Degenerations of Representations and Thin Triangles.

Daryl Cooper*

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Abstract

This paper gives a compactification of the space of representations of a finitely generated group into the groups of isometries of all spaces with Δ-thin triangles. The ideal points are actions on $\mathbb{R}$-trees. It is a geometric reformulation and extension of the Culler-Morgan-Shalen theory concerning limits of (characters of) representations into SL_2C and more generally $SO(n, 1)$.

1 Introduction

The purpose of this paper is to compactify certain spaces of representations. Let G be a finitely generated group and $\mathcal{H}$ any geodesic space having Δ-thin-triangles. We consider a homomorphism ρ of G into the group of isometries of $\mathcal{H}$. The set of all such homomorphims is denoted $Hom_\Delta(G)$. Then ρ defines a length function $L(\rho) : G \longrightarrow \mathbb{R}$. The space of all length functions, with the product topology, is denoted $\mathcal{LF}(G)$. Hence there is a map

$$L : Hom_\Delta(G) \longrightarrow \mathcal{LF}(G).$$

The space of projectivized length functions on G, denoted $\mathcal{PLF}(G)$, gives a superspace $\mathcal{LF}(G) \cup \mathcal{PLF}(G)$. Our first result is that $image(L)$ has compact closure in this superspace, and ideal points are projective classes of length functions of certain actions of G on $\mathbb{R}$-trees.

This may be viewed as a generalization of the compactification of spaces of characters of representations into the Lie groups $SO(n, 1)$ used by Culler, Morgan and Shalen [6, 7, 4, 9] by actions on $\mathbb{R}$-trees. Our point of view is that rescaling negatively curved spaces gives limiting spaces which are $\mathbb{R}$-trees. This idea was used by Bestvina [1] to provide new proofs of some results of Morgan and Shalen [9]. In particular he applied a geometric method to obtain this compactification for the subspace of discrete faithful representations into $SO(n, 1)$. Our contribution is to remove the hypotheses of discrete and faithful, and apply the technique to a wider class of groups than $SO(n, 1)$.

Next we give sufficient conditions on a length function to ensure it is the length function of an action of G on a simplicial tree. This is used to recover some results of Culler and Shalen [6] on compactifications of curves of characters by actions on simplicial trees.

The standard approach to the theorem of Culler and Shalen uses the the Tits-Bass-Serre approach [11] to trees via $SL_2\mathcal{F}$, where $\mathcal{F}$ is a field with a discrete rank-1 valuation. We have reduced the use of valuations to showing that a certain $\mathbb{R}$-tree has a minimal invariant subtree which is simplicial.

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Our starting point in section 2 is the observation that if the vertices of a finite connected graph in a geodesic space with thin triangles, $\mathcal{H}$, are moved so that the diameter of the graph goes to infinity and if the induced metric on the graph is rescaled to keep the diameter bounded, then the metric on the graph converges to a pseudo-metric for which the associated metric space is a tree.

Given a representation $\rho : G \longrightarrow Isom(\mathcal{H})$ we map the complete graph, K, on vertex set G into $\mathcal{H}$ by defining the map on the vertex set G equivariantly. This requires the choice of a single point x in $\mathcal{H}$ and we use a modified version of the notion of the *center* of ρ introduced by Bestvina [1]. Given a sequence of representations for which the translation length of some element goes to infinity, we obtain a subsequence for which K converges to an $\mathbb{R}$-tree, Γ, on which G acts by isometries. This leads to the compactification result. We then show that, if the translation length of every element of G acting on Γ is an integer, and if the action is irreducible, then Γ has an invariant *simplicial* subtree. The argument up to this point is geometric and combinatorial. This is done in section 3.

In section 4, valuations make their appearance in showing that, if the above representations all lie on an algebraic curve in $Hom(G, SL_2\mathbb{C})$, then the translation length of every element is an integer. Here the idea is that when the translation length of an element of G is large, it behaves like the logarithm of a polynomial (the trace) of one variable when that variable is large. The degree of the polynomial then determines the translation length. This is what the valuation says. Our treatment of the valuations necessary is based on resultants and Newton polygons.

We believe that these ideas were in essence known to Culler, Morgan and Shalen and motivated the search for a more algebraic formulation. However it seems to us that the geometric point of view deserves to be more widely known.

2 Rescaling Graphs in Negatively Curved Spaces

The results of this section apply to geodesic spaces satisfying the *thin triangles* property made popular by Gromov in [8]. The main result of this section is (2.5) which states that the limit of certain rescaled metric spaces is an $\mathbb{R}$-tree.

In an arbitrary metric space, not necessarily a Riemannian manifold, (X, d) one defines the *length* of a path $\gamma : [a, b] \longrightarrow X$ to be the supremum over finite dissections of $[a, b]$ of the sum of the distances between endpoints of the dissecting subintervals. A *path-metric* is a metric with the property that the distance between two points is the infimum of the lengths of paths connecting the points. Thus a Riemannian manifold is a path-metric space. A *metric simplicial tree* is a simplicial complex which is a tree and it is equipped with a path-metric. An $\mathbb{R}$*-tree* Γ is a path-metric space with the property that given any two points a, b there is a unique arc γ in Γ with endpoints a, b. Furthermore $length(\gamma) = d(a, b)$, see [5] or [12] for a discussion. Thus every metric simplicial tree is an $\mathbb{R}$-tree.

A *geodesic* in a path-metric space, X, is an arc in X with the property that the length of the arc equals the distance in X between its endpoints. In a simply-connected complete Riemannian manifold of non-positive curvature this is equivalent to the usual definition of geodesic. A path-metric space in which every pair of points are the endpoints of a geodesic is called a *geodesic space.* We denote a choice of geodesic in X with endpoints a, b by $[a, b]$. We allow degenerate geodesics $[a, a]$ consisting of a single point. A *triangle* in a geodesic space consists of three geodesics $[a, b], [b, c], [c, a]$ called the sides of the triangle. Given $\Delta > 0$ a triangle is called Δ*-thin* if for each edge, every point on that edge is within a distance of at most Δ from at least one point on the union of the other two edges. A geodesic space is called *negatively curved* or *(Gromov)-hyperbolic* if there is $\Delta > 0$ such that every triangle is Δ-thin. It is clear that simplicial trees and more generally $\mathbb{R}$-trees have Δ-thin triangles for all $\Delta > 0$. We shall use the term *hyperbolic space* to refer to $\mathbb{H}^n$ which is the complete simply connected Riemannian n-manifold with constant sectional curvature -1.

Theorem 2.1 *(***Thin Triangles***) Triangles in $\mathbb{H}^n$ are Δ-thin with $\Delta = ln\left(1+\sqrt{2}\right)$.*

From now on we will assume that $(\mathcal{H}, d_{\mathcal{H}})$ is a geodesic space with Δ-thin triangles. Often we will write $d(x,y)$ instead of $d_{\mathcal{H}}(x,y)$ if this will not cause confusion.

Corollary 2.2 *(***Thin Polygons***) Suppose that a polygon P in $\mathcal{H}$ has n geodesic sides then every edge of P lies within a distance $(n-2)\Delta$ of the union of the other edges of P.*

Proof The polygon P may be triangulated using $n-2$ triangles by coning from a vertex of P, and the result follows from the thin triangles property. ■

Let K be a *connected graph* in other words a connected 1-dimensional simplicial complex. A map $f : K \longrightarrow \mathcal{H}$ is called a *straight map* if the image of each edge e of K is a geodesic segment or a point, and in addition $f|e$ maps e linearly onto its image. Recall that a *pseudo-metric* satisfies the same conditions as a metric except that the distance between two distinct points is allowed to be zero. There is a pseudo-metric, $d = d_K$, defined on K by pulling back the metric on $\mathcal{H}$

$$d_K(x,y) = d_{\mathcal{H}}(fx, fy).$$

Given a pseudo-metric space (K,d) there is an equivalence relation on K given by $x \sim y$ if $d(x,y) = 0$. Then d induces a metric, also denoted d, on $K/\sim$. We call $(K/\sim, d)$ the metric space *associated* to (K,d). Note that f induces an isometry of $(K/\sim, d)$ with the subset $f(K)$ of $\mathcal{H}$. Note that the metric d_K is not usually a path-metric. This is because two points in $f(K)$ lying on the images of distinct edges are connected by a geodesic which typically is not contained in $f(K)$.

