NOTES ON WORD HYPERBOLIC GROUPS

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Attention (January 2004 (sic!!)) : *I have included the pictures as ps files by enormous public demand (namely Richard Weidmann and Indira Chatterji — two in the same day!) who say the book is difficult to find. There are still MANY misprints — some very annoying ... please let me know and I will correct them. I must add that I for one had a lot of fun all those years ago in those seminars in MSRI. Note that I have not changed the institutions of the authors.*
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This is a record of a series of seminars held at the Mathematical Sciences Research Institute during the spring of 1989, as part of the program on Combinatorial Group Theory and Geometry. This series followed on from, and interacted with, the previous series of seminars on J.W. Cannon, D.B.A. Epstein, D.R. Holt, M.S. Paterson and W.P. Thurston's work on Automatic groups [CEHPT], under the direction of M. Shapiro. In those seminars, M. Gromov's hyperbolic groups were frequently cited as examples (see [BGSS]). Also S. M. Gersten's work on isoperimetric inequalities in groups at M.S.R.I stimulated interest in Gromov's work. These notes were subsequently revised after the meeting in Trieste in March 1990 for inclusion in the proceedings of that meeting.

The object of the seminars was to gain some understanding of the class of groups studied by Gromov in his important (and difficult) paper 'Hyperbolic groups' published in the volume "Essays on Group Theory" [G]. The class of groups studied is defined in geometric terms, usually making reference to the Cayley graph of a finitely generated group. The aim of the theory is to generalise results obtained for the fundamental groups of closed compact hyperbolic manifolds to some larger class, where techniques similar to those used in studying Kleinian groups may be useful. The class includes most of the small cancellation groups which have been subject to much study by some group theorists, and many results from that theory

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hold for all hyperbolic groups. In this way many of the ideas follow on from Dehn's work around 1910 (see Dehn's papers translated into English [De]).

The aim of these notes is to give an accessible introduction to the ideas of hyperbolic groups, accentuating the group theoretic approach. Hopefully these notes can be read by a final year undergraduate or beginning graduate student without too much pain and work. We presuppose some basic knowledge of metric spaces and of groups given by generators and relations, though this goes little deeper than the triangle inequality and the definition of free groups and group presentations.

There has been some discussion about the proper way to refer to this class of groups. In the first preprint version of these notes, we referred to 'negatively curved groups', following the suggestion of Epstein, Thurston, Cannon, Rips and Cooper. However subsequent litterature seems to favour the term 'word hyperbolic' or 'hyperbolic'.

Of course it was possible to work through only a small portion of the 200 or so pages of [G], and we have concentrated on establishing the basic definitions. At many points of [G], details are ommitted or left to the reader; we have tried to complete some of these.

We did not cover the important concepts of quasiisometry and geometric properties. We simply did not have time to cover this very basic idea, which is more than adequately covered in [GH, Chapter 1] and [Gh].

While the seminar in MSRI was being held, we benefitted from access to early versions of the notes being produced at the time by three other groups, usually in hand written form. These were by:

W. Ballman, E. Ghys, A. Haefliger, P. de la Harpe, E. Salem, R. Strebel and M. Troyanov [GH], notes of a series held at Berne, edited by E. Ghys and P. de la Harpe and published as a book recently by Birkhäuser;

B. Bowditch at Warwick [Bow], to appear elsewhere in this volume;

M. Coornaert, T. Delzant and A. Papadopoulos [CDP] at Strasbourg, to appear shortly as a book (Springer–Verlag).

While we did indeed benefit from the use of the above notes (especially in Chapter 4), many of the proofs here are original. Partly this is because we did not have the now complete versions of these notes, partly because we were interested in the group–theoretic, rather than the metric space aspect. Some other articles known to us at the moment on the subject are:

D. Cooper's preprint [C] on automorphisms of hyperbolic groups;

F. Paulin's work [P1], [P2];

J. Alonso's article 'Combings of groups' [A] which grew out of his talk on the Rips complex (section 4) (this contains another definition of a hyperbolic group which is not discussed in these notes — we refer the interested reader to [A] for this definition and the proof of its equivalence to those given here); in [A2] Alonso shows that the type of isoperimetric inequality satisfied by a group is invariant under quasiisometry.

J.W. Cannon's notes from the Trieste 1989 meeting [Can2].

E. Ghys' Bourbaki seminar on hyperbolic groups contains, as well as the excellent main text, an extensive bibliography [Gh].

M. Bestvina and G. Mess' paper on the boundary of hyperbolic groups [BM].

M. Bestvina and M. Feighn's paper on obtaining hyperbolic groups from amalgamating two hyperbolic groups along a subgroup [BF].

S.M. Gersten and H. Short's article [GS] contains as an appendix the proof that the linear isoperimetric inequality implies that a group is hyperbolic, (included here as 2.4 – 2.7); this is also contained in I. G. Lysenok's article [L], as is a proof of the equivalence of the Dehn algorithm definition and some other interesting results.

G. Baumslag, S.M. Gersten, M. Shapiro and H.Short use properties of hyperbolic groups to show that the free product of two hyperbolic groups amalgamated along a cyclic subgroup is automatic.

The paper is divided up as follows:

The first chapter consists of a collection of alternative definitions, both of hyperbolic metric spaces and of hyperbolic groups including Gromov's inner product, slim and thin triangles, Cooper's diverging geodesics, the linear isoperimetric inequality, and Dehn's algorithm. The next chapter consists of proofs of the equivalence of these definitions of a hyperbolic metric space, and of a hyperbolic group. We also show here that the Dehn's algorithm definition gives immediately that there only a finite number of conjugacy classes of torsion elements, and also provides a time efficient algorithm for solving the word problem (a result originally due to Domanski and Anshel [DA]).

Some properties of quasigeodesics are developed in Chapter 3. These are used to establish the fact that the centralizer of an element of a hyperbolic group is cyclic–by–finite, and that thus there are no $\mathbf{Z} \times \mathbf{Z}$ subgroups in a hyperbolic group. We define the boundary of a hyperbolic metric space in Chapter 4, though we do not make much use of the construction to establish properties of hyperbolic groups, as is done say in [GH]. We finally build the Rips complex to show that a hyperbolic group is FP_∞. This gives another proof that there only a finite number of conjugacy classes of torsion elements.

Where possible we have tried to give references to original statements in Gromov's paper and to treatment of the topics in in [CDP] and in [GH].

Main differences between this version and the earlier MSRI preprint version of these notes:

Terminology : *negatively curved* has now become *hyperbolic* or *δ-hyperbolic*.

thin triangle has now become *slim* triangle. This is because we decided to return to Gromov's use of the term thin triangle — A. Haefliger suggested the 'slim' terminology. Thus *fine* triangle has become *thin* triangle.

There is much more work to be done before Gromov's work is properly understood; we hope that these notes be of some help to others working in this area. The various authors would like to also thank the other participants in the seminars for their contributions to the development of these notes. We would also like to thank T. Delzant for his talk in the series, though he wished to be absent from the list of authors.

Misprints and remaining mistakes in this written report on the activities of the seminar series are (mostly) due, of course, to

the editor, Hamish Short

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Chapter 0 Some definitions and notation

We assume that the reader knows what a free group is. If X is a finite set of generators for the group G, then there is a natural surjection $\mu : F(X) \to G$, where $F(X)$ is the free group on X.

We can construct a (geodesic) metric space called the Cayley graph $\Gamma_X(G)$ of G with respect to the generating set X (Dehn also called this the '*Gruppenbild*'). This graph has a vertex for each element of G, and an oriented edge labelled x from g to gx for each element $g \in G$ and each $x \in X$. The group G acts on $\Gamma_X(G)$ by multiplication on the left: the element $g \in G$ defines a map ϕ_g, which maps a vertex $h \in \Gamma_X(G)$ to the vertex gh; the endpoints of an edge go to the endpoints of an edge. Notice that in general multiplication on the right does not define an automorphism of the graph, as the endpoints of an edge are not in general sent to the endpoints of an edge. For more about Cayley graphs see for instance [MKS, 1.6].

A metric is defined by assigning unit length to each edge, and defining the distance between to points to be the minimum length of paths joining them (the space is clearly arc–connected).

With this metric, the left–action of G on $\Gamma_X(G)$ is by isometries.

We define the *length* of an element g of G with respect to the generators X, written $|g|_X$, to be the length of the shortest word in $F(X)$ representing g: i.e. $|g|_X = \min\{\ell(w) \mid w \in F(X),\ \mu(w) = x\}$. The distance between two vertices corresponding to elements $h, h' \in G$ is then $d(h, h') = |h^{-1}h'|_X$; this is called a *word metric* on G. The fact that the left action is by isometries can now be seen by noticing that $d(gh, gh) = |(gh)^{-1}(gh')|_X = |h^{-1}h'|_X$.

Examples

The Cayley graph of a free group with respect to a free basis is a tree.

The Cayley graph of $\mathbf{Z} \times \mathbf{Z}$ with respect to the standard pair of generators x, y is the square grid of horizontal and vertical lines in the plane.

The Cayley graph of the fundamental group of a closed, orientable surface of genus g greater than 1 can be embedded in the hyperbolic plane in an natural way. Take a convex fundamental domain consisting of a regular polygon of $4g$ sides and corner angle $\pi/2g$. The group is generated by reflections in the sides of the polygon, and repeatedly reflecting in the sides fills out the hyperbolic plane. The dual graph to the tiling is the Cayley graph with respect to these generators.

The same phenomenon occurs in higher dimensional manifolds. For more about this and quasiisometries see the first chapter of [GH], and other articles elsewhere in this volume.

Chapter 1. Some Notions of Hyperbolicity

We consider a path-connected metric space with distance function d. Always in mind is the example of the Cayley graph of a finitely generated group. We wish also to be able to talk about geodesic (i.e distance minimizing) paths between two points of the space, and in particular to be able to affirm their existence. We list a number of definitions which will later be shown to be equivalent. As usual different definitions will be useful in different contexts.

We say that a metric space X is a *geodesic* metric space if for all points x, y in X there is an isometric map from the interval $[0, d(x, y)]$ to a path in X joining x and y; that is, there is a path between the point x and y realising the distance $d(x, y)$. We denote an image of such an isometry by $[xy]$, and we use $d(w, [xy])$ to denote the distance of the point w from a geodesic arc $[xy]$ (notice that such a path is not necessarily unique — consider the Cayley graph of $\mathbf{Z} \times \mathbf{Z}$). For any path $\alpha : [0, n] \to X$ such that $\alpha : [0, n] \to \alpha([0, n])$ is an isometry, we call n the *length* of α, denoted by $\ell(\alpha)$. Thus for instance $\ell([xy]) = d(x, y)$ for geodesics $[xy]$.

Examples

A locally finite connected graph where the metric is induced by giving each edge unit length, is a geodesic metric space. The Euclidean plane with the usual metric, but the origin omitted is not a geodesic metric space.

We shall be particularly interested in the case when X is a Cayley graph of a finitely generated group (with respect to a finite generating set).

Definition 1.1 Inner product (Gromov [G, 1.1])

Given a base point $w \in X$, we define an *inner product* on X by

$$(x.y)_w \;=\; \frac{1}{2}(d(x, w) + d(w, y) - d(x, y)) \ .$$

If there is a constant $\delta \geq 0$ such that

$$\forall x, y, z \in X, \quad (x.y)_w \;\geq\; \min\{(x.z)_w, (z.y)_w\} \;-\; \delta.$$

we say that the inner product is *(δ) hyperbolic*.

Remark 1.2

(1) If X is a tree, we may take $\delta = 0$ (in fact this characterises an $\mathbf{R}$-tree (see for instance [GH, 1.6,7,8]).
(2) If w lies on a geodesic $[xy]$, then $(x.y)_w = 0$.
(3) Let $t \in [xy]$ such that $d(t, w) = d(w, [xy])$. Then

$$d(w, t) \;+\; d(t, x) \;\geq d(w, x) \;\text{ and }\; d(w, t) \;+\; d(t, y) \;\geq d(w, y).$$

As $d(x, t) \;+\; d(t, y) \;=\; d(x, y)$, adding these inequalities gives

$$d(w, [xy]) \;=\; d(w, t) \;\geq\; (x.y)_w \ .$$

We shall show that this definition is independent of base point; i.e. if the inner product is (δ) hyperbolic with respect to one base point, then it is (2δ) hyperbolic with respect to any base point.

In the standard hyperbolic plane $\mathbf{H}^2$, triangles do not have the same properties as triangles in the Euclidean plane. For instance in the Euclidean plane, in a large isosceles Euclidean triangle, the mid–point of the hypotenuse is far away from the other two sides. This cannot happen in hyperbolic space. This property gives rise to the following definition (the use of the word 'slim' is suggested by A. Haefliger):

<u>Definition 1.3 Slim Triangles</u> (attributed to Rips)

Given any three points x, y, z in X, we say that a triangle xyz of geodesics joining these points is *δ-slim* if for any point w on $[xy]$ we have that $min(d(w,[xz]), d(w,[yz])) \leq \delta$. We say that *triangles are slim* in X if there is a constant δ such that all geodesic triangles in X are δ-slim.

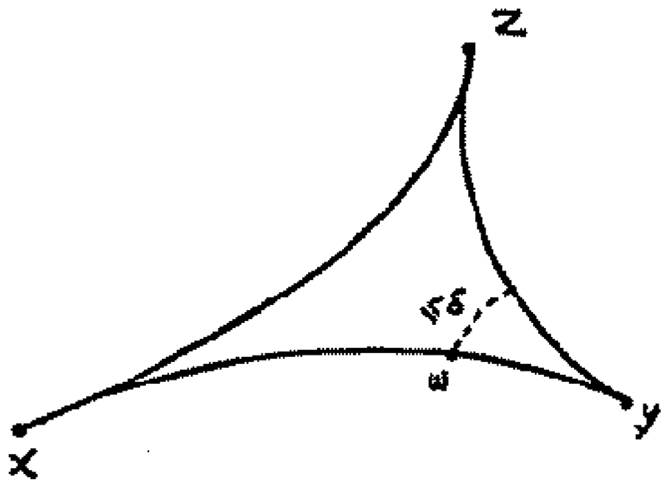

Figure 1.1

Now consider a slim triangle defined by the points x, y, z. Let

$$N^+ = \{p \in [xz] \text{ such that } d(p,[xy]) \leq \delta\}$$

and let

$$N^- = \{q \in [xz] \text{ such that } d(q,[zy]) \leq \delta\}\ .$$

These two closed sets cover $[xz]$, so there is some point $y' \in N^+ \cap N^-$. Then there are points $z' \in [xy]$, $x' \in [yz]$ such that $d(y', z') \leq \delta$ and $d(y', z') \leq \delta$. Thus the set $\{x', y', z'\}$ has diameter at most 2δ. This suggests the following definition:

<u>Definition 1.4 Minsize</u> ('taille minimale' [CDP, 1.3] , [GH, 2.18])

Let xyz be a geodesic triangle and let x', y', z' be points on xyz (x' on the side opposite vertex x etc.). Define the *minsize* of the triangle to be

$$minsize(xyz) = inf\ diam\{x', y', z'\}$$

where the infimum is taken over all triples of points $\{x', y', z'\}$.

