1) \subset \Omega' \subset B^{n-1}(r)$, where $r = R_{\mathcal{B}}(n-1)$.

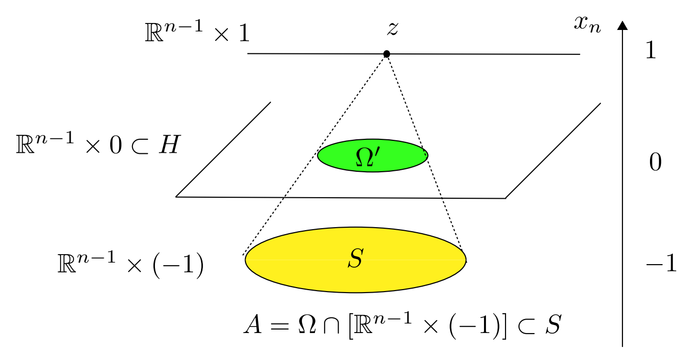

Figure 3. Shadows

There are affine coordinates on $\mathbb{RP}^n \setminus K = \mathbb{R}^n$ so that the affine part of H is $\mathbb{R}^{n-1} \times 0$. In what follows we will apply projective transformations in $\mathrm{PGL}(n+1, \mathbb{R})$ which are the identity on H. This moves Ω while keeping Ω' fixed. The first step is to arrange that

$$\Omega \subset \mathbb{R}^{n-1} \times [-1, 1]$$

and $\partial\Omega$ contains a point $z \in \mathbb{R}^{n-1} \times 1$. Then we may shear so that $z = (0, \cdots, 0, 1)$.

Next consider the one-parameter group $A(t) \in \mathrm{PGL}(n+1, \mathbb{R})$ fixing z and H. As t varies, the points that are not fixed move between z and H. This group preserves the family of affine planes $\{x_n = \text{const}\}$ in $\mathbb{R}^n$. Since it fixes z the affine plane $\mathbb{R}^{n-1} \times 1$ is preserved (though not fixed) by this group. Thus we may move Ω by an element of this group so that it still is contained in $\mathbb{R}^{n-1} \times [-1, 1]$, still contains z, and

$$A = \Omega \cap \left(\mathbb{R}^{n-1} \times (-1)\right)$$

in not empty. Let $C \subset \mathbb{R}^{n-1} \times [-1, 1]$ be the set of points on all lines ℓ passing through z and some point in Ω'. Then $S = C \cap [\mathbb{R}^{n-1} \times (-1)]$ is the shadow from the point z of Ω' on $\mathbb{R}^{n-1} \times (-1)$. Since Ω is convex it follows that $A \subset S$. Since $\Omega' \subset B_r(0)$ it follows that S is contained in the shadow of $B_r(0)$, which is the ball $D \subset \mathbb{R}^{n-1} \times (-1)$ of radius $2r$ center $(0, \cdots, 0, -1)$. Finally, let X be the union of all line segments in $\mathbb{R}^{n-1} \times [-1, 1]$ containing a point of S and B_r. This is contained in the union of the shadows on $\mathbb{R}^{n-1} \times 1$ of $B_r(0) \subset H$ from points in D. This is a ball in $\mathbb{R}^{n-1} \times 1$ of radius $4r$ center z. It lies within distance $1 + r \leq 5r$ from 0. $\square$

Let $\mathcal{S}$ be the set of all $\Omega \subset \mathbb{R}P^n$, which are disjoint from some hyperplane and compact, convex, and with non-empty interior, equipped with the Hausdorff topology. Let $\mathcal{S}_* \subset \mathcal{S} \times \mathbb{R}P^n$ be the set of all pairs (Ω, p) with p in the interior of Ω with the subspace topology of the product topology. There is an action of $\tau \in \mathrm{PGL}(n+1, \mathbb{R})$ on $\mathcal{S}_*$ given by $\tau(\Omega, p) = (\tau\Omega, \tau p)$. The quotient of $\mathcal{S}_*$ by this action is given the quotient topology and denoted $\mathcal{B}$. The flowing is due to Benzécri [3]

Corollary 9.2 (Benzécri's compactness theorem)**.** *$\mathcal{B}$ is compact.*

It follows that there is a compact set of preferred charts centered on a point in a properly convex manifold M. Different preferred charts give Euclidean coordinates around p which vary in a compact family independent of M, depending only on dimension.

References

[1] S. Ballas. Finite Volume Properly Convex Deformations of the Figure Eight Knot. *Geom. Dedicata*, 178:49–73, 2015.