Proposition 2.3 *Given $\Delta > 0$, suppose that $f_n : K \longrightarrow \mathcal{H}_n$ is a sequence of straight maps of a finite connected graph K into geodesic spaces $\mathcal{H}_n$ with Δ-thin triangles. These induce pseudo-metrics d_n on K. Suppose that $\lambda_n \to 0$ and rescale the pseudo-metrics by λ_n. Thus $\tilde{d}_n = \lambda_n d_n$ and suppose that $\{\tilde{d}_n\}$ converges pointwise on K to a pseudo-metric $\overline{d}$. Then the metric space associated to $(K, \overline{d})$ is a metric simplicial tree.*

Proof For notational simplicity we assume that all the spaces $\mathcal{H}_n$ are the same. This does not affect the proof. Since f_n is linear on each edge it is clear that $\overline{d}$ pseudo-metrizes each edge of K as an interval which, however, might have zero length. Let $(\Gamma, \overline{d})$ be the metric space associated to $\overline{d}$, thus Γ is the union of finitely many metric intervals.

The proof proceeds by induction on the number of edges of K. It suffices to consider the case that β is a new edge which is attached to a connected graph K at one or both endpoints. Consider the subset, β_-, of β consisting of all points, x, in β with the property that there exists a point in K which is $\overline{d}$-distance 0 from x. We claim that β_- is a closed subinterval of β.

To prove this, let x', y' be points in β_-. Let x, y be points in K which are $\overline{d}$-distance zero from x', y' respectively. Since K is connected we may choose an embedded arc ℓ in K connecting x, y. Then ℓ is the union of some finite number, m, of intervals each contained in some edge of K. Thus $f_n(\ell)$ is a polygonal path in $\mathcal{H}$ with endpoints $f_n x, f_n y$ made up of m geodesic segments and m does not depend on n. Let γ_n be the geodesic segment in $\mathcal{H}$ with endpoints $f_n x, f_n y$, then every point of γ_n lies within a distance $D = (m-1)\Delta$ of $f_n(\ell)$. This follows by thin polygons (2.2) applied to the polygon $\gamma_n \cup f_n(\ell)$. Now $f_n\beta_-$ is a geodesic arc in $\mathcal{H}$ with endpoints $f_n x', f_n y'$. Since $\overline{d}(x,x') = 0$ it follows that $\lambda_n d_{\mathcal{H}}(f_n x', f_n x) \to 0$ as $n \to \infty$. Similarly $\lambda_n d_{\mathcal{H}}(f_n y', f_n y) \to 0$. If two geodesic arcs in $\mathcal{H}$ start at points within a distance M of each other and end at points within a distance M of each other, then every point on the first arc is within a distance $M+2\Delta$ of some point on the second arc. This follows easily from considering the thin quadrilateral made by these two arcs plus a geodesic arc connecting their start points and another geodesic arc connecting their end points. We apply this to the arcs γ_n and $f_n(\beta_-)$. It follows that given z' on β_- there is z_n on γ_n with $d_{\mathcal{H}}(f_n z', z_n) \leq M + 2\Delta$. Since the point z_n on γ_n is within a $d_{\mathcal{H}}$-distance D of some point on $f_n(\ell)$ it follows that $d_n(z', \ell) \leq M+2\Delta+D$. Thus $\overline{d}(z', \ell) = \lim \lambda_n d_n(z', \ell) = 0$. This proves that β_- is an interval. The obvious argument shows that β_- is closed, establishing the claim.

Since β is attached to K at one endpoint if $\beta - \beta_-$ is not empty then it is an interval. If $\beta_- = \beta$ then it is clear that the tree formed by adding β to K is the same as the tree formed from just K. Let $\gamma = [a', b']$ be the closure of $\beta - \beta_-$. One of the endpoints, say a', of γ is a $\overline{d}$-distance of zero from some point, say a, of K. Thus $\overline{d}(a, a') = 0$. The topological space $\Gamma \equiv (K \cup \gamma)/\sim$ is the identification space formed from the union $K/\sim$ with γ by identifying a' with a. Therefore the result is homeomorphic to a simplicial tree. It remains to show that the metric induced by $\overline{d}$ makes Γ into a metric tree.

Let $x \in \gamma$ and $y \in K$ be arbitrary points then there are arcs $[y, a] \subset K/\sim$ and $[a', x] \subset \gamma$. The union of these arcs, $[y, a] \cup [a', x]$, is the unique arc in Γ from x to y. The proof is completed by the claim that

$$\overline{d}(y, x) = \overline{d}(y, a) + \overline{d}(a, x).$$

Suppose that, given $\mu > 0$ that for all sufficiently large n that there is a point, z, on $[f_n x, f_n y]$ with $d_{\mathcal{H}}(f_n a, z) \leq \mu\lambda_n^{-1} + 2\Delta$. Then

$$\begin{aligned} & \mid d_n(x, y) - (d_n(x, a) + d_n(a, y)) \mid \\ \leq & \mid d_{\mathcal{H}}(f_n x, z) + d_{\mathcal{H}}(z, f_n y) - d_{\mathcal{H}}(f_n x, f_n a) - d_{\mathcal{H}}(f_n a, f_n y) \mid \\ \leq & \mid d_{\mathcal{H}}(f_n x, z) - d_{\mathcal{H}}(f_n x, f_n a) + d_{\mathcal{H}}(z, f_n y) - d_{\mathcal{H}}(f_n a, f_n y) \mid \\ \leq & \mid d_{\mathcal{H}}(z, f_n a) + d_{\mathcal{H}}(z, f_n a) \mid \\ \leq & \ 2(\mu\lambda_n^{-1} + 2\Delta). \end{aligned}$$

Multipying both sides by λ_n and using that $\lambda_n \to 0$, the claim follows in this case.

We consider the thin triangle with sides $[f_n x, f_n a]$ $[f_n x, f_n a']$ and $[f_n a, f_n a']$. Since $\overline{d}(a, a') = 0$ it follows that given $\epsilon > 0$ for sufficiently large n we have $d_n(a, a') \leq \epsilon\lambda_n^{-1}$. Hence for all points q_n on $[f_n x, f_n a]$

$$d_{\mathcal{H}}(q_n, [f_n x, f_n a']) \leq \epsilon\lambda_n^{-1} + \Delta.$$

We may assume that there is some $\mu > 0$ such that for arbitrarily large n that $d_{\mathcal{H}}(f_n a, [f_n x, f_n y]) > \mu\lambda_n^{-1} + 2\Delta$. Define p_n to be the point on $[y, a]$ with $d_n(p_n, a) = \mu\lambda_n^{-1}$ and q_n to be the point on $[f_n x, f_n a]$ with $d_{\mathcal{H}}(q_n, f_n a) = \mu\lambda_n^{-1}$. Now:

$$\begin{aligned} d_{\mathcal{H}}(f_n p_n, [f_n y, f_n x]) & \geq d_{\mathcal{H}}(f_n a, [f_n y, f_n x]) - d_{\mathcal{H}}(f_n a, f_n p_n) \\ & \geq \mu\lambda_n^{-1} + 2\Delta - \mu\lambda_n^{-1} \\ & = 2\Delta. \end{aligned}$$

Then thin triangles applied to the triangle with vertices $f_n a, f_n x, f_y$ gives

$$d_{\mathcal{H}}(f_n p_n, [f_n a, f_n x]) \leq \Delta.$$

Let r_n be a point on $[f_n a, f_n x]$ with $d_{\mathcal{H}}(f_n p_n, r_n) \leq \Delta$. Then

$$\mid d_{\mathcal{H}}(f_n a, r_n) - d_{\mathcal{H}}(f_n a, f_n p_n) \mid \leq \Delta.$$

Since r_n and q_n are both on $[f_n a, f_n x]$ and $d_n(a, p_n) = d_{\mathcal{H}}(f_n a, q_n)$ it follows that $d_{\mathcal{H}}(r_n, q_n) \leq \Delta$. Thus

$$d_{\mathcal{H}}(f_n p_n, q_n) \leq d_{\mathcal{H}}(f_n p_n, r_n) + d_{\mathcal{H}}(r_n, q_n) \leq 2\Delta.$$

Combining this with the bound on $d_{\mathcal{H}}(q_n, [f_n x, f_n a'])$ gives

$$d_{\mathcal{H}}(f_n p_n, [f_n x, f_n a']) \leq d_{\mathcal{H}}(f_n p_n, q_n) + d_{\mathcal{H}}(q_n, [f_n x, f_n a']) \leq 2\Delta + \epsilon\lambda_n^{-1} + \Delta.$$

As $n \to \infty$ there is a subsequence of p_n which converges to a point p in $[a, y]$. We get $\overline{d}(p, [x, a']) \leq \epsilon$. Since $\epsilon > 0$ is arbitrary this gives $\overline{d}(p, [x, a']) = 0$. Now $\overline{d}(p, a) = \mu > 0$ so $p \neq a$ but this contradicts that a is the only point on $[a, y]$ zero distance from $[x, a']$. This proves the claim. ■

Lemma 2.4 *Suppose that (Γ, d) is a metric space which it the union of an increasing sequence of subspaces T_n, each of which is a metric simplicial tree. Then Γ is an $\mathbb{R}$-tree.*

Proof Suppose that x, y are two distinct points in Γ then there is some T_m which contains both of them. There is a unique arc $[x, y]$ in T_m. Suppose that there is a different arc α in Γ with the same endpoints. The intersection of α with $[x, y]$ is closed. We may assume that $[x, y]$ intersects α only in the points x, y. For otherwise we may replace $[x, y]$ by a suitable subarc.