Thus, if all geodesic triangles are δ-slim, then all geodesic triangles have $minsize \leq 2\delta$. We shall establish the converse below.

Definition 1.5 Thin triangles ([G, 6.3], 'fins' [CDP, 1.3], [GH, 2.16])

Given a geodesic triangle $\Delta = xyz$ in X, let $\Delta' = x'y'z'$ be a Euclidean comparison triangle with sides of the same lengths (i.e. $d_E(x', y') = d_X(x, y)$ etc., where d_E is the standard Euclidean metric). There is a natural identification map $f : \Delta \to \Delta'$. The maximum inscribed circle in Δ' meets the side $[x'y']$ (respectively $[x'z']$, $[y'z']$) in a point c_z (resp. c_y, c_x) such that

$$d(x', c_z) = d(x', c_y) \, , d(y', c_x) = d(y', c_z) \, , d(z', c_y) = d(z', c_x) \; .$$

We call the points c_x, c_y, c_z the *internal points* of xyz (here we are identifying c_x with $f^{-1}(c_x)$ etc.).

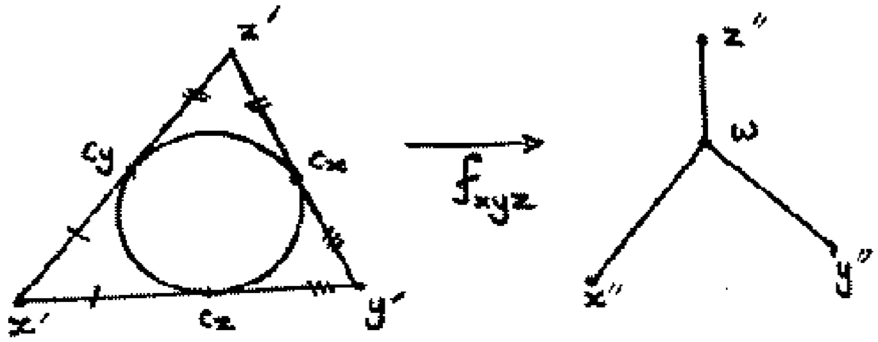

Figure 1.2

Notice that

$$d(x', c_z) \;=\; \frac{1}{2}(d(x', c_z) \;+\; d(x', c_y)) \;=\; \frac{1}{2}(d(x', z) \;+\; d(x', y) \;-\; d(z', y)) \; .$$

There is a unique isometry t_Δ of the triangle Δ' onto a *tripod* T_Δ, a tree with one vertex w of degree 3, and vertices x'', y'', z'' each of degree one, such that $d(w, z'') = d(z, c_y) = d(z, c_x)$ etc.. Let f_Δ be the composite map $f_\Delta = t_\Delta \circ f : \Delta \to T_\Delta$.

We say that xyz is *δ-thin* if the fibres of f_Δ have diameter at most δ in X. In other words, for all p, q in Δ,

$$f_\Delta(p) \;=\; f_\Delta(q) \;\Rightarrow\; d_X(p, q) \;\leq\; \delta \, .$$

We say that *triangles are thin* if there is a constant δ such that all geodesic triangles in X are δ-thin.

Definition 1.6 insize ([G, 6.5], 'taille interne' [CDP, 1.3], [GH, 2.18])

We define the *insize* of xyz to be

$$insize(xyz) \;=\; diam\{c_x, c_y, c_z\} \, .$$

Remark

(1) It is immediately clear that $minsize(xyz) \;\leq\; insize(xyz)$.
(2) If a triangle is δ-thin then its insize is $\leq \delta$.

Another way of characterising hyperbolic geometry is by the way in which infinite rays emmanating from a point diverge. In euclidean space, rays diverge linearly, while in hyperbolic space, rays diverge exponentially. To make this precise requires a rather complicated–looking definition. Consider two people walking along two geodesic rays at unit speed, starting at the same point. The distance between them at time t is at most $2t$ in any metric space, by the triangle inequality. But what we are interested in is the distance between them by following a path which is outside of the ball of radius t around the start point. What characterises a hyperbolic space is that once the distance between the two travellers crosses a certain threshold, the length of the path outside the ball of radius t grows exponentially in t. Here is the detailed definition:

For $\rho > 0$ and $x \in X$, let $B_\rho(x)$ denote the ball of radius ρ about the point x in X. Recall that we consider a path α in X of length $\ell(\alpha)$ as a local isometry $\alpha : [0, \ell(\alpha)] \to X$.

Definition 1.7 Geodesics Diverge

We say that $e : \mathbf{N} \to \mathbf{R}$ is a *divergence function* for X if for all points $x \in X$, and all geodesics $\gamma = [xy]\,, \gamma' = [xz]$, the function e satisfies the following condition.

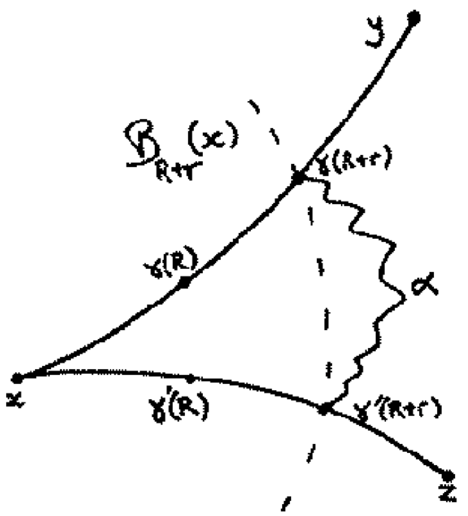

Figure 1.3

For all $R, r \in \mathbf{N}$ such that $R{+}r < min(\ell([xy]), \ell([xz]))$, if $d(\gamma(R), \gamma'(R)) > e(0)$, and α is a path in $\overline{X - B_{R+r}(x)}$ from $\gamma(R{+}r)$ to $\gamma'(R{+}r)$, then we have $\ell(\alpha) > e(r)$.

We say that *geodesics diverge exponentially* if there is an exponential divergence function.

Notice that in the Euclidean plane, there is no divergence function.

D. Cooper has suggested a variation of this, where we say that *geodesics diverge supralinearly* if there is a divergence function $e(r)$ such that $\lim_{r\to\infty} e(r)/r = \infty$. We shall establish the rather surprising fact that supralinear divergence is equivalent to exponential divergence, and we shall just say that *geodesics diverge.*

Brian Bowditch has an interesting variant on this definition ([Bow]).

Some Definitions Relevant to Word Hyperbolic Groups

The case in which we are interested here is when the geodesic metric space under consideration is the Cayley graph $\Gamma_X(G)$ of a group G with respect to a finite generating set X.

Definition 1.8 Word hyperbolic groups

We say that a group G is *word hyperbolic* (often abbreviated to *hyperbolic*) if it has a finite set of generators X such that the corresponding Cayley graph $\Gamma_X(G)$ is a geodesic metric space with a δ-hyperbolic inner product, for some δ.

It follows from the definition that hyperbolic groups are finitely presented: this is shown in 2.18.

Yet another difference between hyperbolic and Euclidean geometries is the ratio of area to circumference of a circle (or polygon); in the Euclidean plane the area is a quadratic function of the circumference, whereas in the hyperbolic plane it is a linear function. This gives a further characterisation of a hyperbolic group, once we formulate a concept of area in a group.

Definition 1.9 Linear Isoperimetric Inequality (see [G, 2.3], [CDP, chap. 6])

Let $\langle X; R\rangle$ be a finite presentation of the group G with X finite. If w is a freely reduced word w in $F(X)$ of length $\ell(w)$, the free group on X, and $\overline{w} = 1$ in G, then there are words $p_i \in F(X)$, relators $r_i \in R$, and $\epsilon = \pm 1$ such that

$$w = \prod_{i=1}^{N} p_i r_i^{\epsilon_i} p_i^{-1} \quad \text{in} \quad F(X).$$

If there is a constant K such that for all such words w, $N < K.\ell(w)$, we say that G *satisfies a linear isoperimetric inequality* . (Compare [G].)

The reason for the restriction to the finitely presented case is that otherwise one could just throw in as relators of the presentation all words in $F(X)$ which represent the trivial element of G. This would give a very uninteresting linear isoperimetric inequality for any group.

Isoperimetric inequalities are more fully discussed in 2.4 – 2.7, where it is shown that a finitely presented group which satisfies a linear isoperimetric inequality is word hyperbolic. S.M. Gersten has developped a more general study of isoperimetric inequalities in [Ge]. If some finite presentation of a group satisfies a linear isopermetric inequality, then all finite presentations do, as can be seen by applying Tietze transformations (see [Ge]). (Care is required: notice that a free group satisfies a zero isoperimetric inequality with respect to a free basis, but a linear term is added when the basis is changed.) This shows that the definition is independent of generating set. More generally, Alonso has shown [A2] that the type of isoperimetric inequality satisfied by a group is invariant under quasiisometry.

It is clear that a free group has a linear isoperimetric inequality.

It is pointed out in [BGSS] that it follows from Newman's spelling theorem (see e.g. [LS]) that one relator groups with torsion are word hyperbolic.

One of the reasons Gromov gives for his study of hyperbolic groups is a desire to generalize small cancellation theory ([G, 0.4]). This latter theory has its origins in Dehn's solution of the word problem for the fundamental groups of surfaces, which was generalized by Greendlinger and Lyndon in the 1960s (see [LS, Chapter V] or

R. Strebel's appendix to [GH] for more details). We give the conditions used by this theory here. Their main utility resides in the fact that they give easily checked conditions on a presentation which ensure hyperbolicity. Unfortunately the class of groups so defined is somewhat limited: a torsion–free C(7) small cancellation group has cohomological dimesion 2. Gromov indicates in [G, 0.2] that the class of hyperbolic groups is much larger than the class of C(7) small cancellation groups, while at the same time stating that hyperbolic presentations are generic. It would be interesting to formalise and understand some idea of the genericity of hyperbolic groups sketched by Gromov in these statements.

Definition 1.10 Small Cancellation Conditions

Given a finite presentation $\mathcal{P} = \langle X; R \rangle$, let $\mathcal{R}$ denote the cyclic closure of R, i.e. the set of cyclic conjugates of elements of R and their inverses. A *piece* is a non-trivial word $v \in F(X)$ such that there are two different relators $r_1, r_2 \in \mathcal{R}$ such that $r_1 = vr_1'$ and $r_2 = vr_2'$.

We say that $\mathcal{P}$ satisfies the $C(p)$ condition if no element of $\mathcal{R}$ is a product of fewer than p pieces. We say that $\mathcal{P}$ satisfies the $C'(1/p)$ if for each piece v occurring in the relator r, $p\ell(v) < \ell(r)$. Thus if the C$'(1/p)$ condition holds, then so does the C$(p+1)$ condition.

Example The surface group of genus $g > 1$ has a presentation

$$\langle a_1, \ldots, a_g, b_g, \ldots b_g \mid \prod_{i=1}^{i=g} a_i b_i a_i^{-1} b_i^{-1} \rangle.$$

A maximal piece consists of a single letter, and so this presentation satisfies the condition C$(4g)$, and also the condition C$'(\frac{1}{4g-1})$.

The presentation satisfies the T(q) condition if for any sequence $r_1, r_2, \ldots, r_k$ of elements of $\mathcal{R}$ with $k < q$, such that

$$r_1 = a_1 r_1' a_2^{-1}, r_2 = a_2 r_2' a_3^{-1}, \ldots, r_k = a_k r_k' a_1^{-1},$$

where $a_i \in X \cup X^{-1}$, for some $j \le k$, we have that $r_j = r_{j+1}^{-1}$ in $F(X)$ (suffices considered mod k). Notice that the T(3) condition is void.

Example The presentation $\langle x, y \mid xyx^{-1}y^{-1} \rangle$ of the group $\mathbf{Z} \times \mathbf{Z}$ satisfies the conditions $C(4) - T(4)$.

It is "well known" that a group which satisfies one of the C(7),C(5)–T(4),C(4)–T(5),C(3)–T(7) small cancellation conditions satisfies a linear isoperimetric inequality (see for instance [GS]). In the metric case (i.e. when for instance C$'(1/6)$, C$'(1/4)$–T(4) or C$'(1/4)$–T(7) are satisfied), the main lemma of small cancellation theory states that a word $w \in F(X)$ which represents the trivial element of the group contains a subword which is more than half of some cyclic conjugate of a relator.

It follows that fundmental groups of compact surfaces of genus greater than 1 are hyperbolic.

In a series of papers studying the word problem for surface groups at the beginning of this century, Max Dehn studied the connection between hyperbolic geometry and surface groups (see [De]). There he gave a solution to the word problem, which we shall generalise here:

Definition 1.11 Dehn's Algorithm

A *Dehn presentation* for the group G is a finite presentation $\langle X; R \rangle$ such that any non-trivial word in $F(X)$ which represents the identity element of G contains more than half of some word in R. That is, if $w \in F(X)$ is a reduced word, and $\mu(w) = 1$ in G, then there is a relation $r = r_1 r_2 \in R$ with $\ell(r_1) > \ell(r_2)$, such that $w = w_1 r_1 w_2$.

A group is said to have a Dehn's algorithm if it has a Dehn presentation.

It is clear that a group with a Dehn's algorithm satisfies a linear isoperimetric inequality (with multiplicative constant 1).

We shall show (2.12) that a group is hyperbolic if and only if it has a Dehn's algorithm. This was also established by Lysenok [L] and Cannon (see [Can2]).

Chapter 2. The Equivalence of the definitions

We now show the equivalence of several of the initial definitions concerning δ-hyperbolic metric spaces. We begin by showing the equivalence of some of the properties of geodesic triangles. Most of the proofs here are elementary and based on pictures of triangles. We have not tried to optimize the different values of δ involved.