[2] S. Ballas, D. Cooper, and A. Leitner. A classification of generalized cusps in projective manifolds. *In Preparation*, 2015.

[3] J.-P. Benzécri. Sur les variétés localement affines et localement projectives. *Bull. Soc. Math. France*, 88:229–332, 1960.

[4] S. Choi. Geometric structures on orbifolds and holonomy representations. *Geom. Dedicata*, 104:161–199, 2004.

[5] S. Choi. The convex real projective manifolds and orbifolds with radial ends: the openness of deformations. *ArXiv e-prints*, Nov. 2010.

[6] S. Choi. The classification of ends of properly convex real projective orbifolds II: Properly convex radial ends and totally geodesic ends. *ArXiv e-prints*, Jan. 2015.

[7] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni. *The Ricci flow: techniques and applications. Part III. Geometric-analytic aspects*, volume 163 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[8] D. Cooper, J. Danciger, and A. Wienhard. Limits of geometries. *ArXiv e-prints*, Aug. 2014.

[9] D. Cooper and D. D. Long. A generalization of the Epstein-Penner construction to projective manifolds. *Proc. Amer. Math. Soc.*, 143(10):4561–4569, 2015.

[10] D. Cooper, D. D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

[11] M. Crampon and L. Marquis. Un lemme de Kazhdan-Margulis-Zassenhaus pour les géométries de Hilbert. *Ann. Math. Blaise Pascal*, 20(2):363–376, 2013.

[12] J. Faraut and A. Korányi. *Analysis on symmetric cones*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1994. Oxford Science Publications.

[13] W. Goldman. *Projective geometry on manifolds*. Versions 11 March 1988 and 2 June 2009 downloaded from http://www2.math.umd.edu/∼wmg/, 1988–2009.

[14] W. M. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 169–198. Amer. Math. Soc., Providence, RI, 1988.
[15] R. E. Greene and H. Wu. Approximation theorems, C^∞ convex exhaustions and manifolds of positive curvature. *Bull. Amer. Math. Soc.*, 81:101–104, 1975.
[16] R. E. Greene and H. Wu. C^∞ convex functions and manifolds of positive curvature. *Acta Math.*, 137(3-4):209–245, 1976.
[17] M. W. Hirsch. *Differential topology*. Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, No. 33.
[18] M. W. Hirsch and B. Mazur. *Smoothings of piecewise linear manifolds*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 80.
[19] M. Koecher. Positivitätsbereiche im R^n. *Amer. J. Math.*, 79:575–596, 1957.
[20] J.-L. Koszul. Variétés localement plates et convexité. *Osaka J. Math.*, 2:285–290, 1965.
[21] J.-L. Koszul. Déformations de connexions localement plates. *Ann. Inst. Fourier (Grenoble)*, 18(fasc. 1):103–114, 1968.
[22] A. Leitner. A Classification of subgroups of $SL(4, R)$ Isomorphic to R^3 and Generalized Cusps in Projective 3–Manifolds. *ArXiv e-prints*, July 2015.
[23] E. L. Lima. On the local triviality of the restriction map for embeddings. *Comment. Math. Helv.*, 38:163–164, 1964.
[24] R. Schoen and S.-T. Yau. *Lectures on differential geometry*. Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994. Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu, Translated from the Chinese by Ding and S. Y. Cheng, Preface translated from the Chinese by Kaising Tso.
[25] H. Shima. *The geometry of Hessian structures*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
[26] H. Shima and K. Yagi. Geometry of Hessian manifolds. *Differential Geom. Appl.*, 7(3):277–290, 1997.
[27] P. A. N. Smith. Counterexamples to smoothing convex functions. *Canad. Math. Bull.*, 29(3):308–313, 1986.
[28] L.-F. Tam. Exhaustion functions on complete manifolds. In *Recent advances in geometric analysis*, volume 11 of *Adv. Lect. Math. (ALM)*, pages 211–215. Int. Press, Somerville, MA, 2010.
[29] W. P. Thurston. *The Geometry and Topology of Three-Manifolds*. Princeton Univ. Math. Dept. Available from http://msri.org/publications/books/gt3m/, 1979.
[30] È. B. Vinberg. The theory of homogeneous convex cones. *Trudy Moskov. Mat. Obšč.*, 12:303–358, 1963.
[31] J. H. C. Whitehead. On C^1-complexes. *Ann. of Math. (2)*, 41:809–824, 1940.

DC & DL: Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

ST: School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia

E-mail address: cooper@math.ucsb.edu
E-mail address: long@math.ucsb.edu
E-mail address: tillmann@maths.usyd.edu.au