We define a nearest-point projection

$$\pi : \Gamma \longrightarrow [x, y]$$

as follows. Given a point z in Γ there is an $n > m$ such that T_n contains z. Define πz to be the unique point on $[x, y]$ which is closest to z in T_n. Since the trees T_n are an increasing sequence, it is clear that this definition of πz is independent of n. We claim that π is continuous. For if $d_\Gamma(z, z') < \epsilon$ then if n is chosen large enough that T_n contains both z and z' then $d_{T_n}(z, z') < \epsilon$. But the restriction of π to the simplicial tree T_n clearly does not increase distance thus $d_{T_n}(\pi z, \pi z') < \epsilon$. This proves the claim.

We claim that $\pi\alpha = x$. Let β be the maximal subarc of α containing x such that $\pi\beta = x$. Since $\pi x = x$ it follows that β is not empty. Since π is continuous it follows that β is closed. Writing $\beta = [x, z]$ choose $n > m$ with T_n containing z. Assuming $z \neq x, y$ then $d_{T_n}(z, \pi z) = d_{T_n}(z, [x, y]) = \delta > 0$. Now $\pi y = y$ so $\beta \neq \alpha$ thus we may choose z' a point in $\alpha - \beta$ with $\pi z' \neq x$. Now choose $n > m$ such that T_n contains both z, z'. By choice of z' we have that $[z, z'] = [z, \pi z] \cup [\pi z, \pi z'] \cup [\pi z', z']$. Therefore $d_{T_n}(z, z') \geq d_{T_n}(z, \pi z) = \delta$. Since z' can be arbitrarily close to z this is a contradiction. Hence $z = x$.

It follows that there is a unique arc in Γ between any pair of distinct points. Since there is some T_n containing both these points, this arc is metrized as an interval. Hence Γ is an $\mathbb{R}$-tree. ■

Combining these two results together gives:

Theorem 2.5 *Suppose that $\Delta > 0$ is given and suppose that $\mathcal{H}_n$ are geodesic spaces with Δ-thin triangles. Suppose that f_n is a sequence of straight maps of a connected graph K with countably many edges into $\mathcal{H}_n$. Let d_n be the pull-back via f_n of the metric on $\mathcal{H}_n$ to K. Suppose that $\lambda_n \to 0$ and rescale the pseudo-metrics d_n by λ_n. Thus $\tilde{d}_n = \lambda_n d_n$ and suppose that $\{\tilde{d}_n\}$ converges pointwise on K to a pseudo-metric $\overline{d}$. Then the metric space, Γ, associated to $(K, \overline{d})$ is an $\mathbb{R}$-tree.*

3 Limiting Actions on R-trees.

In this section we consider an action of a finitely generated group G acting by isometries on a geodesic space $\mathcal{H}$ with Δ-thin triangles. We show that if a sequence of such representations blows up, then there is a limiting action of G on an $\mathbb{R}$-tree. In addition we show that under certain circumstances, the $\mathbb{R}$-tree may be replaced by a simplicial tree.

Suppose that (X, d) is a metric space and τ is an isometry of this space then we define the *translation length* of τ by

$$t(\tau) = \inf\{d(x, \tau x) : \ x \in X \ \}$$

thus $t : Isom(X) \longrightarrow \mathbb{R}$.

Let G be a finitely generated group and let $K \equiv K(G)$ be the complete graph with G as vertex set; this is a graph with one edge between every pair of vertices. Choose a finite generating set S for G. Given x in $\mathcal{H}$ and a representation $\rho : G \longrightarrow Isom(\mathcal{H})$ define

$$r_S(x) = \sup\{d(x, (\rho\alpha)x) : \ \alpha \in S \ \}.$$

An *approximate center* of the representation ρ is a choice of x in $\mathcal{H}$ for which $r_S(x)$ is at most 1 greater than the infimum over all choices for x, compare [1]. Intuitively, an approximate center is approximately where in $\mathcal{H}$ the representation "lives". It is used to overcome the problems associated with changing a representation by conjugacy.

Suppose that we are given a sequence of representations $\rho_n : G \longrightarrow Isom(\mathcal{H})$ which are *blowing up* which means that there is $\alpha \in G$ such that $\lim_{n\to\infty} t(\rho_n\alpha) = \infty$, see [3]. Choose an approximate center x_n for ρ_n and define $\iota_n : K \longrightarrow \mathcal{H}$ on the vertices of K by:

$$\iota_n(\alpha) = (\rho_n\alpha)x_n$$

and now extend ι_n linearly over each edge of K so that ι_n is a straight map. Let d_n be the pull back pseudo-metric on K. Now G acts on the vertices of K by multiplication on the left and extending this action linearly over the edges of K we get an action $\sigma : G \longrightarrow Isom(K, d_n)$ by isometries on the pseudo-metric space (K, d_n) (it would be natural to use the term pseudo-isometry here, but this term has a different meaning, and iso-pseudometry lacks linguistic appeal!). It is clear that ι_n is a G-equivariant isometry from K onto its image.

Now define the *rescaling factor*

$$\lambda_n = (\max\{d_n(1_G, \beta) : \ \beta \in S \ \})^{-1}$$

and the *rescaled pseudo-metric* by $\tilde{d}_n = \lambda_n d_n$. Since ρ_n is blowing up it is easy to see that $\lambda_n \to 0$.

Proposition 3.1 *Suppose that G is a finitely generated group and ρ_n is a sequence of representations of G into $Isom(\mathcal{H})$ which blows up. Let $\tilde{d}_n$ be the rescaled metrics on K defined above. Then there is a subsequence $\tilde{d}_{n_i}$ which converges pointwise to a pseudo-metric $\overline{d}$ on K. Furthermore, the metric space associated to $(K, \overline{d})$ is an $\mathbb{R}$-tree.*

Proof Define K_S to be the subgraph of K which is the Cayley graph corresponding to the generating set S. Thus for each $s \in S$ there is an edge in K_S connecting g to gs. This condition is preserved by multiplication on the left, hence K_S is preserved by G. Thus K_S is a connected graph on which G acts. The rescaling factor was chosen so that the $\tilde{d}_n$-length of every edge in K_S adjacent to the identity is at most 1. Since G acts transitively on vertices, and by $\tilde{d}_n$-isometries, it follows that the $\tilde{d}_n$-length of every edge in K_S is at most 1. Hence, for every pair of vertices x, y in K_S we have that $\tilde{d}_n(x, y)$ is bounded by the number of edges in a path from x to y. Since $\tilde{d}_n$ metrizes every edge of K as an interval, it follows that for every pair of points x, y in K there is a bound on $\tilde{d}_n(x, y)$

Choose a countable dense subset, D, of K. For x, y in D we have that $\tilde{d}_n(x, y)$ is bounded independently of n. Hence there is a subsequence, n_i, so that for all x, y in D then $\tilde{d}_{n_i}(x, y)$ converges. We claim for all x, x' in K that $\tilde{d}_{n_i}(x, x')$ converges. The pointwise-limit is clearly a pseudo-metric, $\tilde{d}$, on K. In what follows we assume the sequence has been replaced by the subsequence.