Proposition 2.1.
The following are equivalent for a geodesic metric space X.

(1) *Triangles are slim.*
(2) *Triangles are thin.*
(3) *There is a global bound on the insize of geodesic triangles.*
(4) *There is a global bound on the minsize of geodesic triangles.*
(5) *The inner product on X is hyperbolic with any choice of base point.*

Definition We say that a geodesic metric space is *hyperbolic* if it satisfies one of the above equivalent conditions.

Proof. It is clear that (2) implies (3) which in turn implies (4). (Also (2) immediately implies (1).)
(1) implies (3)
Suppose that all geodesic triangles in X are δ-slim. Let xyz be a geodesic triangle, and let c_x, c_y, c_z be the internal points.

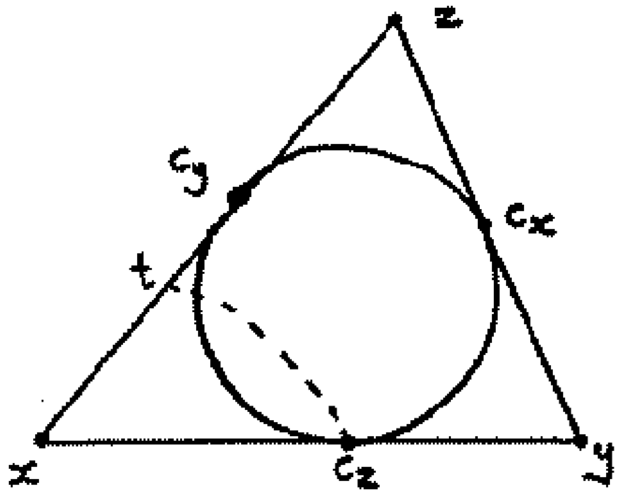

Figure 2.1

Consider the point c_z on $[xy]$; there is a point t on $[xz] \cup [yz]$ such that $d(c_z, t) \leq \delta$. Without loss of generality, suppose that t lies on $[xz]$. Then

$$d(x,t) + \delta \geq d(c_z, x) = d(c_y, x) \quad \text{and} \quad d(x,t) \leq d(x, c_z) + \delta$$

and so $d(t, c_y) \leq \delta$, and $d(c_z, c_y) \leq 2\delta$.
A similar argument shows that c_x is at distance not more than 2δ from one of c_z and c_y. It follows that $diam\{c_x, c_y, c_z\} \leq 4\delta$, and (3) holds.

(1) implies (2)

Let u be a point on $[xc_y]$ and v a point on $[xc_z]$ such that $d(u,x) = d(v,x)$. As geodesic triangles are δ-slim,

$$d(u, [xc_z] \cup [c_y c_z]) \leq \delta \ .$$

If there is a point $t \in [xc_z]$ such that $d(u,t) \leq \delta$, then $d(t,v) \leq \delta$, so that $d(u,v) \leq 2\delta$. Thus if $d(u,v) > 2\delta$, it follows that there are points $t_u, t_v \in [c_y c_z]$ such that $d(u,t_u) \leq \delta$ and $d(v,t_v) \leq \delta$, and $d(u,v) \leq 6\delta$, and (2) holds.

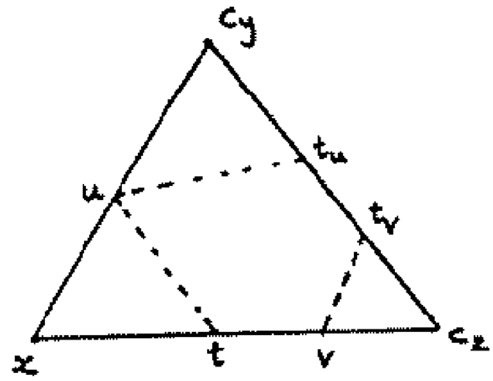

Figure 2.2

(4) implies (1)

Let x', y', z' be points on $[yz], [xz], [xy]$ such that $diam\{x', y', z'\} \leq \delta$. This reduces the problem to studying three geodesic triangles, each with base $\leq \delta$.

Suppose that there is a point $t \in [xz']$ such that $d(t, [xy']) > 2\delta$.

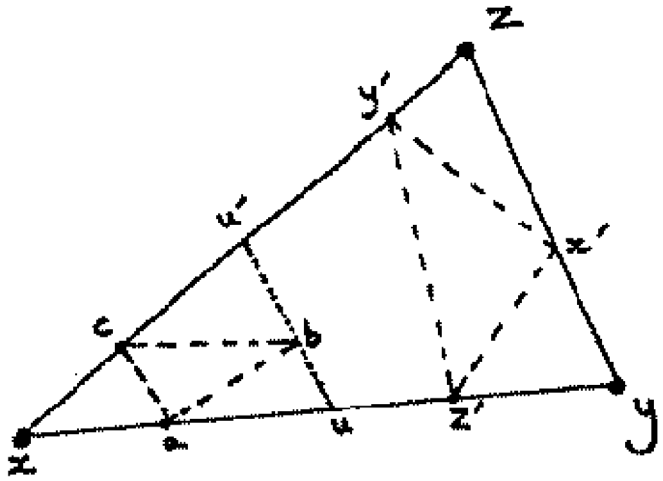

Figure 2.3

Let u be the point in $[z't]$ nearest to t such that $d(u,u') = 2\delta$ for some point $u' \in [xy']$. Now consider the geodesic triangle $uu'x$. The bound on the *minsize* of triangles implies that there are points a, b, c on the three sides of $uu'x$ such that $diam\{a,b,c\} \leq \delta$. The point a on $[xu]$ does not lie in $[tu]$ by supposition, and $d(u,a) \leq 3\delta$, so $d(t,u') \leq 3\delta$ or $d(t,c) \leq 3\delta$.

(2) implies (5)

Consider a geodesic triangle wxy, with internal points c_x, c_y, c_w. The inner product with base point w is

$$(x.y)_w = \frac{1}{2}(d(w,x) + d(w,y) - d(x,y))$$

and we must show that $\forall z \in X$, $(x.y)_w \geq min((x.z)_w, (y.z)_w) - \delta$.

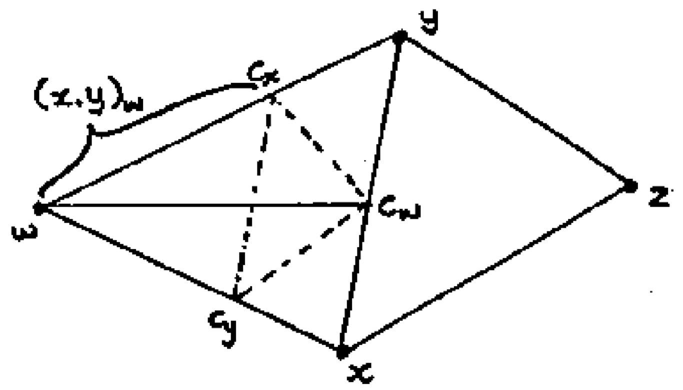

Figure 2.4

Recall that $d(w, c_x) = (x.y)_w$, and notice that

(1) $d(w, c_w) \leq d(w, c_x) + d(c_x, c_w) \leq (x.y)_w + \delta$
(2) $(x.y)_w \leq d(w, [xy])$.

Let z be another point in X. Then

$$(x.y)_w + 2\delta \geq d(w, c_w) + \delta \geq min(d(w, [xz]), d(w, [yz]))$$

But $d(w, [xz]) \geq (x.z)_w$ so that

$$(x.y)_w + 2\delta \geq d(w, c_w) + \delta \geq min((x.z)_w, (y.z)_w)$$

and the inner product condition holds.

(5) implies (1)

We first show:

Claim: If the inner product with base point w is hyperbolic, then for any geodesic triangle wxy,

$$(x.y)_w \leq d(w, [xy]) \leq (x.y)_w + 2\delta .$$

The left hand inequality follows immediately as in the remarks after definition 1.1. Now let c_w, c_x, c_y be the internal points of the triangle wxy on the side $[xy]$. Now consider the internal points d_x, d_w, d_y (resp. e_y, e_w, e_x) of geodesic triangles xwc_w (resp. ywc_w), where $d_y \in [wx]$ and $e_x \in [wy]$.

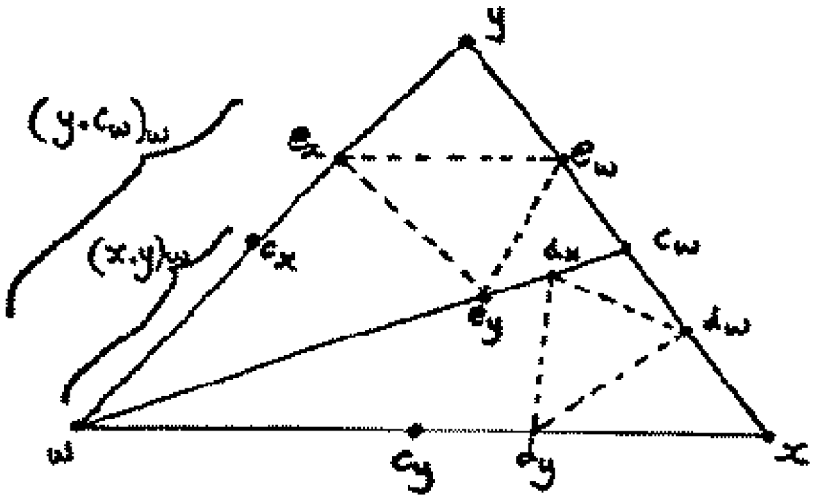

Figure 2.5

As d_w lies in $[xc_w]$, $d(x,d_y) = d(x,d_w) \leq d(x,c_y)$. It follows that $d(w,d_y) > d(w,c_y)$. Similarly $d(w,_x\ ) > d(w,c_x)$. Thus

$$(x.c_w)_w \geq (x.y)_w \quad \text{and} \quad (c_w.y)_w \geq (x.y)_w \, .$$

Without loss of generality suppose that $(c_w.y)_w \leq (x.c_w)_w$; then by (5) (with c_w in place of z):

$$\delta \geq (c_w.y)_w - (x.y)_w = d(c_x,e_x) = d(e_w,c_w) \, .$$

But

$$\begin{aligned} d(w,c_w) &= d(w,e_y) + d(e_y,c_w) = d(w,e_x) + d(c_w,e_w) \\ &= d(w,c_x) + 2d(c_x,e_x) \leq 2\delta + (x.y)_w \end{aligned}$$

and it follows that

$$d(w,[xy]) \leq d(w,c_w) \leq (x.y)_w + 2\delta \, .$$

This completes the proof of the claim.

Now suppose that the inner product is δ-hyperbolic with respect to any base point. Let xyz be a geodesic triangle, and let w be a point on the side $[xy]$. By the above,

$$0 = (x.y)_w \geq min\{(x.z)_w, (z.y)_w\} - \delta \, .$$

Without loss of generality suppose that $(x.z)_w \leq (z.y)_w$. Then

$$\delta \geq (x.z)_w \geq d(w,[xz]) - 2\delta$$

and so

$$d(w,[xz]) \leq 3\delta$$

and the triangle xyz is 3δ-slim, as required.

This completes the proof of proposition 2.1. □

We now give Gromov's direct proof that for inner products, the property of being hyperbolic is independant of base point.

Proposition 2.2. ([G, 1.1B])
If X is δ-hyperbolic with inner product based at w and $t \in X$ then X is 2δ-hyperbolic with inner product based at t.

This means that the reference to base point can be removed from the definition: we say that X is hyperbolic if there is a positive constant δ such that $\forall w, x, y, z \in X$,

$$(x.y)_w \geq min((x.z)_w, (z.y)_w) - \delta .$$

To prove the proposition we first show:

Lemma 2.3. ([G, 1.1A])

$$(x.y)_w + (z.t)_w \geq min((x.z)_w + (y.t)_w, (t.x)_w + (y.z)_w) - 2\delta$$

Proof. We remove reference to the base point w.

$$\begin{aligned} x.y + z.t &\geq min(x.t, t.y) + z.t - \delta \\ &= min(x.t + z.t , t.y + z.t) - \delta \\ &\geq min(x.t + min(z.y, y.t) , t.y + min(z.x, x.t)) - 2\delta \\ &= min(x.t + z.y , x.t + y.t , t.y + z.x) - 2\delta \end{aligned}$$

and this achieves a unique minimum value of $x.t + y.t$ if and only if $y.t < z.y$ and $x.t < z.x$.

Similarly

$$\begin{aligned} x.y + z.t &\geq min(x.z, z.y) + z.t - \delta \\ &= min(x.z + z.t , z.y + z.t) - \delta \\ &\geq min(x.z + min(z.y, y.t) , z.y + min(z.x, x.t)) - 2\delta \\ &= min(x.z + z.y , x.z + y.t , z.y + x.t) - 2\delta. \end{aligned}$$

This achieve a unique minimum value of $x.z + z.y$ if and only if $z.y < y.t$ and $x.z < x.t$. Both of these cannot be true at the same time, so the result holds. □

Corollary 2.4.
If X is δ-hyperbolic, then for all $t, x, y, z \in X$,

$$d(x,y) + d(z,t) \leq \max\{d(x,z) + d(y,t), d(x,t) + d(y,z)\} + 2\delta$$

Proof of Proposition 2.2. We wish to establish a lower bound for

$$\min\{(x.z)_t, (z.y)_t\} \; - \; (x.y)_t$$

$$= \min\{d(x,t) \; + \; d(z,t) \; - \; d(x,z)\, , \, d(z,t) \; + \; d(y,t) \; - d(z,y)\} \\ + \; d(x,y) \; - \; d(x,t) \; - \; d(y,t)$$

$$= \min\{-d(y,t) \; - \; d(x,z)\, , -d(x,t) \; - \; d(z,y)\} \; + \; d(x,y) \; + \; d(z,t) \; .$$

Adding $d(x,w)+d(y,w)+d(z,w)+d(t,w)$ inside and subtracting from the outside of the minimum gives

$$\min\{(y,t)_w \; + \; (x.z)_w \; , \; (x.t)_w \; + \; (z.y)_w\} \; - \; (z.t)_w \; - \; (x.y)_w$$

it follows , by Lemma 2.3 that

$$\min\{(x.z)_t \; , \; (z.y)_t\} \; - \; (x.y)_x \; \leq \; 2\delta$$

□

Linear isoperimetric inequality implies hyperbolic

We shall now show that a finitely presented group which satisfies a linear isoperimetric inequality is hyperbolic. The converse will be shown in 2.10, and again, using different methods in 2.12. We need first to develop some of the language of disc diagrams.

Singular Disc Diagrams

Let $F(X)$ denotes the free group on X. When R is a finite set of cyclically reduced words in $F(X)$, $\mathcal{P} = \langle X; R\rangle$ denotes a finite presentation of a group G; we use $\mathcal{R}$ to denote the cyclic closure of R, consisting of all elements r in R, their inverses, and all cyclic conjugates of $r^{\pm}$.

We form a 2-complex $K(\mathcal{P}) = K$ whose fundamental group is G in the standard way: K has one vertex, one labelled, directed edge for each generator, and one 2-cell for each relator. The 2-cell D_i corresponding to the relator r_i is glued to the 1-skeleton $K^{(1)}$ via a continuous map which identifies the boundary ∂D_i with a loop representing the word r_i.

A freely reduced word w in the generators is equal to the identity in G if an only if there is a continuous map from a disc $(D, \partial D)$ to $(K, K^{(1)})$ taking the boundary to a loop representing the word w. After a homotopy, the cell decomposition of K induces a cell decomposition of a simply connected complex, which we also call D, consisting of a set of discs joined by arcs or vertices. The vertices map to the vertex of K, the interiors of the 1-cells of D, called *edges*, map homeomorphically to the interiors of the 1-cells of K, and the interiors of the 2-cells, called *regions*, map homeomorphically to the interiors of the discs D_i of K. We can orient and label the edges according to the generating loop in $K^{(1)}$ to which they map, in such a way that reading the labels on the edges around the boundary of D gives the word w. The complex D we call a *singular disc diagram* (or Van Kampen or Dehn diagram) for $w = 1$ in G (for more details see [**LS**], chapter V). Regarding the singular disc diagram as a topological space, each component of the interior is a topological disc.

A singular disc diagram is *unreduced* if there are two regions R_1 and R_2 in D whose boundaries have an edge e in common, such that the labels on their boundaries, reading around from the edge e, clockwise on R_1 and anti-clockwise on R_2, are the same. It is not hard to see how to remove two such neighbouring regions in an unreduced singular disc diagram without changing the boundary label, so that we may concentrate our attention on reduced singular disc diagrams; *from now on all diagrams are assumed to be reduced.*

Let $\mathcal{P} = \langle X; R\rangle$ be a finite presentation of the group G. Suppose that $f : \mathbf{N} \to \mathbf{R}$ is a function with the property that if w is a freely reduced word of length n in the free group $F(X)$ and $\overline{w} = 1$ in the group G, then there is a singular disc diagram for w with at most $f(n)$ regions. Following S.M. Gersten [Ge], we say that f is a *Dehn function* for $\mathcal{P}$.

Notice that as $w = 1$ in G, there are words p_i in $F(X)$, and r_i in R, such that

$$w \;=\; \prod_{i=1}^{N} p_i r_i^{\epsilon_i} p_i^{-1} \;,\; \epsilon_i = \pm 1.$$

The Dehn function tells us that there is such a product with $N \leq f(n)$ relators. In addition, as we have a bound on the total length of the 1-skeleton of the diagram, we may take

$$\ell(p_i) < n + f(n)M \quad \text{where} \quad M = \max_{r\in R}(\ell(r))\,.$$

Tietze transformations transform lengths by scalar multiples together with the addition of new trivial words, and so transform Dehn functions by scalar multiples and the addition of linear terms [Ge]. Thus if a group G has a presentation with a linear (quadratic, exponential) Dehn function, then we say that G *satisfies a linear (quadratic, exponential) isoperimetric inequality* .

If X is a finite set of generators, and R is a recursive set of relations, then finding a recursive Dehn function for $\mathcal{P}$ solves the word problem for $\mathcal{P}$ [Ge].

Theorem 2.5.

If G is a finitely presented group satisfying a linear isoperimetric inequality, then there is a constant δ such that all geodesic triangles in the Cayley graph Γ are δ-slim.

Proof. We argue by contradiction: suppose that there is no constant δ such all geodesic triangles are δ-slim.

Let K be the constant associated to the linear Dehn function for the finite presentation $P = \langle\, X;\, R\,\rangle$ for the group G, and let ρ be the maximum length of a relator in R. We can suppose that $\rho > 1$, else G is free and so hyperbolic (with radius of curvature 0). We can also assume that $K \geq 1$.

Then for each $r > 0$, there is a geodesic triangle xyz in Γ and a point $w \in [xy]$ such that

$$(\star) \qquad min(d(w, [yz])\,,\, d(w, [xz])) \;>\; 2r\,.$$

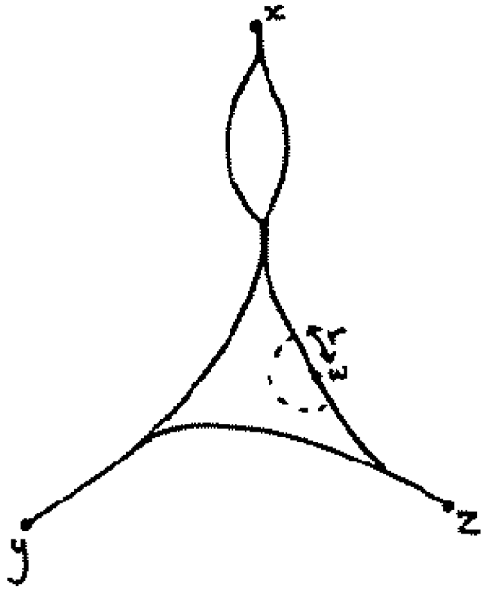

Figure 2.6

If xyz is degenerate (i.e. if $[xy] \cap [yz] - \{y\} \neq \emptyset$ or $y = z$ say) then xyz contains a non-degenerate geodesic triangle (or bigon) where $(\star)$ holds. So it suffices to consider non-degenerate geodesic triangles and bigons.

Let B_r be the ball of radius r (in Γ) with centre w.

Let ϵ be a constant, and suppose that $r > 6\epsilon$. We first cut off the corners of xyz such that the remaining segments are all at distance at least 4ϵ from each other, and the cut-off arcs are of length exactly 4ϵ. This will give one of three cases:

(1) a non-degenerate hexagon H with three sides of length 4ϵ.
(2) a non-degenerate quadrilateral with two sides of length 4ϵ.
(3) a degenerate hexagon.

As $r > 6\epsilon$, the length α of the side $[x'y']$ containing w is at least $2r$. Without loss of generality, we suppose that the cases are as shown below (figure 2.7).

Consider first case (1), with $H = x'x''z'z''y''y'$ and let α, β and γ be the number of edges in the segments $[x'y']$, $[x''z']$ and $[y''z'']$.

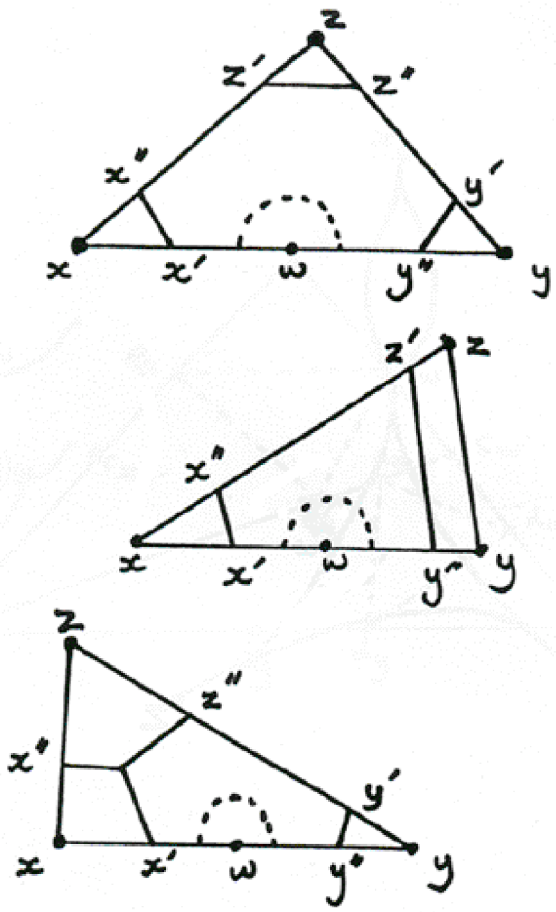

Figure 2.7

Let D be a minimal disc diagram for the word represented by the hexagon H. We shall consider D both as a cell complex and as the underlying topological space, and we identify ∂D with H in Γ. As H is a simple closed curve in Γ, the diagram D is a topological disc, and each 1-cell which is not contained in ∂D is on the boundary of two 2-cells in D. If T is a subcomplex of D, we define $star_D(T)$ to be the set of all cells which intersect T. If θ is one of the geodesic arcs $[x'y']$, $[x''z']$, $[z''y'']$ we use $N(\theta)$ to denote the subcomplex of D obtained by iterating the star operation $[\epsilon/\rho]+1$ times ($[\epsilon/\rho]$ denotes integer part), starting from the arc θ in D. Let $\ell(\theta)$ denote the number of 1-cells in θ; thus $\ell([x'y']) = \alpha$. We need the following 2 lemmas to complete the proof. Their proofs are deferred.

Lemma 2.6. *If $\epsilon > \rho$, then there is a constant C_1 depending solely on ϵ, such that the number of 2-cells in $N(\theta)$ is at least $\ell(\theta)\epsilon/\rho^2 - C_1$.*

Let $A(D)$ be the number of 2-cells in the diagram D.

Lemma 2.7. *If $\epsilon > \rho$, there is a constant C_2 depending solely on ϵ such that*

$$A(D) > (\alpha+\beta+\gamma)\epsilon/\rho^2 - C_2 + 2r/\rho$$

We now use this last result to complete the proof of the main theorem.
The linear isoperimetric inequality implies that

$$A(D) \leq (\alpha + \beta + \gamma)K + 12K\epsilon$$

Combining this inequality, and the result of Lemma 2.7, we have

$$(\alpha+\beta+\gamma)\epsilon/\rho^2 - C_2 + 2r/\rho \leq (\alpha+\beta+\gamma)K + 12K\epsilon\,.$$

Now set $\epsilon = K\rho^2$ (as $\rho > 1$ and $K \geq 1$, it follows that $\epsilon > \rho$). We thus obtain

$$2r/\rho - C_2 \leq 12K^2\rho^2$$

which is clearly a contradiction for sufficiently large values of r.

This completes the proof in case (1).

It remains to prove the lemmas. First notice that the map $\partial D \to H$ extends naturally to a map $h : D^{(1)} \to \Gamma$ from the 1-skeleton of D to the Cayley graph. The 1-cells of D are the inverse images of the edges of Γ, so there may well be vertices of degree 2 in the interior of D.

Proof of Lemma 2.6. Suppose that $\theta = [x'y']$; the other cases follow exactly analogously. Let γ_0 denote the segment θ in ∂D, and let $N_1 = star_D(\gamma_0)$.

Then γ_0 contains α 1-cells, each of which lies in the boundary of a 2-cell. Each 2-cell has at most ρ 1-cells in its boundary, so there are at least $\ell(\theta)/\rho$ 2-cells in N_1. Now $N_1 \cap D^{(1)} \cap \overline{(int(D) - N_1)}$ is a path γ_1 from the segment $[x'x'']$ to $[y'y'']$, and maps to a path in Γ lying in a ρ-neighbourhood of $[x'y']$.

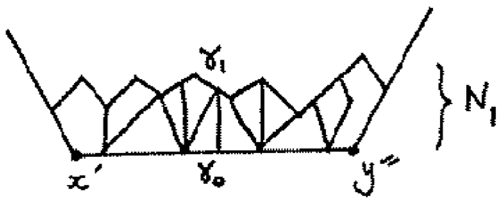

Figure 2.8

It follows that γ_1 contains at least $(\ell(\theta) - 2\rho)$ 1-cells. Continue this process $[\epsilon/\rho] + 1$ times; i.e. let

$$N_i = star(N_{i-1}) \text{ and let } \gamma_i = N_i \cap D^{(1)} \cap \overline{(int(D) - N_i)}\,.$$

The number of 2-cells in $N_i - N_{i-1}$ is at least $(\ell(\theta) - 2(i-1)\rho)/\rho$. Repeating $[\epsilon/\rho] + 1$ times gives at least

$$\alpha\epsilon/\rho^2 - \epsilon(\epsilon+\rho)/\rho^2$$

2-cells in $N([x'y'])$.

This concludes the proof of Lemma 2.6, setting $C_1 = \epsilon(\epsilon+\rho)/\rho^2$. □

Proof of Lemma 2.7. We first show that

$$\begin{aligned} N([x'y']) \cap N([y''z'']) &= N([x'y']) \cap N([x''z']) \\ &= N([x''z']) \cap N([y''z'']) = \emptyset\,. \end{aligned}$$

In the construction of the proof of A.1, we have that γ_i maps to a path in Γ which lies in a $(i\rho)$-neighbourhood of $[x'y']$. It follows that $N([x'y'])^{(1)}$ maps into a $(\epsilon+\rho) < 2\epsilon$ neighbourhood of $[x'y']$. By the construction of H, $p \in [x'y']$ and $q \in [x''z']$ implies that $d(p,q) \geq 4\epsilon$. Hence the 2ϵ neighbourhoods of $[x'y']$ and of $[x''z']$ in Γ do not intersect, and so $N([x'y']) \cap N([x''z']) = \emptyset$. The other cases follow analogously.

Let ϕ' be the set of 1-cells in $N([x'y']) \cap \overline{(int(D) - N([x'y']))}$, and let ϕ be the subset of ϕ' which maps into B_r (via the map h). A point $p \in \phi$, maps to a point at distance at most r from w, and so lies at least $r - 2\epsilon > 0$ from $N([x''z'] \cup N([y''z''])$. Also $h(\phi)$ contains an arc of length at least $2r - 4\epsilon$, so that ϕ contains at least $(2r - 4\epsilon - 2)$ 1-cells. Each of these 1-cells lies in the boundary of a 2-cell which does not lie in $N([x'y'])$, so there are at least $(2r - 4\epsilon - 2)/\rho$ 2-cells in D which do not lie in $N([x'y']) \cup N([y''z'']) \cup N([x''z'])$.