To prove the claim, consider an edge e of K, then the map ι_n is linear on e. The limit pseudo-metric restricted to e metrizes e as a closed interval. Given a point, x, on e we may choose x_1, x_2 in $D \cap e$ with x between them and with the limiting distance between x_1 and x_2 very small so that for all n sufficiently large $\tilde{d}_n(x, x_1) < \epsilon$. Given another point x' on some other edge e' we may similarly choose x_1', x_2' with limiting distance between them ϵ'. Then for n sufficiently large:

$$\begin{aligned} |\tilde{d}_n(x,x') - \tilde{d}_n(x_1,x_1')| &\leq \tilde{d}_n(x,x_1) + \tilde{d}_n(x_1',x') \\ &\leq \tilde{d}_n(x_2,x_1) + \tilde{d}_n(x_1',x_2') \\ &\leq 2\epsilon. \end{aligned}$$

Since $\tilde{d}_n(x_1, x_1')$ converges it follows that $\tilde{d}_n(x, x')$ converges as $n \to \infty$. It follows from (2.5) that the metric space associated to $(K, \overline{d})$ is an $\mathbb{R}$-tree. ■

Definition 3.2 *Given an isometry, τ of an $\mathbb{R}$-tree Γ the* **core** *of τ is the set of points in Γ which are moved the minimal distance by τ :*

$$C(\tau) = \{ \ x \in \Gamma \ : \ d_\Gamma(x, \tau x) = t(\tau) \ \}.$$

It is clear that this a closed convex set (ie $x, y \in core(\tau)$ implies $[x, y]$ is in $core(\tau)$) which is invariant under τ. An isometry, τ, of Γ is called **elliptic** *if τ has a fixed point. It is called* **loxodromic** *if $t(\tau) > 0$ and the core of τ is isometric to a line. This line is called the* **axis** *of τ.*

Lemma 3.3 *Every isometry of an $\mathbb{R}$-tree is either elliptic or loxodromic.*

Proof Choose a point x in Γ and consider the intersection of $[x, \tau x]$ with $\tau^{-1}([x, \tau x])$. This is a geodesic $[x, y]$. If $y = \tau y$ then τ is elliptic. Otherwise, from the definition of y it follows that the intersection of $[y, \tau y]$ with $\tau^{-1}[y, \tau y]$ is the single point y. Because any other point of intersection would also give an extra point of intersection of $[x, \tau x]$ with $\tau^{-1}([x, \tau x])$. Thus $\ell = \bigcup_n \tau^n[y, \tau y]$ is isometric to $\mathbb{R}$ and invariant under τ. Now $[x, \tau x] = [x, y] \cup [y, \tau y] \cup [\tau y, \tau x]$ and so $d(x, \tau x) \geq d(y, \tau y)$ with equality if and only if x is on ℓ. Thus the points moved the minimal distance by τ are exactly those points on ℓ. Hence τ has axis ℓ and is thus loxodromic. ■

Lemma 3.4 *Suppose that α, β are isometries of an $\mathbb{R}$-tree and that $core(\alpha)$ is disjoint from $core(\beta)$. Then $t(\alpha\beta) = t(\alpha) + t(\beta) + 2d(core(\alpha), core(\beta))$.*

Proof Let a be a point in $core(\alpha)$ and b a point in $core(\beta)$ such that $d(a, b)$ is minimal. To see that such points exist, choose points a' in $core(\alpha)$ and b' in $core(\beta)$. Define a by $[a', a] = [a', b'] \cap core(\alpha)$ and b by $[b, b'] = [a', b'] \cap core(\beta)$. Given any arc, γ, with one endpoint in $core(\alpha)$ and the other endpoint in $core(\beta)$ then γ contains $[a, b]$. For otherwise, since cores are convex we can join to γ an arc in $core(\alpha)$ connecting a to γ and another arc in $core(\beta)$ connecting b to γ. Observe that the arcs joined to γ meet $[a, b]$ only at a or b. The union of these three arcs is a path from a to b which contains a subset which is an embedded arc from a to b. But $[a, b]$ is the unique arc in the $\mathbb{R}$-tree with these endpoints. Hence γ contains $[a, b]$ as asserted. Hence $length(\gamma) \geq d(a, b)$. Thus a, b have the claimed property.

Define

$$\ell = [\alpha(b), \beta^{-1}(b)] = \alpha([b, a]) \cup [\alpha(a), a] \cup [a, b] \cup [b, \beta^{-1}(b)].$$

Observe that $[\alpha(a), a]$ is contained in the $core(\alpha)$ and $[b, \beta^{-1}(b)]$ is contained in the $core(\beta)$. Also $[a, b]$ intersects $core(\alpha) \cup core(\beta)$ only at a, b. It follows from this that the four arcs on the right hand side only intersect at endpoints. Since $a \in core(\alpha)$ we have $t(\alpha) = d(a, \alpha(a))$ and similarly $t(b) = d(\beta(b), b) = d(b, \beta^{-1}(b))$. Thus

$$\begin{aligned} length(\ell) &= d(\alpha(b), \alpha(a)) + d(\alpha(a), a) + d(a, b) + d(b, \beta^{-1}(b)) \\ &= d(\alpha(b), \alpha(a)) + t(a) + d(a, b) + t(b) \\ &= t(a) + 2d(a, b) + t(b) \\ &= t(a) + 2d(core(\alpha), core(\beta)) + t(b). \end{aligned}$$

We claim that $\ell \cap \alpha\beta(\ell) = \alpha(b)$, from which it follows that ℓ is contained in the axis of $\alpha\beta$ and is a fundamental domain for $\alpha\beta$. The computation of $length(\ell)$ above then completes the proof.

Now $[b, \beta^{-1}(b)]$ is contained in $core(\beta)$. Since $core(\beta)$ is invariant under β it follows that $\beta([b, \beta^{-1}(b)])$ is contained in $core(\beta)$. Since $[b, a] \cap core(\beta) = b$ it follows that $[b, a] \cap \beta([b, \beta^{-1}(b)]) = b$. If the claim is false then $\ell \cap \alpha\beta(\ell)$ is an interval of positive length containing $\alpha(b)$. Now $\alpha(b)$ is the start of ℓ and $\alpha([b, a])$ is a subinterval at the start of ℓ. Therefore this interval at the start of ℓ intersects the interval $\alpha\beta([b, \beta^{-1}(b)])$ at the end of $\alpha\beta(\ell)$ in an interval of positive length. Thus $\alpha([b, a]) \cap \alpha\beta([b, \beta^{-1}(b)])$ has positive length. But this implies $[b, a] \cap \beta([b, \beta^{-1}(b)]) = b$ has positive length, a contradiction. ■

Lemma 3.5 *If every element of a finitely generated group of isometries of an $\mathbb{R}$-tree is elliptic then there is a point in the tree fixed by every element of the group.*

Proof If α and β are elliptic then

$$t(\alpha\beta) = 2d(core(\alpha), core(\beta)).$$

Thus if $\alpha\beta$ is also elliptic then $core(\alpha)$ intersects $core(\beta)$. It follows that the intersection of the cores of a finite number of elements which generate the group is non-empty and each point in the intersection is fixed by every element in the group. ■

It is clear that the action of G on K discussed above induces an action $\sigma : G \longrightarrow Isom(\Gamma)$ on Γ by isometries.

Addendum 3.6 *Assume the hypotheses of (3.1). Then G acts isometrically on the $\mathbb{R}$-tree Γ and G fixes no point of Γ. Thus some element of G acts on Γ as a loxodromic.*

Proof Suppose that G fixes the point x in Γ. Then given $\epsilon > 0$, for n sufficiently large, there is a point x'_n in $(K, \tilde{d}_n)$ with image x in Γ such that x'_n is moved a $\tilde{d}_n$-distance at most ϵ by all elements of S. Taking $\epsilon = 1/2$ this means that for all sufficiently large n and all $\alpha \in S$ that:

$$d_{\mathcal{H}}(\iota_n x'_n, \alpha\iota_n x'_n) \leq \lambda_n/2.$$

Thus $r_S(\iota_n x'_n) \leq \lambda_n/2$ which contradicts the definition of λ_n and of the center of ρ_n. The conclusion that some element of G acts as a loxodromic follows from lemma (3.5). ■

Lemma 3.7 *Suppose that G acts by isometries on an $\mathbb{R}$-tree Γ. Assume that some element of G is loxodromic. Let Γ_- be the intersection of all connected subsets of Γ containing the axis of every loxodromic. Then Γ_- is the unique G-invariant subtree of Γ which is* **minimal** *in the sense that it contains no proper, G-invariant subtree.*

Proof Let g be a loxodromic element of G with axis ℓ. Then $t(g) > 0$. Suppose that Γ' is a G-invariant subtree of Γ then the restriction of g to Γ' is an isometry with translation length in Γ' at least as large as $t(g)$. By (3.3) it follows that $g|\Gamma'$ is a loxodromic therefore it has an axis, ℓ', which is invariant under $g|\Gamma'$. It is easy to check that the only line preserved by a loxodromic is its axis. Hence $\ell' = \ell$. Hence every invariant subtree contains the axis of every loxodromic. Therefore Γ' contains Γ_-. Also Γ_- is clearly G-invariant and connected therefore an $\mathbb{R}$-tree. ■

We say that an action of G on an $\mathbb{R}$-tree Γ is *irreducible* if there are two elements a, b of G which are both loxodromic and whose translation axes are disjoint. This can be shown to be equivalent to the statement that there is no point in $\Gamma \cup \partial\Gamma$ which is fixed by all of G. Here, $\partial\Gamma$ is the *boundary* of the $\mathbb{R}$-tree Γ. It can be described as either the set of ends of Γ, or as the boundary, in the sense of Gromov, of the negatively curved space Γ. We will not need this description. If two loxodromics α, β, have axes whose intersection is compact, then for large n the axes of α and $\beta^n\alpha\beta^{-n}$ are disjoint.

Proposition 3.8 *Let G be a finitely generated group which acts on an $\mathbb{R}$-tree Γ. Suppose that the action is minimal ie there is no invariant proper subtree. Suppose that the translation length of every element of G is an integer. Also assume that the action is irreducible. Then Γ is a simplicial tree.*

Proof We call a point v in Γ a *vertex* if $\Gamma - v$ has more than 2 components. We will show that the distance between vertices in Γ is half of an integer. An inductive argument now implies that Γ is a simplicial tree.

Given two loxodromic isometries a, b in G define d to be the distance between their axes. If $d > 0$ then by (3.4) we have $t(ab) = t(a) + t(b) + 2d$. Since all translation lengths are assumed to be integers this implies that d is a half integer.

Given two distinct vertices u_1, u_2 in Γ we claim that, for $i = 1, 2$, there are loxodromics τ_i with axis ℓ_i such that $\ell_i \cap [u_1, u_2] = u_i$. Then $[u_1, u_2]$ is the shortest arc between ℓ_1 and ℓ_2 and therefore by the previous remark $d(u_1, u_2)$ is half an integer.

Let P be the component of $\Gamma - interior[u_1, u_2]$ containing u_1. Since u_1 is a vertex it follows that $P - u_1$ is not connected. Let Q, R be the closures of two distinct components of $P - u_1$. Since the action of G is irreducible, there are loxodromics α, β with disjoint axes. Our subclaim is that there is a loxodromic, τ_Q, with axis, ℓ_Q, contained in $Q - u_1$. Let A be the axis of α and B the axis of β. If the orbit of A under G is disjoint from Q then there is an invariant subtree given by taking the intersection of all subtrees containing the orbit of A. This subtree does not contain Q contradicting minimality. Thus we may assume that A intersects Q. Then $A \cap Q$ is an infinite half-line, since the axis A can only enter or leave Q at the point u_1. We may replace α by α^{-1} if necessary to ensure that $\alpha(A \cap Q) \subset A \cap Q$. The axis of $\alpha^n\beta\alpha^{-n}$ is $\alpha^n(B)$. For n large $\alpha^n(B)$ contains points in Q and does not contain u_1 therefore it is contained in $Q - u_1$. This proves the subclaim. Similarly there is a loxodromic, τ_R, with axis, ℓ_R, contained in $R - u_1$.

Suppose that δ is an arc connecting ℓ_Q to ℓ_R. Then $interior(\delta)$ contains u_1 because $Q \cap R = u_1$. Furthermore δ is contained in $Q \cup R$. Hence δ intersects $[u_1, u_2]$ only at u_1. Choose δ to be the shortest arc connecting the axes of τ_Q and τ_R. The proof of (3.4) shows that the axis, ℓ_1, of the loxodromic $\tau_Q\tau_R$ contains δ. Therefore ℓ_1 intersects $[u_1, u_2]$ in the single point u_1. Otherwise the intersection is an interval which contains a subinterval of positive length in δ and this is impossible. Similarly one obtains a loxodromic with axis intersecting $[u_1, u_2]$ in the single point u_2. This proves the claim ■

4 Length Functions and Compactifications.

In this section we show how spaces of representations can be compactified by mapping the representation space into the space of length functions on the group and compactifying this latter space projectively.

Using the ideas in the previous section, any action, ρ, of a group G by isometries on a geodesic space $\mathcal{H}$ and any choice of a point, x, in $\mathcal{H}$ gives an equivariant straight map

$$\iota : K(G) \longrightarrow \mathcal{H}.$$

Let d be the pseudo-metric on $K(G)$ obtained by pull-back of the $d_{\mathcal{H}}$ metric. This gives an action, σ, of G on $K(G)$ by d-isometries. Given g in G the translation length $t(\sigma g)$ is at least as large as the translation length $t(\rho g)$. It may be larger, for example if $\rho(g)$ is loxodromic and no point on the axis of $\rho(g)$ in $\mathcal{H}$ is in $image(\iota)$. However there is a bound on the difference in the translation lengths of ρg and σg which depends only on the thin triangles constant Δ. We now prove this.

Lemma 4.1 *Suppose that $\mathcal{H}$ has Δ-thin triangles and suppose that τ is an isometry of $\mathcal{H}$ and x is any point in $\mathcal{H}$. Then there is a point y in $[x, \tau x]$ such that*

$$|d_{\mathcal{H}}(y, \tau y) - t(\tau)| \leq 28\Delta.$$

Proof If $d(x, \tau x) = t(\tau)$ then $y = x$ works. Otherwise choose z in $\mathcal{H}$ such that

$$d(z, \tau z) \;<\; \min\left(d(x, \tau x), t(\tau) + \Delta\right).$$

Every point in the orbit $\{\ \tau^n z\ \}$ of z under the group generated by τ is moved by τ the same distance that z is moved. Thus we may assume that z is chosen in this orbit so that it is almost the closest point in the orbit to x in the sense that $d(z, x) < d(\tau^n z, x) + \Delta$ for all n.

The first case is that there are points y in $[x, \tau x]$ and w in $[z, \tau z]$ such that $d(y, w) \leq 2\Delta$. Now

$$d(w, \tau w) \leq d(w, \tau z) + d(\tau z, \tau w) = d(w, \tau z) + d(z, w) = d(z, \tau z).$$

Also

$$d(y, \tau y) \leq d(y, w) + d(w, \tau w) + d(\tau w, \tau y) = 2d(y, w) + d(w, \tau w).$$

Combining these gives

$$d(y, \tau y) \leq 4\Delta + d(z, \tau z) \leq 4\Delta + t(\tau) + \Delta.$$

This proves that this y works in this case.

Next we prove that if the first case does not hold then $d(z, \tau z) \leq 20\Delta$. Assume the contrary. Let a, b be the points on $[z, \tau z]$ with $d(z, a) = d(b, \tau z) = 5\Delta$. Let a' be the point on $[x, z]$ with $d(z, a') = 5\Delta$ then we claim $d(a, a') \leq 4\Delta$. In the thin quadrilateral with vertices $x, \tau x, \tau z, z$ we have that every point on $[z, \tau z]$ is within a distance of 2Δ of the union of the two sides $[x, z]$ and $[\tau x, \tau z]$. This is because the first case does not hold thus no point on $[z, \tau z]$ is within 2Δ of $[x, \tau x]$. In particular there is some point, v say, in $[z, x]$ or $[\tau z, \tau x]$ with $d(a, v) \leq 2\Delta$. Refer to Figure 2 in what follows. First suppose that v is in $[\tau z, \tau x]$ then:

$$\begin{aligned} d(v, \tau z) &\geq d(a, \tau z) - d(a, v) \\ &= d(z, \tau z) - d(z, a) - d(a, v) \\ &\geq 20\Delta - 5\Delta - 2\Delta \\ &= 13\Delta. \end{aligned}$$