So using Lemma 2.6, there are at least

$$(\alpha + \beta + \gamma)\epsilon/\rho^2 - 3C_1 - (4\epsilon + 2)/\rho + 2r/\rho$$

2-cells in D.
This concludes the proof of Lemma 2.7, setting $C_2 = 3C_1 + (4\epsilon + 2)/\rho$. □

The cases (2) and (3) of Theorem 2.5 remain to be considered
In case (2) Lemma 2.6 holds for the arcs $[x'y']$, $[x''z']$. Lemma 2.7 now holds for a minimal disc bounded by the quadrilateral $Q = x'x''z'y'$, with $\gamma = 0$ (and with $C_2 = 2C_1 + (4\epsilon + 2)/\rho$). The proof of the theorem is concluded by obtaining a contradiction as before.
In (3), look at the simple closed curve $P = x'\overline{z}z''y''y'$. Lemma 2.6 holds as before for the arcs $[x'y']$, $[y''z'']$, and Lemma 2.7 follows with $\beta = 0$ and $C_2 = 2C_1 + (4\epsilon{+}2)/\rho$. Note that the sum of the lengths of the segments $[x'\overline{z}]$ and $[\overline{z}z'']$ lies between 4ϵ and 8ϵ. The side $[y'y'']$ has length 4ϵ, and the side $[y''z'']$ has length $\gamma > 2r - 12\epsilon > 0$.
The proof is concluded by obtaining a contradiction as before.

We shall give now give two proofs that a group with a linear isoperimetric inequality has slim (or thin) triangles. The first follows Gromov [**Gr**, 1.7C](see also [CDP, §5.3.1]). A second proof is given in theorems 2.15, 2.16.

A presentation of a group is said to be *triangular* if each relation has length three.

Every finitely presented group has a triangular presentation, and to such a presentation there is an associated simply–connected, locally finite 2 dimensional simplicial complex X, where each cell is isometric to a chosen standard 2-cell in $\mathbf{R}^2$. For a simplicial curve $\mathcal{C}$ in X, let $L(\mathcal{C})$ denote the length of $\mathcal{C}$, i.e. the number of 1-cells in $\mathcal{C}$, and let $A(\mathcal{C})$ denote the area bounded by $\mathcal{C}$, i.e. the minimal number of 2-cells in a singular disc embedded in X bounded by $\mathcal{C}$.

Proposition 2.10.
If geodesic triangles in X are slim then X satisfies a linear isoperimetric inequality.

Proof. Let $d = 10\delta$ and let

$$A_0 = max\{A(\mathcal{C}); L(\mathcal{C}) \leq 3d\}.$$

Let $N(\mathcal{C})$ denote $int(L(\mathcal{C}))/d) + 1$. We shall show by induction on N that $A(\mathcal{C}) \leq 3N(\mathcal{C})A_0$.

The result clearly holds for $N = 1$.

Now suppose that the result holds for all curves $\mathcal{C}'$ such that $N(\mathcal{C}') \leq n$.

Let $\mathcal{C}$ be curve such that $N(\mathcal{C}) = n + 1$ and choose a base point x on $\mathcal{C}$. Choose y_0 on the curve $\mathcal{C}$ farthest from x, and choose points y_1, y_2 on $\mathcal{C}$ at distance d from y.

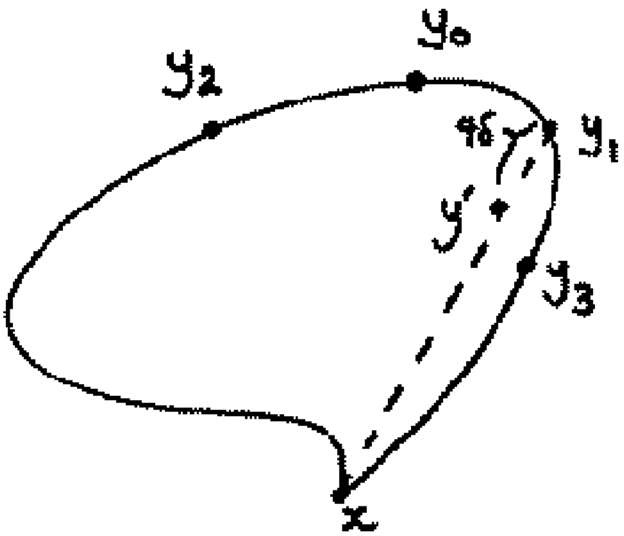

Figure 2.9

If $d(y_1, y_2) \leq d$ then the result holds, by applying the induction hypothesis to the curve obtained from $\mathcal{C}$ by omitting the segment between y_1 and y_2 which contains y_0, and replacing it by a geodesic $[y_1 y_2]$. This curve has length at most $L(\mathcal{C}) - d$, and the induction hypothesis applies. The curve $\mathcal{C}$ then bounds a disc obtained from this one by adding on a disc bounded by the curve $y_1 y_0 y_2$ which has length at most $3d$ and the induction argument is complete in this case.

We are thus left with the case that $d(y_1, y_2) > d$; it follows that $d(x, y_0) > 5\delta$.

Without loss of generality, suppose that $d(x, y_1) \geq d(x, y_2)$; it follows that $d(x, y_1) > 5\delta$.

Let y' be a point on the geodesic arc $[xy_1]$ at distance 4δ from y_1, and let y_3 be a point on $\mathcal{C}$ at distance d from y_1 , in the segment from x to y_1 not containing y_0.

To complete the argument, we use the following:

Lemma 2.11.
For $i = 0, 1, 2, 3$, we have $d(y', y_i) \leq d$.

Given this, we see that the argument is complete, as then we have a disc bounded by $\mathcal{C}$, made up of a disc bounded by a curve which is shorter than $\mathcal{C}$ by at least d, and three other discs, each bounded by curves of length at most $3d$.

Proof of Lemma 2.11. We apply Corollary 2.4 three times; firstly to the points x, y_0, y_1, y_2:

$$d(x,y_0)+d(y_1,y_2)\le max\{d(x,y_1)+d(y_0,y_2)\,,\ d(x,y_2)+d(y_0,y_1)\}+2\delta\,.$$

But $d(y_0,y_i)\le d$ for $i=1,2$, and $d(y_1,y_2)>d$ by assumption, and $d(x,y_2)\le d(x,y_1)$, so we get

$$(\star)\qquad d(x,y_0)\le d(x,y_1)+2\delta\,.$$

Now consider the points x, y_1, y', y_i where $i=0,1,3$:

$$d(x,y_1)+d(y',y_i)\ \le max\{d(x,y')+d(y_1,y_i)\,,\ d(x,y_i)+d(y_1,y')\}\ +\ 2\delta\,.$$

But by $(\star)$, and the definition of y', we get

$$d(x,y_1)\ +\ d(y',y_i)\ \le\ max\{d(x,y_1)-4\delta+d\,,\ d(x,y_i)+4\delta\}\ +\ 2\delta\,.$$

But $d(x,y_i)\le d(x,y_0)\le d(x,y_1)+2\delta$, so we get

$$d(y',y_i)\le 8\delta,\quad \text{for}\quad i=0,1,3$$

and the lemma holds for these three points. It remains to show that $d(y',y_2)\le 10\delta$. Consider the points x, y_0, y', y_2. We have

$$d(x,y_0)\ +\ d(y',y_2)\ \le\ max\{d(y',x)+d(y_0,y_2)\,,\ d(x,y_2)+d(y',y_0)\}\ +\ 2\delta\,.$$

But $d(y',x)\le d(y_0,x)-4\delta$, and $d(y',y_0)\le 8\delta$, so

$$d(y',y_2)\ \le 10\delta\,.$$

This completes the proof of the Lemma. □

We now give an alternative proof of the fact that if triangles are thin in the Cayley graph of a group G, then G satisfies a the linear isoperimetric inequality. The proof will also show that G is finitely presented. Our plan follows the basic outline of Cannon's paper [Can] on co-compact hyperbolic groups; we subsequently discovered that a similar proof is given by Cannon in [Can2]. We first define *local geodesics*, and show that these follow near their corresponding geodesics and are comparable to them in length. From this, we are able to show the existence of a *Dehn's algorithm* (see definition 1.11), that is, a finite collection of relators such that any word which represents the trivial element contains more than half of one of the relators, and so may be shortened by use of one relator from this list. This was also procedby Lysenok [L]. The linear isoperimetric inequality then follows immediately.

Let G be a group with finite generating set X. As usual, we regard a word u as a path $[0,\ell(u)]\to\Gamma_X(G)$.

We shall now show that:

Theorem 2.12.

Let G be hyperbolic, in the sense that geodesic triangles in the Cayley graph $\Gamma_X(G)$ are δ-thin. Let

$$R = \{w \in F(X) \mid \ell(w) \leq 8\delta \text{ and } \mu(w) = 1\}.$$

Then $\langle X \mid R\rangle$ is a Dehn presentation for G.

Corollary.

Hyperbolic groups are finitely presented.

When $\delta = 0$, G is a free group and X is a set of free generators, the theorem follows immediately. In what follows we assume $\delta \geq 1$, and that δ is an integer, so that the points in the Cayley graph which are considered are all vertices.

Definition 2.13

We will say that a path p is a *k-local geodesic* if each sub-path u of p of length at most k is geodesic.

Thus paths which are not k–local geodesics can be shortened locally.

Lemma 2.14.

Let $k = 4\delta$, and let u be a k-local geodesic, let v be a geodesic with $\mu(u) = \mu(v)$. Assume that each of these has length at least 2δ. Let r and s be the points in $\Gamma_X(G)$ on u and v at distance 2δ from $\mu(u)$ and $\mu(v)$. Then $d(r, s) \leq \delta$.

Proof. Assume inductively that this is true when $\ell(u) \leq N$. We now take u, v such that $N \leq \ell(u) \leq N + k$. Let p be the vertex at distance k from $\mu(u)$ along u (recall k is integral). Let w be a geodesic from 1 to $\mu(w) = p$. Let q, t, and s be the points on u and w at distance 2δ from p as shown.

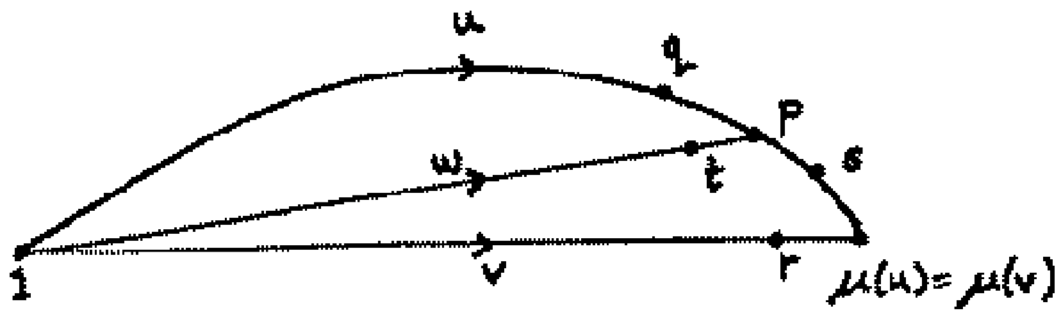

Figure 2.10

The segment of u from q to s has length k and thus is a geodesic, so that $d(q, s) = k = 4\delta$. By the induction hypothesis, $d(q, t) \leq \delta$. Hence, $d(t, s) \geq 3\delta$. But w, v, and the final length k segment of u form a geodesic triangle, and hence t lies close to v, and, in fact, is within δ of the point of v distance 2δ from $\mu(v)$. □

Theorem 2.15.
If u is a k-local geodesic, and v a geodesic with $\mu(u) = \mu(v)$, then u lies in a 3δ-neighbourhood of v.

Proof. Let z be a point on u at least distance 2δ from the ends of u. (Otherwise there is nothing to prove.) Let x and y be geodesics from 1 to z and from z to $\mu(u)$. Let a, b be the points at distance 2δ from z along u, and let r and s be the points at distance 2δ from z along x and y, all as shown.

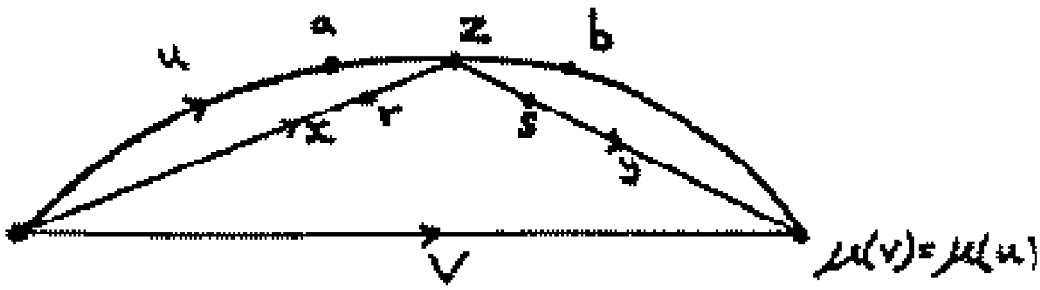

Figure 2.11

Since u is a k-local geodesic, $d(a, b) = 4\delta$, while by the lemma, $d(a, r) \leq \delta$ and $d(b, s) \leq \delta$. Consequently, $d(r, s) \geq 2\delta$. But x, y, and v form a geodesic triangle, so that $d(r, v) \leq \delta$. Hence $d(z, v) \leq 3\delta$. □

Proof of Theorem 2.12. We must show that if $w \in F(X)$ and $\mu(w) = 1$ then w may be shortened by using a single relator from R, that is, that w contains a subword r_1, and R contains a relator $r_1 r_2$ with $\ell(r_1) > \ell(r_2)$. Thus $w = w_1 r_1 w_2$ and $\mu(w) = \mu(w_1)\mu(r_2)^{-1}\mu(w_2)$.

If w is not a k-local geodesic, then w has a sub-path p of length at most k which is not geodesic. This means that w can be shortened as required.

Suppose now that w is a k-local geodesic. Then w stays within distance 3δ of any geodesic for w. But since $\mu(w)$ is the identity element of G, a geodesic for $\mu(w)$ is the empty word. Thus the path w lies in the 3δ–neighborhood of the identity element in $\Gamma_X(G)$. But then w must be of length at most 3δ, for if w has an initial subpath of length $3\delta + 1$, then this subpath is geodesic and hence strays distance $3\delta + 1$ from the identity, which is a contradiction. In particular, $w \in R$. □

The proof of the following is immediate.

Theorem 2.16.
If G has a Dehn's algorithm, then G satisfies a linear isoperimetric inequality.

The existence of a Dehn presentation has a simple consequence concerning elements of finite order in a hyperbolic group:

Corollary 2.17. ([GH,3.13])
In a hyperbolic group, there are only finitely many conjugacy classes of elements of finite order.

Proof. Consider the presentation giving a Dehn's algorithm. Let g be an element of finite order, and let $[g]$ be the set of all conjugates of g in G. Let w be a word in $F(X)$, chosen to be shortest over all $v \in F(X)$ such that $\mu(v) \in [g]$. Let n be the order of $\mu(w)$. As $\mu(w^n) = 1$, there must be some subword of w^n which is more than half of a word $r \in R$. This implies that $\ell(w) < \ell(r)$, else w, or some cyclic conjugate of w, can be shortened. (In fact w is contained in a cyclic conjugate of a relator.) Thus the number of conjugacy classes of elements of finite order is less than the number of elements of length at most $\max_{r \in R} \ell(r)$. □

This result contrasts with Gromov's result that in a hyperbolic group which is not a finite extension of a cylic group, there are an infinite number of conjugacy classes of prime (i.e. not proper powers) non-torsion elements [G, 5.1.B]. In particular an infinite torsion group is not hyperbolic (an alternative proof of this is to be found in [GS2]).

Solving the Word Problem in Linear Time

B. Domanski and M. Anshel have shown [DA] that if a group has a Dehn presentation, then there is an algorithm for solving the word problem in a length of time bounded linearly in terms of the length of the input word. To accomplish this requires a Turing machine with more than one tape. The way the machine works is to read through a given word, and on finding a place wher the word can be shortened, it does so. This may affect introduce a shortening in the part of the word already read, but not too far away. Here are the details.

Since the longest relator in our Dehn presentation has length $2k = 8\delta$, we know that we may shorten any word w representing the trivial element by replacing subwords of w of length at most $2k$ by shorter words of length at most k. Since each such replacement reduces the length of w by at least one, at most $\ell(w)$ replacements are required. We carry this procedure out on a Turing machine with an input tape, T and two pushdown stacks, S and S'. We also assume that there is enough internal memory to remember the final $2k$ letters of S at each step. We start with the word $w = a_1, \dots, a_n$ written on tape T, and the two stacks S and S' empty.

Assume now that the first j letters from T have been read, and the contents of S and S' are respectively $x_1 \dots x_p$ and $y_1 \dots y_q$.

First we consider the case when a shortening of the last $2k$ letters is possible. Suppose that $1 \leq p - r + 1 \leq 2k$ and the final $p - r + 1$ letters $x_r \dots x_p$ of S can be shortened to $v = v_1 \dots v_s$. We now read v onto S' in reverse order, together with letters from S , until $2k$ letters have been transferred (if possible). The contents of S and S' are now $x_1 \dots x_{t-1}$ and $y_1 \dots y_q v_s \dots v_1 x_{r-1} x_t$ where $t = \max\{1, r - 2k - 1\}$.

Now suppose that the final $\max\{p, 2k\}$ letters of S do not form a word which can be shortened. According to the values of p, q, j, one of the following happens. If $q \geq 1$, read the last letter of S' onto S, so that their new contents are $x_1 \dots x_p y_q$ and $y_1 \dots y_{q-1}$. If $q = 0$ (i.e. the S' tape is empty) and $j < n$, one letter is read from T onto S, so that $j + 1$ letters have now been read from T, and the word on S is $x_1 \dots x_p a_{j+1}$; S' remains empty. If $q = 0$ and $j = n$ but $p \neq 0$, then all of T has been read, but w does not reduce to the empty word, and thus w is rejected by the machine as it cannot represent the trivial element of the group. Finally if $p = q = 0$ and $j = n$, the word w is accepted by the machine as we have reduced w to the empty word, so that $\mu(w) = 1$.

It is easy to see that this procedure halts after a length of time proportional to

$n = \ell(w)$. To see this, notice that we read each letter of w once from T and move at most $4k$ letters from S to S' and back again for each replacement we make. Since each replacement reduces length, we make at most n replacements.

One might ask when the stack S' can be replaced by a finite amount of memory. In this case, the word problem can actually be solved by a pushdown automaton. This is equivalent to saying that the word problem for the group is a *context free grammar*. That is to say, the collection of all words representing the identity element of the group may be generated by a simple set of replacement rules (for more about languages and automata see for instance [HU]). It is a result of Muller and Schupp [MS] (together with Dunwoody's accessibility result) that this can be done if and only if the group is virtually free.

Now it is easy to see how to extend this procedure to a group G which is a direct product of finitely many hyperbolic groups, say $G_1, \ldots, G_m$. One simply takes a Turing machine with a tape T and m pairs of pushdown stacks, $S_1, S'_1, \ldots, S_m, S'_m$. Then as each letter a_i is read off of T, one may rewrite it in terms of the generators of $\{G_j\}$ and sort these out to the appropriate stacks. Since this rewriting requires only a linear amount of time, this procedure solves the word problem in G in linear time.

Finally, if H is a finitely generated subgroup of G, the restriction of this procedure to elements of H solves the word problem in H in linear time. We have shown the following

Theorem 2.18.

Let H be a finitely generated subgroup of a direct product of hyperbolic groups. Then the word problem in H is soluble in linear time.

Such groups can be fairly complicated. They include, for example, the finitely generated subgroups of the direct product of two free groups of rank 2. Many of these are not finitely presented. In fact, it is a theorem of G. Baumslag and J. Roseblade that these groups are finitely presented if and only if they contain a subgroup of finite index which is itself a direct product [BR].

Any problem which can be solved in linear time is also solved in linear space. Such problems correspond to the so-called *context sensitive grammars*. Such languages are also characterized by a set of rules for generating the words of the language (see [HU]). Clearly the class of groups with context sensitive word problem is much larger than the class of groups with context free word problem! In fact, though we will not show it here, the automatic groups of [**CEHPT**]all have context sensitive word problem, and hence so do their finitely generated subgroups.

Divergence of geodesics

Here we shall show (2.20) that in a geodesic metric space, non–linear divergence of geodesics implies that the space is hyperbolic. (The definition of divergence functions is given in 1.7.) We first show the more elementary result (2.19) that in a hyperbolic space geodesics diverge exponentially, leading to the remarkable fact that exponential and non–linear divergence are equivalent in geodesic metric spaces.

Theorem 2.19.
In a hyperbolic metric space geodesics diverge exponentially.

Proof. Suppose that all geodesic triangles are δ-thin. Let γ and γ' be geodesics of length $R+r$ beginning at the point x such that $d(\gamma(R), \gamma'(R)) > \delta$. We thus set $e(0) = \delta$. Let p be a path from $\gamma(R+r)$ to $\gamma'(R+r)$ lying in the closure of the complement of $B_{R+r}(x)$ We will show there is an exponential function $e(r)$ independent of the choice of γ and γ' such that $\ell(p) \geq e(r)$.

Let α be a geodesic from $\gamma(R+r)$ to $\gamma'(R+r)$. Let b be a binary sequence of length s (possibly 0), and suppose α_b to have been chosen (the geodesic α thus corresponds to the empty sequence). Let m_b be the midpoint of the segment of p between the ends of α_b. We choose α_{b0} to be a geodesic from $\alpha_b(0)$ to m_b, and α_{b1} to be a geodesic from m_b to $\alpha_b(1)$.

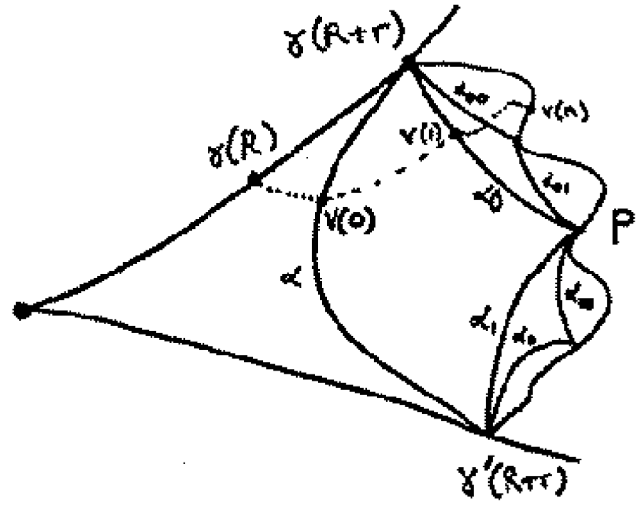

Figure 2.12

We continue this subdivision until each α_b in our final subdivision has length between $\frac{1}{2}$ and 1. This ensures that these last α_b's are contained in p. This also means that we have divided the original path α into n pieces, where

$$\log_2(\ell(p)) \leq n \leq \log_2(\ell(p)) + 1 \,.$$

Notice that for each b, the segments α_b, α_{b0} and α_{b1} form a geodesic triangle.

As $d(\gamma(R), \gamma'(R)) > \delta$, there is a point $v(0)$ on α wth the property that $(d(v, \gamma(R)) \leq \delta$.

For each i, if a point $v(i)$ lies on α_b, then as the triangle with sides $\alpha_b, \alpha_{b0}, \alpha_{b1}$ is δ-thin, there is a point $v(i+1)$ on α_{b0} or α_{b1} such that $d(v(i), v(i+1)) < \delta$.

Thus we may find a path from x to a point $v(n)$ on p whose length is at most $R + \delta(\log_2(\ell(p)) + 2)$. But as the path p lies outside of the ball of radius $R + r$, $d(x, v(n)) \geq R + r$, so that

$$R + \delta(\log_2(\ell(p)) + 2)) \; > \; R + r$$

$$\Leftrightarrow \ell(p) > 2^{\frac{r}{\delta} - 2},$$

and we see that e is an exponential function as required. □

We now establish the opposite direction of the equivalence.

Proposition 2.20.
If X is a geodesic metric space with a non–linear divergence function, then geodesic triangles are δ-slim for some δ.

Proof. Let e be a divergence function for X, and let xyz be a geodesic triangle in X. Consider the edge $[xy]$ (resp. $[xz]$) as an isometric embedding α_1 (resp α_2) $: [0, n] \to X$ based at x.

Let T be the maximum value of $t \in [0, n]$ such that

$$\forall t \in [0, T],\; d(\alpha_1(t), \alpha_2(t)) \leq e(0)\,,$$

and let $x_1 = \alpha_1(T)$, $x_2 = \alpha_2(T)$. Similarly define points z_1, z_2, y_1, y_2.
Claim 1 If $[xx_1] \cap [y_2y] \neq \emptyset$, then there is a bound on $max\{d(z_2, y_1), d(z_1, x_2)\}$, and so the triangle is δ-slim with $\delta = \frac{K}{2} + 2e(0)$.

If $[xx_1] \cap [y_2y] \neq \emptyset$ then there are points $x_3 \in [xx_2]$ and $y_3 \in [yy_1]$ such that $d(x_3, y_3) < 2e(0)$. Applying the divergence function to $[zy_3]$ and $[zx_3]$ bounds the lengths of $[z_1x_3]$ and $[z_2y_3]$, and hence the lengths of $[z_1x_2]$ and $[z_2y_1]$.

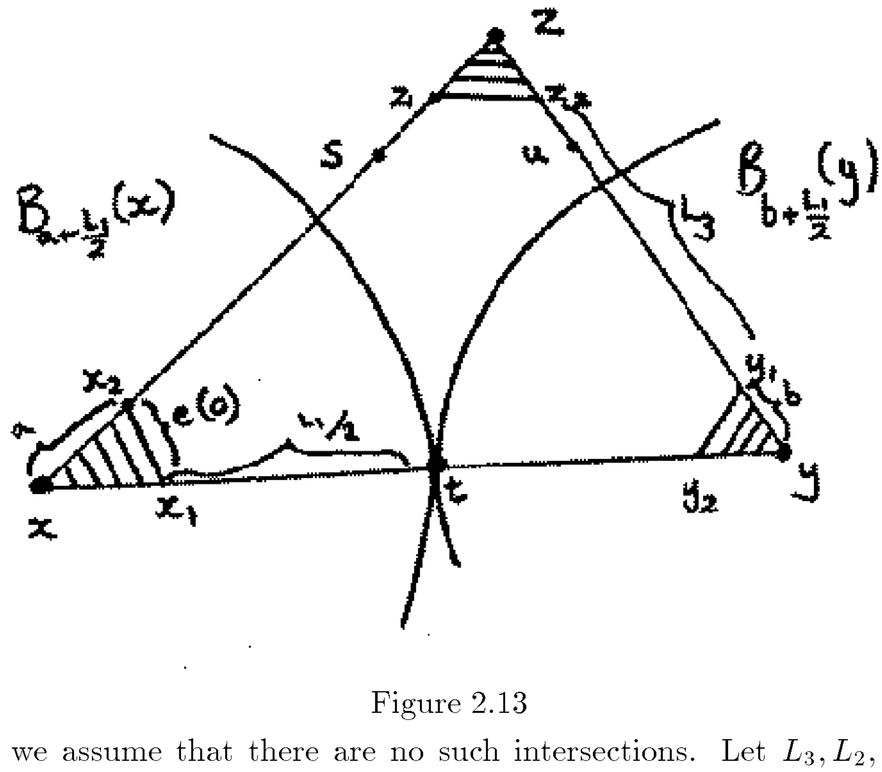

Figure 2.13

Hence we assume that there are no such intersections. Let L_3, L_2, L_1 be the lengths $d(z_2, y_1), d(x_2, z_1), d(y_2, x_1)$, and suppose that $L_1 = max\{L_3, L_2, L_1\}$. It suffices to show that L_1 is bounded by some constant K, for then xyz will be $(K/2 + e(0)$-thin).

Let t be the midpoint of $[x_1 y_2]$.

Let $a = d(x, x_1)$ and $b = d(y, y_1)$; then $t \in B_{a+L_1/2}(x) \cap B_{b+L_1/2}(y)$, though these balls (call them B_1, B_2) have disjoint interiors.

Without loss of generality say $L_3 \geq L_2$ (i.e. $d(x_2, z) \leq d(y_1, z)$).

Claim 2 $[x_2 z] \cap int(B_2) = \emptyset$.