Hence

$$\begin{array}{rcl} d(\tau^{-1}z,x) & = & d(z,\tau x) \\ & \leq & d(z,a)+d(a,\tau x) \\ & \leq & 5\Delta + d(a,v)+d(v,\tau x) \\ & \leq & 5\Delta+2\Delta+(d(\tau z,\tau x)-d(v,\tau z)) \\ & \leq & 7\Delta + d(z,x) - 13\Delta \\ & = & d(z,x)-6\Delta. \end{array}$$

This implies that $d(z,x) > d(\tau^{-1}z,x)+\Delta$ which contradicts the choice of z, proving the subclaim that v is in $[z,x]$. Since $d(a,z)=5\Delta$ and $d(a,v)\leq 2\Delta$ it follows that $3\Delta \leq d(v,z) \leq 7\Delta$. Since v and a' are both on $[z,x]$ and $d(a',z)=5\Delta$ it follows that $d(v,a')\leq 2\Delta$. Hence $d(a,a')\leq 4\Delta$ as claimed. Now define $b'=\tau a'$ thus b' is the point on $[\tau x,\tau z]$ with $d(\tau z,b')=5\Delta$. A similar argument shows that $d(b,b')\leq 4\Delta$. It now follows that

$$\begin{array}{rcl} d(a',\tau a')=d(a',b') & \leq & d(a',a)+d(a,b)+d(b,b') \\ & \leq & 4\Delta+(d(z,\tau z)-d(z,a)-d(\tau z,b))+4\Delta \\ & \leq & 8\Delta+(d(z,\tau z)-10\Delta) \\ & \leq & d(z,\tau z)-2\Delta \\ & < & (t(\tau)+\Delta)-2\Delta \\ & < & t(\tau). \end{array}$$

It is impossible that τ moves a' less than $t(\tau)$ thus $d(z,\tau z)\leq 20\Delta$. It remains to show y exists with this hypothesis.

Refer to Figure 3 in what follows. Since the first case does not hold it follows that every point on $[x,\tau x]$ is within 2Δ of the union of $[x,z]$ and $[\tau x,\tau z]$. Hence there is some point y on $[x,\tau x]$ which is within a distance of 2Δ of a point c on $[x,z]$ and also of a point e on $[\tau x,\tau z]$. This is proved by an easy continuity argument. Now

$$\begin{array}{rcl} \mid d(e,\tau z)-d(c,z)\mid & \leq & d(e,c)+d(\tau z,z) \\ & \leq & d(e,y)+d(y,c)+20\Delta \\ & \leq & 24\Delta. \end{array}$$

This choice of y works because:

$$\begin{array}{rcl} d(y,\tau y) & \leq & d(y,e)+d(e,\tau c)+d(\tau c,\tau y) \\ & = & d(y,e)+d(e,\tau c)+d(c,y) \\ & \leq & 2\Delta+d(e,\tau c)+2\Delta \\ & = & 4\Delta+|d(e,\tau z)-d(\tau c,\tau z)| \\ & = & 4\Delta+|d(e,\tau z)-d(c,z)| \\ & \leq & 4\Delta+24\Delta. \end{array}$$

Observe that we use that $e,\tau z, tauc$ all are on $[\tau x,\tau z]$ to obtain the second equality. ■

Corollary 4.2 *Suppose that G is a group and K the complete graph with vertex set G. Suppose that the geodesic space $\mathcal{H}$ has Δ-thin triangles. The action of G on itself by left multiplication extends to an action, σ, on K. Suppose that ρ is an action of G by isometries on $\mathcal{H}$. Suppose that $\iota : K \longrightarrow \mathcal{H}$ is a G-equivariant straight map. Then using the pull-back under ι of the metric on $\mathcal{H}$ gives a pseudo-metric, d, on K, and σ is an action by d-isometries. Then for all g in G*

$$|t(\sigma g)-t(\rho g)|\leq 28\Delta.$$

Proof Given g in G since ι is an isometry onto its image it follows that $t(\rho g) \leq t(\sigma g)$. Consider the edge $[1, g]$ in K. Then $\iota([1, g]) = [x, \rho(g).x]$ where $x = \iota(1)$. The lemma (4.1) gives a point $y = \iota(z)$ such that

$$t(\sigma g) \leq d_K(z, \sigma(g).z) = d_{\mathcal{H}}(y, \rho(g).y) \leq t(\rho g) + 28\Delta.$$

■

A *length function* on a group G is a function $L : G \longrightarrow \mathbb{R}$. The set of length functions on G is denoted $\mathcal{LF}(G)$ and equals $\prod_{g \in G} \mathbb{R}$ with the product topology. Thus a sequence L_n in $\mathcal{LF}(G)$ converges to L if for all g in G we have that $L_n(g)$ converges to $L(g)$.

It is clear that conjugate isometries of X have the same translation length, thus some authors regard a length function as a real valued map from *conjugacy classes.* Given a representation $\rho : G \longrightarrow Isom(X)$ we define the *length function* $L_\rho : G \longrightarrow \mathbb{R}$ associated to ρ by

$$L_\rho(g) = t(\rho g)$$

for g in G. This in turn defines a map

$$L : Hom(G, Isom(X)) \longrightarrow \mathcal{LF}(G)$$

by $L(\rho) = L_\rho$. We will consider the cases that X is $\mathcal{H}$ or an $\mathbb{R}$-tree.

In the previous section we introduced rescaling factors to obtain limiting $\mathbb{R}$-trees. Thus we wish to discuss convergence of length functions in a projective sense. Note that $\mathbb{R} - 0$ acts on $\mathcal{LF}(G)$ by multiplication. The space of *projectivized length functions* on a group G, denoted $\mathcal{PLF}(G)$ is the set of projective equivalence classes of non-zero length functions

$$\mathcal{PLF}(G) \equiv \frac{\mathcal{LF}(G) - 0}{\mathbb{R} - 0}.$$

This is given the quotient topology (of the subspace topology on $\mathcal{LF}(G) - 0$). Thus a sequence of length functions L_n define projective classes which converge to the projective class of a length function L if there are $\lambda_n \neq 0$ such that for all g in G

$$\lim \ \lambda_n L_n(g) \ = \ L(g).$$

We assume here that none of these length functions is identically zero.

The following is the main result of this section:

Proposition 4.3 *Suppose that G is a finitely generated group and $\mathcal{H}_n$ is a geodesic space with Δ-thin triangles. Suppose that $\rho_n : G \longrightarrow Isom(\mathcal{H}_n)$ is a sequence of representations which blows up. Let K be the complete graph with vertex set G. Let x_n be an approximate center of ρ_n and $\iota_n : K \longrightarrow \mathcal{H}_n$ an equivariant straight map such that $\iota_n(g) = \rho_n(g).x_n$. Let d_n be the metric on K obtained by pulling-back via ι_n the metric $d_{\mathcal{H}_\backslash}$. Then there is a subsequence, still denoted $\{\rho_n\}$, and $\lambda_n \to 0$ so that $\lambda_n d_n$ converges pointwise to a pseudo-metric $\overline{d}$ on K. Furthermore, the metric space associated to $(K, \overline{d})$ is an $\mathbb{R}$-tree Γ. The action of G on K descends to an action $\sigma : G \longrightarrow Isom(\Gamma)$ such that some element of G has non-zero translation length on Γ. Furthermore $\lim_{n\to\infty} L(\rho_n) = L(\sigma)$ in $\mathcal{PLF}(G)$. Suppose, in addition, that σ is irreducible and that $L(\sigma)$ is integer-valued. Then the minimal G-invariant subtree of Γ is a simplicial tree on which G acts with the same length function as σ.*

Proof The existence of the $\mathbb{R}$-tree Γ comes from (3.1) and (3.6) gives the action, σ, of G on Γ and guarantees that for some element g of G that σg is loxodromic. Thus $t(\sigma g) > 0$ so $L(\sigma) \neq 0$ defines a (non-zero !) projective length function. Now (4.2) implies that $L(\rho_n) \to L(\sigma)$ in $\mathcal{PLF}(G)$. To see this, let

$$\sigma_n : G \longrightarrow Isom(K, \lambda_n d_n)$$

be the action action of G on K, with K regarded as a pseudo-metric space using the rescaled pseudo-metric $\lambda_n d_n$. Then $|t(\sigma_n g) - \lambda_n t(\rho_n g)| \leq \lambda_n 28\Delta$. Now $\lambda_n 28\Delta \to 0$ and so $L(\sigma_n) - \lambda_n L(\rho_n) \to 0$ in $\mathcal{LF}(G)$. Now $L(\sigma_n) \to L(\sigma)$ in $\mathcal{LF}(G)$ hence in $\mathcal{PLF}(G)$. Also $\lambda_n L(\rho_n)$ has the same limit in $\mathcal{PLF}(G)$ as $L(\rho_n)$. This proves the claim. If, in addition, σ is irreducible and $L(\sigma)$ is integer valued, then (3.8) supplies the G-invariant simplicial sub-tree. ■