Suppose not, and let $s \in [x_2 z] \cap int(B_2)$. As $s \notin B_1$, it follows that $d(s, x_2) \geq L_1/2$. Since $L_3 \geq L_2$, there is a point $u \in [y, z]$ such that $d(u, z) = d(s, z)$. It follows that

$$\begin{aligned} L_1/2 \leq d(s, x_2) &= d(x_2, z) - d(z, s) \\ &= d(x_2, z_1) + d(z_1, z) - d(z, s) \\ &\leq d(z_2, y_1) + d(z_1, z) - d(z, u) \\ &= d(z, y_1) - d(z, u) = d(u, y_1) \end{aligned}$$

Hence $u \notin int(B_2)$; but

$$d(z, y) = d(z, u) + d(u, y) \leq d(z, s) + d(s, y)$$

and so

$$b + L_1/2 \leq d(u, y) \leq d(s, y) < L_1/2 + b.$$

This contradiction establishes claim 2.

Let v be a point on the edge $[yz]$ such that $d(y, v) = b + L_1/2$. There is a path from t to v in the complement of B_2 of length at most

$$d(t, x_1) + 3e_0 + L_3 + 3e_0 + d(z_2, v) \leq L_1/2 + 6e_0 + L_1 + L_1/2.$$

Hence $e(L_1/2) \leq 2L_1 + 6e_0$, giving the required bound for L_1. $\square$

Chapter 3 Quasigeodesics

In this chapter we shall study the infinite cyclic subgroup $\langle g\rangle$ generated by a non-torsion element g in a word hyperbolic group. We shall see that the set of vertices in the Cayley graph $\Gamma_X(G)$ which correspond to $\langle g\rangle$ form something like a geodesic (a *quasigeodesic*). This in turn is used to obtain various results which are analogous to existing results about groups acting discretely and cocompactly on hyperbolic space. For instance, we show that an abelian subgroup of a hyperbolic group is a finite extension of a cyclic group (i.e. there are no $\mathbf{Z}\times\mathbf{Z}$ subgroups in a hyperbolic group). Most of the treatment here is due to Mihalik and Lustig.

In a geodesic metric space, we can define the length of an arc α between two points as $d_\alpha(x,y) = sup\sum d(x_i,x_{i+1})$ over finite sets of points x_i on α between the points x, y.

Or, working in the domain, in order to give a precise sense to 'between' :

let $\alpha : [0,1] \to X$ be a path from x to y in X. We define the length of the arc to be $d_\alpha(x,y) = \lim_{n\to\infty}\sum_{i=0}^{n-1} d(\alpha(\frac{i}{n}),\alpha(\frac{i+1}{n}))$.

Definition 3.1

An arc α in a geodesic metric space X is called a (λ,ϵ)-*quasigeodesic* if there are positive constants $\lambda > 1, \epsilon \geq 0$ such that for all points x, y on α,

$$d_\alpha(x,y) \leq \lambda d(x,y) + \epsilon.$$

We must first prove the following technical result, which is essential to all that follows.

Proposition 3.2. ([G, 8.1.D], [GH,8.21])

Let g be an element of infinite order in a hyperbolic group G, and let Γ be the Cayley graph with respect to some finite generating set. Let α be a path from the vertex corresponding to the identity element to the vertex corresponding to g. Then the bi-infinite path

$$(\ldots, g^{-1}\alpha, \alpha, g\alpha, \ldots)$$

is a quasigeodesic.

Proof. Suppose that the positive integer R is given, and choose k such that $d(g^k,1) > 8R + 2\delta$, Let β be the geodesic from 1 to g^k, and let y be the midpoint of β. Let I be the subinterval of β of length R centered on y. Recall that $B_r(z)$ denotes the ball of radius r about the point z in Γ. In what follows, by 'midpoint'of a geodesic we mean a vertex at distance at most 1/2 from the actual midpoint of the arc.

Claim 1 If $p \in B_R(1)$ and $q \in B_R(g^k)$, then the midpoint m_1 of the geodesic arc $[pq]$ is in $N_{2\delta}(I)$, i.e. $d(m_1, I) < 2\delta$.

Note that if $a \in B_R(1)$ and $b \in B_r(g^k)$ then the midpoint of $[ab]$ is at least distance R from balls of radius $R+\delta$ about 1 and g^k, by the choice of k.

If m_2 is the midpoint of $[p, g^k]$ then as $|\ell([pg^k])-\ell([pq])| < R$, we have $|d(m_1,p)-d(m_2,p)| < R/2$.

By assumption geodesic triangles are δ-thin in Γ. Considering the geodesic triangle pqg^k, we see that the internal points are all inside the ball $B_{R+\delta}(g^k)$, so that there is a point $m_1' \in [pg^k]$ such that

(1) $d(m_1,p) = d(m_1',p)$,

(2) $d(m_1', m_2) < R/2$,
(3) $d(m_1, m_1') \leq \delta$,
(4) $d(m_1', 1) > 3/2R + \delta$.

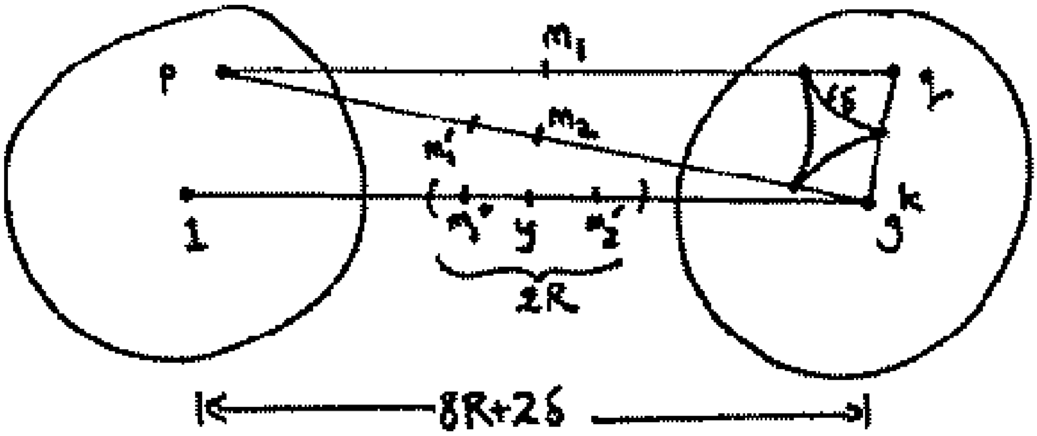

Figure 3.1

Similarly the internal points of the geodesic triangle $1pg^k$ are all inside the ball $B_{R+\delta}(1)$, and there are points m_2', m_1'' on $[1g^k]$ satisfying

(1) $d(m_2', g^k) = d(m_2, g^k)$ and $d(m_1'', g^k) = d(m_1', g^k)$,
(2) $d(m_2', m_2) \leq \delta$ and $d(m_1'', m_1') \leq \delta$,
(3) $d(m_1'', m_2') = d(m_1', m_2)$,
(4) $d(m_2', y) < R/2$

It follows from (2) that $d(m_1'', m_1) \leq 2\delta$, and from (3) and (4) that $d(y, m_1'') < R$. The claim is thus established.

Let N be the number of distinct vertices of Γ in the ball $B_{2\delta}(1)$; the 2δ neighbourhood of the interval I then contains at most RN vertices. Now consider the midpoints of each translate of the arc $\alpha = [1g^k]$ by each of the elements $1, g, g^2, \dots g^{NR}$. These midpoints are all distinct (else some power of g would act on Γ fixing a point, and thus would have finite order), and there are $1 + NR$ of them. Hence there is a number $p(R) \leq NR$ such that $g^{p(R)} \notin B_R(1)$ (and so $g^{k+p(R)} \notin B_R(g^k)$). Notice that $p(R) \geq R/\ell(g)$.

Claim 2 $\ell(g^{NR}) \geq R$ for all R.

Suppose that the claim is false, and thus that there is some R_0 with $\ell(g^{NR_0}) < R_0$. For all $s > NR_0$, let $s = nNR_0 + R_1$, with $n \in \mathbf{Z}^+$ and $0 \leq R_1 < NR_0$ Then

$$\ell(g^s) \leq \ell(g^{nNR_0}) + \ell(g^{R_1}) \leq n\ell(g^{NR_0}) + \ell(g^{R_1}) < n(R_0 - \epsilon) + \ell(g^{R_1})$$

$$< nR_0 \quad \text{when} \quad n\epsilon > \ell(g^{R_1})$$

But this means that for all sufficiently large values of n, we have that $\ell(g^s) < nR_0$. If we choose a value of R such that $p(R) > NR_0$, then by claim 1, we have that $\ell(g^{p(R)}) \geq R$. But the above says that $ell(g^{p(R)}) < p(R)/N \leq R$, giving the required contradiction.

We now show that the bi-infinite arc $\beta = (\ldots, g^{-1}\alpha, \alpha, g\alpha, \ldots)$ is a quasigeodesic. Let γ be a geodesic arc between two points x, y on β. By construction, the initial and final points of γ are within $N\ell(g)$ of vertices g^{aN} and g^{bN} for some integers a, b, and so the length $d_\beta(x, y)$ of the subarc of β between the points x, y is at most $(|b-a|+2)\ell(g)N$. But by the above, $d(g^{aN}, d^{bN}) \geq |b-a|$, and so

$$d(x, y) \geq |b-a| - 2\ell(g)N.$$

This means that

$$d_\beta(x, y) \leq |b-a|\ell(g)N + 2\ell(g)N \leq \ell(g)Nd(x, y) + 2\ell(g)^2N^2 + 2\ell(g)N$$

and it follows that β is a (λ, ϵ)-quasigeodesic for

$$\lambda = \ell(g)N \quad \text{and} \quad \epsilon = 2\ell(g)^2N^2 + 2\ell(g)N.$$

□

We now show that geodesic arcs between points on a quasigeodesic lie near the quasigeodesic. More precisely, let $N_r(U)$ denote the r neighborhood of the subset U of X.

Proposition 3.3. ([G, 7.2.A], [CDP,3.1.3], [GH, 5.6, 5.11])

Let x, y be points in the hyperbolic metric space X. If α is a (λ, ϵ) quasigeodesic between the points x, y, there are integers $L(\lambda, \epsilon)$, $M(\lambda, \epsilon)$ such that if γ is any geodesic $[xy]$, then $\gamma \subset N_L(\alpha)$ and $\alpha \subset N_M(\gamma)$.

Proof. Let $e : \mathbf{N} \to \mathbf{R}^+$ be an exponential divergence function for X. We first show the existence of the bound L.

Let $D = sup_{x \in \gamma}\{d(x, \alpha)\}$, and choose a point $p \in \gamma$ where this supremum is reached. Then

$$IntB_D(p) \cap \alpha = \emptyset$$

Let a, b be points on γ at distance D from p, and let a', b' be points on γ at distance $2D$ from p (or the points x or y if these are at distance $\leq 2D$ from p). There are points u, v on α such that $d(a', u) \leq D$, $d(b', v) \leq D$. Notice that $([a', u] \cup [b', v]) \cap IntB_D(p) = \emptyset$. Following a path via a', p and b', we see that $d(u, v) \leq 6D$, and as α is a (λ, ϵ) quasigeodesic, we have $d_\alpha(u, v) \leq 6\lambda D + \epsilon$. Hence there is a path of length $\leq 4D + 6\lambda D + \epsilon$ from a to b which does not meet $IntB_D(p)$.

But the divergence function e says that the length of such a path is exponential in $D - (e(0)/2)$. This gives us the bound $L(\lambda, \epsilon)$ for D. (Notice again that we have only used the fact that the divergence function is nonlinear.)

Now suppose that $\alpha \not\subset N_L(\gamma)$. Then a component of $cl(\alpha - N_L(\gamma))$ is a path ξ with endpoints u, v at distance L from points a, b say on γ.

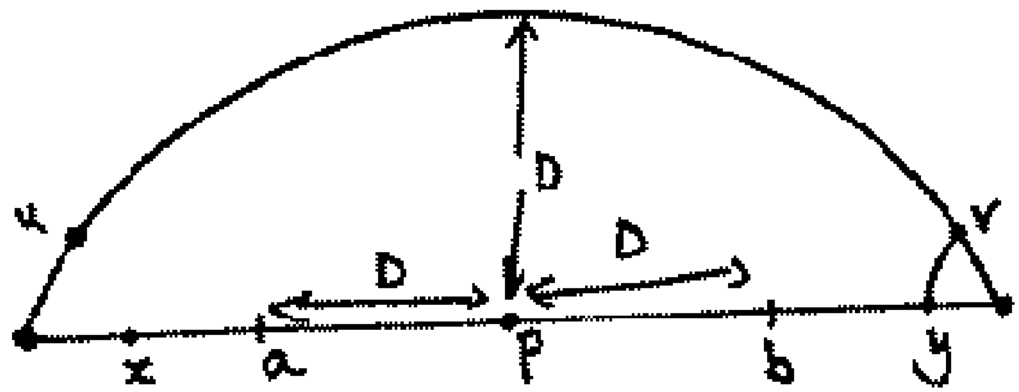

Figure 3.2

If α is an arc $[0,1] \to X$, and $\xi = \alpha([t_1, t_2])$, then each point on γ between a and b is within distance L of some point of $\alpha([0,t_1]) \cup \alpha([t_2,1]) = \alpha_1 \cup \alpha_2$ by the first part of the proof. Then there is some point z on γ between a and b which is at distance $\leq L$ from a point u_1 on α_1 and from a point u_2 on α_2. But then $d(u_1, u_2) \leq 2L$, so $d_\alpha(u_1, u_2) \leq 2\lambda L + \epsilon$, bounding the length of the arc ξ. It thus follows that every point on ξ is at distance at most $L + \lambda L + \epsilon/2$ from γ, and the proposition holds. □

Notice that when the geodesic space under consideration is the Cayley graph of a hyperbolic group, the above results say

Corollary 3.4. *If g is an element of infinite order in a hyperbolic group, then there is a constant L such that for any point x on a geodesic arc $[g^i g^j]$ there is an integer k such that $d(x, g^k) < L$.*

We now use these results on quasigeodesics to obtain information about subgroups of a hyperbolic group. In particular we shall show that an abelian subgroup of a hyperbolic group is cyclic-by-finite. This follows from:

Proposition 3.5. ([CDP, 10.7.2], [GH, 8.35])

Let H be a hyperbolic group, and let g be an element of G of infinite order. Let $C(g)$ denote the centralizer of g, and let $\langle g \rangle$ denote the infinite cyclic group generated by g.

Then $C(g)/\langle g \rangle$ is finite.