We can embed $\mathcal{LF}(G)$ into

$$\overline{\mathcal{LF}(G)} \equiv \mathcal{LF}(G) \cup \mathcal{PLF}(G)$$

topologized so that a sequence $L_n \in \mathcal{LF}(G)$ converges to $[L] \in \mathcal{PLF}(G)$ if and only if there are $\lambda_n \neq 0$ such that $\lambda_n L_n \to L$ in $\mathcal{LF}(G)$. Then (4.3) gives:

Corollary 4.4 *Given a finitely generated group, G, and $\Delta > 0$ define $Hom_\Delta(G)$ to be the set of all homomorphisms of G into $Isom(X)$ for all geodesic spaces X with Δ-thin triangles. Then the image of $L : Hom_\Delta(G) \longrightarrow \mathcal{LF}(G)$ has compact closure in $\overline{\mathcal{LF}(G)}$ with ideal points consisting of the projective classes of all actions which have no global fixed point of G on $\mathbb{R}$-trees .*

Proof That ideal points are actions on $\mathbb{R}$-trees follows from (4.3).

It was shown in [6] that if a sequence of representations of G into $SL_2(\mathbb{C})$ does not blow up then there is a subsequence which, after suitable conjugacy, converges to a representation of G into $SL_2\mathbb{C}$. See [3] corollary (2.1) for a geometric proof of this fact. Specializing (4.3) to this situation, if the sequence blows up then there is a subsequence which converges, in $\mathcal{PLF}(G)$, to an action of G on an $\mathbb{R}$-tree:

Theorem 4.5 *Suppose that G is a finitely generated group and $\rho_n : G \longrightarrow SL_2(\mathbb{C})$ is a sequence of representations. Then either there is a subsequence which converges in $Hom(G, SL_2\mathbb{C})$. Otherwise the sequence blows up and there is an action σ by isometries of G on an $\mathbb{R}$-tree Γ with the following properties.*

- *There is no point of Γ fixed by G; equivalently $L(\sigma)$ is not identically zero.*
- *There is a subsequence ρ_{n_i} such that $\lim L(\rho_{n_i}) = L(\sigma)$ in $\mathcal{PLF}(G)$.*

If in addition $L(\sigma)$ is integer valued, and σ is irreducible, then Γ may be chosen to be a simplicial tree.

The hypothesis that the translation length function take integer values, and that σ is irreducible, will be established using valuations in the next section under certain circumstances.

5 Length Functions and Valuations

In this section we show that given a curve, C, in $Hom(G, SL_2\mathbb{C})$ satisfying certain mild hypotheses and given an end ϵ of C then the rescaled length functions of representations approaching ϵ converge to an integer valued function. This is the translation length function of a certain irreducible action on a simplicial tree.

The translation length of an element A of $SL_2\mathbb{C}$ having large trace is approximately twice the logarithm of the absolute value of its trace. Now trace is a polynomial function on the representation variety and thus we are concerned with the logarithm of a polynomial when the variables are large. The logarithm of a polynomial of one complex variable is approximately the degree of the polynomial times the logarithm of the variable when the variable is large.

For a polynomial p of several independent variables the asymptotic behaviour of $log|p|$ may be rather complicated. However the restriction of $log|p|$ to a *complex curve* behaves in this respect like a polynomial function of one variable. The notion of the degree of a polynomial of one variable is replaced by evaluating a certain *valuation* on the polynomial. This valuation gives the rate of growth of the polynomial on the curve as one goes to infinity along the curve towards a particular topological end of the curve. Of course all this is standard elementary commutative algebra.

We will now give a geometric interpretation of the valuations we need. Let C be a complex curve in $\mathbb{C}^n$ and let $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a linear projection such that $z = (\pi|C)$ is proper. It is not hard to see that such a projection always exists. Then C is a smooth 2-dimensional manifold except at finitely many points (see [10]) and the derivative of z is non-zero except at finitely many points. Let d be the degree (as a map) of z, which is the same as the number of pre-images of a regular value since z is complex differentiable.

Proposition 5.1 *Let C be a complex affine algebraic curve in $\mathbb{C}^n$. Then there is an integer $d > 0$ with the following property. Let $y : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial and let $\gamma : [0, \infty) \longrightarrow C$ be a proper arc. Then* $\lim_{t\to\infty} log|y(\gamma t)|/log|z(\gamma t)| = b/a$ *for some integers a, b with $|a| \leq d$.*

Proof The closure of the image of C in $\mathbb{C}^2$ under the polynomial map (y, z) is a complex curve K, (see [10]). A curve in $\mathbb{C}^2$ is the zero set of a polynomial, thus K is the zero set of a polynomial $p(y, z)$. The *Newton polygon, $Newt(p)$*, of p is the convex hull of the finite set of points in the plane:

$$\{(m, n) : \text{the coefficient of } \mathrm{y^m z^n} \text{ in } \mathrm{p(y,z)} \text{ is not zero }\}.$$

Since $z = (\pi|C)$ is proper, for t large $|z(\gamma t)|$ is large. Thus in looking at the order of magnitude of a term in $p(y(\gamma t), z(\gamma t))$ with t large, we may ignore the modulus of the coefficients in p. The order of magnitude of $y^m z^n$ is $m\ log|y| + n\ log|z| \equiv \theta(m, n)$ which is a linear function of (m, n). Thus for a given large t the order of magnitude of the term $a.y^m z^n$ in $p(y, z)$ is approximately given by the linear map $\theta(m, n)$. Now $p(y(\gamma t), z(\gamma t)) = 0$ hence there cannot be a single term of $p(y, z)$ which is an order of magnitude larger than all other terms in $p(y, z)$. The terms of approximately largest magnitude in $p(y, z)$ nearly lie along a level set for the linear function θ. This implies that the terms of largest order of magnitude are those terms lying along some edge, e, of $Newt(p)$, and that this edge is almost contained in a level set of θ. As $t \to \infty$ the level sets of θ become more nearly parallel to the edge e. We claim that for all t sufficiently large, the edge e is the independent of t. For otherwise, as t increases and the edge e changes to a new edge e' and there is a value of t when the terms along both edge e and edge e' are the same order of magnitude, and this is impossible.

Let $p_e(y, z)$ be the *edge polynomial* consisting of the sum of those terms of p which lie along e. Now $p_e(y, z) = y^r z^s q(y^a z^b)$ where a, b are coprime integers such that e has slope b/a, and q is a polynomial of one variable with $q(0) \neq 0$. Since the terms in p_e are larger than any other terms in p it follows that $\lim_{t\to\infty} q(y^a(\gamma t)z^b(\gamma t)) = 0$. Hence $\lim_{t\to\infty} y^a z^b = \eta$ where η is a (necessarily non-zero) root of q. Now $log|y|, log|z| \to \pm\infty$ along γ so that $\lim_{t\to\infty} log|y(\gamma t)|/log|z(\gamma t)| = -b/a$. For fixed large z the equation $q(y^a z^b) = 0$ has $|a|$ distinct solutions, and thus $p(y, z) = 0$ also has $|a|$ distinct solutions. But since the degree of z is d we have $|a| \leq d$ as claimed. ■

Thus γ determines a map $D_\gamma : \{rational\ functions\ on\ \mathbb{C}^n\} \longrightarrow \mathbb{Q}$ by

$$D_\gamma(y) = \lim_{t\to\infty} log|y(\gamma t)|/log|z(\gamma t)|.$$

Take the smallest integer p such that $V_\gamma(y) = pD_\gamma(y)$ is an integer for all polynomials y. Observe that such p exists and that $p \leq d!$ by the above. We claim that changing the arc γ by a proper homotopy does not change V_γ. Suppose that γ' is properly homotopic to γ. Then the argument in (5.1) shows that there is an edge e for γ containing the terms of largest order of magnitude for large t, and $D_\gamma(y)$ is determined by this edge. Similarly there is an edge e' for γ'. The proper homotopy gives a 1-parameter family, γ_s, of proper arcs with $\gamma_0 = \gamma$ and $\gamma_1 = \gamma'$. Thus we get a 1-parameter family of edges e_s for these arcs. Now all the edges in this family are the same, by the same reasoning in (5.1) that the edge e was indepenent of t. This proves the claim.

Thus the map $V_\epsilon \equiv V_\gamma$ depends only on the end, ϵ, of C to which γ converges. Now ends of C correspond to ideal points of the smooth model $\tilde{C}$ of the projectivization of C and all we have done is to construct the (negative of the) valuation associated to an ideal point of $\tilde{C}$. Note that our sign convention means that $V_\epsilon(y) > 0$ iff $\lim_{t\to\infty} |y(\gamma t)| = \infty$. We say that a function $f(t)$ *blows up* as $t \to \infty$ if $\lim_{t\to\infty} |f(t)| = \infty$. Thus a rational function values positive if it blows up as you go out to the end of the curve in question.

Let G be a finitely generated group and $Hom(G, SL_2\mathbb{C})$ the affine algebraic set of representations $\rho : G \longrightarrow SL_2\mathbb{C}$, following common usage we will call this the *representation variety.* Let C be a curve in $Hom(G, SL_2\mathbb{C})$ and ϵ an end of C such that there is some element $\alpha \in G$ for which $trace(\rho\alpha) \to \infty$ as $\rho \to \epsilon$. Let V_ϵ be the associated valuation. It is an observation of Culler and Shalen that that if one works with a curve of characters instead of representations, then every end has this property eg. by [3] Corollary 2.1.

We next show that for a representation near an end, ϵ, of C the translation lengths of elements of G are approximately certain integer multiples of a large parameter. The integer assigned to an element α of G is the twice the valuation associated to ϵ of the trace of α. The formula for translation length of an element A of $SL_2\mathbb{C}$ is

$$t(A) = 2cosh^{-1}(|trace(A)|/2).$$

For $trace(A)$ large this is approximately $2ln|trace(A)|$. Our primary concern is with elements of G whose translation length goes to infinity as one goes out along C towards ϵ. Such elements act on the limiting tree with non-zero translation length given by this integer. However in a subsequent paper we will also be concerned with elements of G whose translation length is constant or approaches zero near ϵ. We will be interested in how fast the translation length goes to zero as one goes out into the end ϵ.

This is why, following Culler and Shalen, we introduce the function

$$f_\alpha(\rho) = (trace\ \rho\alpha)^2 - 4$$

then

$$t(\rho\alpha)/ln|f_\alpha(\rho)| \approx 1$$

when $trace(\rho\alpha)$ is large. If $trace(A) \approx \pm 2$ this measures how fast the translation length goes to zero.

Let ρ_n be a sequence converging to an end ϵ of C. Using the above approximation to $t(\rho_n\alpha)$ and proposition (5.1) there are λ_n such that for all α in G

$$\lim_{n\to\infty} \lambda_n t(\rho_n\alpha) = \max\{V_\epsilon(f_\alpha), 0\}.$$

The corresponding formula of Morgan and Shalen [9] Proposition [II.3.15] has a negative sign because our V is the negative of the valuation that they use. By (4.5) there is a subsequence which converges to an action $\sigma : G \longrightarrow Aut(\Gamma)$ on an $\mathbb{R}$-tree Γ. Hence

Proposition 5.2 *Let G be a finitely generated group and let C be a an affine algebraic curve of representations of G into $SL_2\mathbb{C}$. Suppose that ϵ is an end of C and that ρ_n is a sequence in C which blows up and converges to ϵ. Define a length function on G by $p(g) = \max\{V_\epsilon(f_g), 0\}$ for each g in G. Then there is an action, σ, of G on an $\mathbb{R}$-tree such that $\lim_{\rho\to\epsilon} L(\rho) = L(\sigma) = L(p)$ in $\mathcal{PLF}(G)$.*

Proof By (4.3) there is a subsequence ρ_{n_i} and an action σ of G on an $\mathbb{R}$-tree such that $\lim_{i\to\infty} L(\rho_{n_i}) = L(\sigma)$ in $\mathcal{PLF}(G)$. From the above discussion $\lim_{i\to\infty} L(\rho_{n_i}) = \lim_{\rho\to\epsilon} L(\rho) = L(p)$. ■

In particular this means that we may scale metric on the $\mathbb{R}$-tree $(\Gamma, \overline{d})$ obtained as a limit of the representations ρ_n so that the translation length function for it is integer valued.

Lemma 5.3 *Suppose that the representations $\rho_n : G \longrightarrow SL_2\mathbb{C}$ lie on a curve, C, in the representation variety and suppose that α is an element of G and that $trace(\rho_n\alpha) \to \infty$. Suppose that there is a representation in C with image which is not virtually solvable. Supppose that σ is an action of G on an $\mathbb{R}$-tree, Γ, such that no point of Γ is fixed by G. Suppose that $L(\rho_n)$ converge to $L(\sigma)$ in $\mathcal{PLF}(G)$. Then σ is irreducible.*

Proof We may suppose that a subsequence of ρ_n has been chosen which converges to some end ϵ of C. Since C contains a representation which is not virtually solvable then for ρ near ϵ we have that ρ is not virtually solvable. The Tits alternative says that a subgroup of a linear group is either virtually solvable or else contains a non-abelian free subgroup. (The proof of this for $SL_2\mathbb{C}$ is elementary.) Thus there is a conjugate, β, of α such that the fixed points of $\rho\beta$ in S^2_∞ are both distinct from those of $\rho\alpha$. Conjugate ρ so that

$$A = \rho(\alpha) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \qquad B = \rho(\beta) = \begin{pmatrix} b+c & b^2 - c^2 - 1 \\ 1 & b - c \end{pmatrix}.$$

This may be done so that $a \to \infty$ as $\rho \to \epsilon$. Then

$$t_p \equiv trace(A^pBA^{-p}B^{-1}) = (1 - b^2 + c^2)(a^{2p} + a^{-2p}) + 2(b^2 - c^2).$$

Observe that $trace(\rho\beta) = 2b$ thus b may be regarded as a polynomial function on C. Similarly $trace(\rho\alpha\beta) = a(b + c) - (b - c)/a$ so that c may also be regarded as a rational function on C.

We claim that for all sufficiently large p that $t_p \to \infty$ near ϵ. To see this, note that if $(1-b^2+c^2) = 0$ then B is lower triangular hence fixes $0 \in S^2_\infty$. But A fixes 0 also, and this contradicts the choice of β. Let V_ϵ be the valuation assoicated to ϵ. Given two functions, one of which grows faster than the other, then the rate of growth of their sum equals the rate of growth of the larger one, thus

$$V_\epsilon(f) > V_\epsilon(g) \quad \Longrightarrow \quad V_\epsilon(f + g) = V_\epsilon(f).$$

For p sufficiently large we have

$$V_\epsilon((1 - b^2 + c^2)(a^{2p} + a^{-2p})) = V_\epsilon(1 - b^2 + c^2) + V_\epsilon(a^{2p} + a^{-2p}) > V_\epsilon(2(b^2 - c^2)).$$

This is because $a \to \infty$ and so $V_\epsilon(a^{2p} + a^{-2p}) \geq 2p$. Thus for p large, $V_\epsilon(t_p) > 0$ so $t_p \to \infty$ near ϵ as claimed.

It follows that $\sigma\alpha^p$ and $\sigma\beta$ are loxodromics and that their commutator is also loxodromic. Thus the axes of these two elements intersect in at most a compact interval. Then $\sigma\alpha^p$ and a conjugate of it by any sufficiently large power of $\sigma\beta$ have disjoint axes. Hence σ is irreducible. ■

In summary we have the main theorem about compactifying curves of representations:

Theorem 5.4 *Let C be a complex curve in $Hom(G, SL_2\mathbb{C})$ such that*

- *There is a representation on C which is not solvable.*
- *There are two representations on C which are not conjugate.*

Given an end ϵ of C then there is an irreducible action, σ, of G by isometries on a simplicial tree with the following property. Suppose that ρ_n is any sequence in C converging to ϵ. Then $\lim L(\rho_n) = L(\sigma)$ in $\mathcal{PLF}(G)$.

We remark that if all the representations on C are virtually solvable then it is easy to show that there is an action σ of G by isometries on $\mathbb{R}$ such that $\lim L(\rho_n) = L(\sigma)$.

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Math Dept
University of California
Santa Barbara
CA. 93106
U.S.A.

email : coopermath.ucsb.edu