Proof. Let Γ be the Cayley graph of G with respect to some finite generating set, and suppose that geodesic triangles are δ thin in Γ. Let L be the constant (guaranteed by the above proposition) such that the geodesic $[1g^n]$ lies in a L-neighborhood of the set $\{1, g, g^2, \ldots, g^n\}$. Let $s \in C(g)$, and choose m such that

$d(1, g^m) > 2\ell(s) + 2\delta$ (it is here that we use the fact that g has infinite order). Consider the geodesic 4-gon (parallelogram) with vertices $1, g^m, s, sg^m$ and sides $[s(sg^m)] = s[1g^m]$, $[g^m(sg^m)] = g^m[1s]$, $[1g^m]$ and $[1s]$.

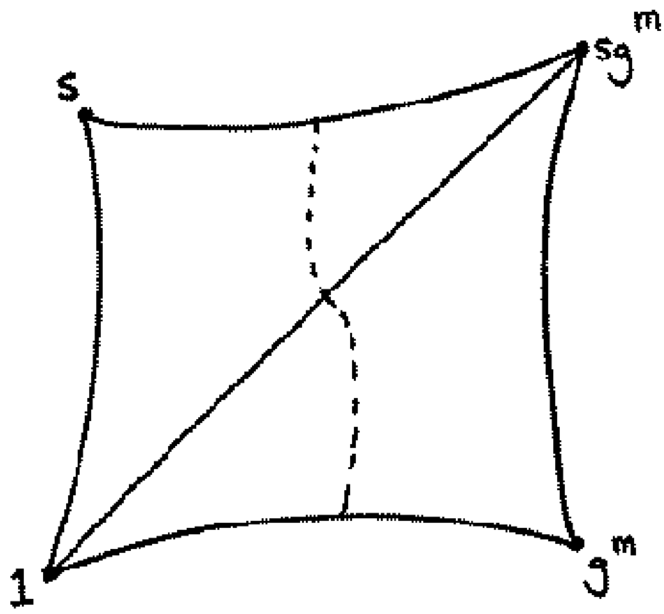

Figure 3.3

By considering the two (δ-thin) geodesic triangles with vertices $1, g^m, sg^m$ and $1, s, sg^m$ we see that either some point of $[1s]$ is at distance $\leq 2\delta$ from some point of $[g^m(sg^m)]$, or there is a point on $[1g^m]$ which is at distance $\leq 2\delta$ from some point of $[s(sg^m)]$. By the choice of m, the former cannot occur, else there is a path of length at most $2\ell(s) + 2\delta$ from 1 to g^m. So let $a \in [1g^m]$ and $b \in [s(sg^m)]$ such that $d(a, b) \leq 2\delta$. But there are integers $i, j < m$ such that $d(a, g^i) \leq L$ and $d(b, sg^j) \leq L$, so $d(g^i, sg^j) \leq 2L + 2\delta$. But then $d(1, sg^{j-i}) \leq 2L + 2\delta$, and so every coset $s\langle g \rangle$ in $C(g)$ intersects the ball of radius $2L + 2\delta$ about the identity. $\square$

Corollary 3.6.

An abelian subgroup of a hyperbolic group which contains an element of infinite order is finite-by-cyclic.

Definition 3.7 ([G, 5.3 page 139, and 7.3 page 191], [CDP, chap. 10])

If X is a geodesic metric space then a subset A is ϵ-*quasiconvex* if for all geodesics $[ab]$ with endpoints $a, b \in A$, $[ab] \subset N_\epsilon(A)$. A subgroup of a finitely generated group is said to be quasiconvex if the vertices in the subgroup form a quasiconvex subset of the Cayley graph.

It is not hard to see that this last definition is independent of the finite set of generators chosen (for more about quasiconvexity see [S]).

It follows from 3.3 that an infinite cyclic subgroup of a hyperbolic group defines a quasiconvex subset of the Cayley graph.

Also, a subgroup of finite index is quasiconvex. A finitely generated subgroup of a finitely generated free group is quasiconvex, as can be seen by thinking of the Cayley graph as a tree.

Rips has shown, [R], that a small cancellation group may have finitely generated subgroups which are infinitely related. This cannot happen for quasiconvex subgroups:

Lemma 3.8. ([CDP, 10.4.2])
A quasiconvex subgroup H of a hyperbolic group G is finitely generated, and in fact is also hyperbolic .

Proof. Let X be finite set of generators for G. Let $w = a_1 \dots a_n$, $a_i \in X \cup X^{-1}$ be a geodesic in the Cayley graph $\Gamma_X(G)$ which ends at a vertex which lies in the subgroup H. The quasiconvexity condition implies that there for each $i = 1, \dots, n$ there is a word $v(i) \in F(X)$ of length at most ϵ such that $a_1 \dots a_i v(i)$ represents an element of H. Thus $w = \prod_{i=1}^n v(i-1)^{-1} a_i v(i)$ where $v(0) = v(n) =$ the empty word, and the set Y of words $v \in F(X)$ of length at most $2\epsilon + 1$ is a finite set of generators for H.

Now let $d_Y(h, h')$ denote the distance between the vertices $h, h' \in H$ in $\Gamma_Y(H)$, and $d_X(g, g')$ denote distance between the vertices $g, g' \in G$ in $\Gamma_X(G)$, and $d_{X\cup Y}(g, g')$ denote the distance between the vertices g, g' in $\Gamma_{X\cup Y}(G)$.

Then by rewriting the elements of Y as words in $F(X)$, we see that

$$(2\epsilon + 1) d_Y(h, h') \geq d_X(h, h') \geq d_{X\cup Y}(h, h').$$

Also, the way the generators Y were found shows that $d_Y(h, h') \leq d_X(h, h')$. Again by rewriting words in $F(X \cup Y)$ as words in $F(X)$ we see that $(2\epsilon+1) d_{X\cup Y}(g, g') \geq d_X(g, g')$ Thus

$$\frac{1}{2\epsilon + 1} d_{X\cup Y}(h, h') \leq d_Y(h, h') \leq (2\epsilon + 1) d_{X\cup Y}(h, h')$$

and a geodesic word for an element $h \in H$ (in terms of the generators Y) is a $(2\epsilon + 1)$–quasigeodesic for h in terms of the generators $X \cup Y$. Thus a geodesic triangle in $\Gamma_Y(H)$ has quasigeodesic sides when regarded as a triangle in $\Gamma_{X\cup Y}(G)$, and these quasigeodesics lie close to geodesics, which are close in $\Gamma_{X\cup Y}(G)$, as G is hyperbolic. Thus the original geodesic triangle was slim.

□

Proposition 3.9.
If $1 \to N \to G \to B \to 1$ is a short exact sequence of infinite groups, with G hyperbolic, then N is not quasiconvex.

Proof. Let Γ be the Cayley graph of G with respect to some finite generating set, and suppose that N is quasiconvex in Γ. Let $p : \Gamma \to \Lambda$ be the quotient map of the action of N on Γ. When $x, y \in \Lambda$ and α is a geodesic of length n from x to y, for any preimages $a \in p^{-1}(x)$ and $b \in p^{-1}(y)$, we have that $d(a, b) \geq n$. Since B is infinite and finitely generated, Λ is a locally finite, infinite 1-complex. Choose a geodesic β of length $K > 2\epsilon + 2\delta$ with initial point $p(1)$ and endpoint b, and let $\tilde{\beta}$ be a lift of β based at 1, with endpoint c. Note that $\tilde{\beta}$ is also geodesic of length K. Choose $u \in N$ such that $\ell(u) > 2K + 2\delta$ and consider the geodesic 4-gon in Γ with sides $[1u]$, $\tilde{\beta}$, $u\tilde{\beta}$, $[c(uc)]$.

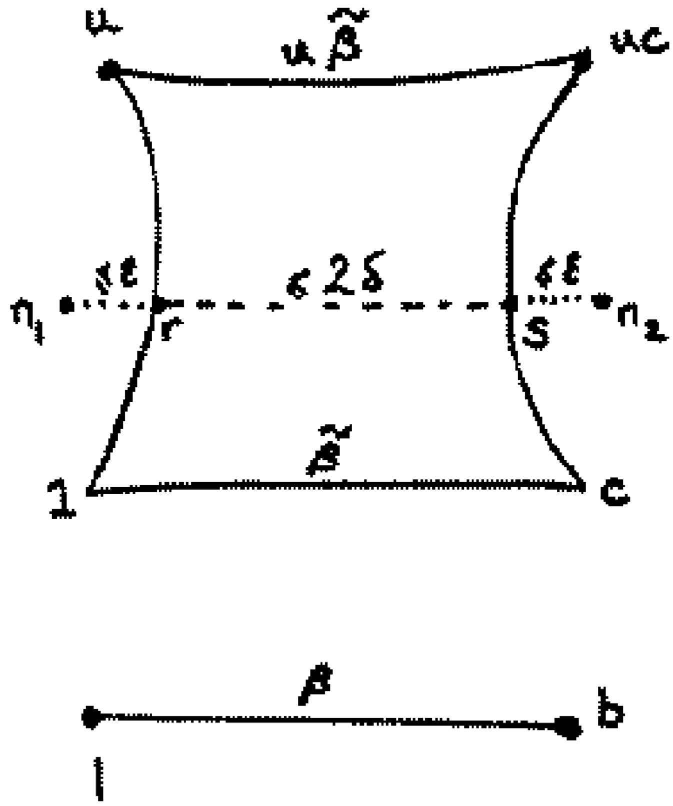

Figure 3.4

As we choose u to be long, no point of $u\tilde{\beta}$ is within 2δ of a point of $\tilde{\beta}$. As in the proof of proposition 3.5, some point r of $[1u]$ is within 2δ of a point s of $[c(uc)]$. As N is normal, $c^{-1}uc \in N$ so $uc = c(c^{-1}uc) \in cN = p^{-1}(b)$; also we have that $1, u \in N = p^{-1}(1)$. Since N is ϵ-quasiconvex for some ϵ, some point n_1 of N is within ϵ of r and some point n_2 of cN is within ϵ of s. But then n_1 is within $2\epsilon + 2\delta$ of n_2, contradicting the restriction on the length of β. □

S. M. Gersten has pointed out that the above proof shows:

Corollary 3.10.

If G is a hyperbolic group, and H is an infinite quasiconvex subgroup of G, then each element of the factor group $N_G(H)/H$ is of finite order.

In general, it does not immediately follow that $N_G(H)/H$ is finite.
<u>Remark</u> M. Mihalik and W. Towle have since then used the methods of the above proof to show that if H is a quasiconvex subgroup of a hyperbolic group, then H has finite index in $N_G(H)$, the normalizer of H in G, and this index is bounded by a function depending on δ, ϵ and the number of generators for G. Furthermore they show that if H is a quasiconvex subgroup of the hyperbolic group G and $x \in G$ such that xHx^{-1} is contained in H, then $xHx^{-1} = H$.

When G is a finitely generated free group, all finitely generated subgroups are quasiconvex, so that 3.8 reduces to the result that a non–trivial finitely generated normal subgroup of a finitely generated free group is of finite index (due originally to Schreier).

Using results from [BGSS], it is shown in [GS2] that $C_G(H)/H$ is finite.

We finish off this chapter by relating quasigeodesics with the existence of a Dehn presentation and local geodesics (see definition 1.11 and 2.12–2.17). Recall that a path is a k-local geodesic if all subpaths of length at most k are geodesic.

Theorem 3.11. (*cfr.* [Can2])

Let X be a geodesic metric space where all geodesic triangles are δ-thin.

If u is a 10δ-local geodesic, then u is a $(12, 7\delta)$-quasigeodesic.

Proof. If v is a geodesic path in X, and $x \in X$, let $p(x)$ be a point on v at minimum distance from x.

Lemma 3.12.

Let u be a 10δ-local geodesic, and v a corresponding geodesic.

If $d(p(u(t)), p(u(t+a))) \leq \delta/2$ then $|a| \leq 13\delta/2$.

Proof of Lemma 3.12. Recall that Theorem 2.15 shows that a 4δ-local geodesic lies in a 3δ-neighborhood of a corresponding geodesic. It follows that $d(u(t), p(u(t))) \leq 3\delta$, and $d(u(t+a), p(u(t+a))) \leq 3\delta$, so that $d(u(t), u(t+a)) \leq 13\delta/2$.

Let u_0 be the segment of u between $u(t)$ and $u(t+a)$, and let v_0 be a geodesic between the same points. As before, u_0 lies in a 3δ neighborhood of v_0 and moreover v_0 lies in a $13\delta/2$ neighborhood of $u(t)$. It follows that u_0 lies in a $19\delta/2$ neighborhood of $u(t)$. If $\ell(u_0) \geq 10\delta$, then an initial 10δ segment is geodesic, so leaves the ball of radius $19\delta/2$, which is not possible. Thus $\ell(u_0) < 10\delta$ and, as it is geodesic, $\ell(u_0) \leq 13\delta/2$, and $|a| \leq 13\delta/2$. □

Returning to the proof of the theorem, let v be a geodesic corresponding to u (i.e. a geodesic between the endpoints of u). Divide v into subintervals $v_1, v_2, \ldots, v_n$ where $\ell(v_i) \leq \delta/2$ and $n \leq 1 + 2\ell(v)/\delta$.

Let $u_i = p^{-1}(v_i) \cap u$; then $diam(u_i) \leq 13\delta/2$ by the lemma. It follows that

$$\ell(u) \leq \sum_{i=1}^{n} diam(u_i) \leq n\frac{13\delta}{2} \leq \frac{13\delta}{2}(\frac{2\ell(v)}{\delta} + 1) = 13\ell(v) + \frac{13\delta}{2}$$

and u is therefore a $(13, 7\delta)$-quasigeodesic. □

By more careful bookkeeping, we can improve this to say that an 8δ-local geodesic is a $(8, 7\delta)$-quasigeodesic.

Chapter 4 The boundary

The aim of this section is to define a boundary ∂X for a hyperbolic metric space X, which gives a compactification $\hat{X} = X \cup \partial X$ when X is complete and locally compact. We follow the plan indicated by Gromov in his section 1.8, [G], using some of the ideas of [CDP]; the structure of the boundary is much more extensively developped in [CDP] and in [GH].

As usual we use $(x.y)$ to denote the inner product on the metric space X with respect to some basepoint w, i.e.

$$(x.y) = \frac{1}{2}\{d(x,w) + d(y,w) - d(x,y)\}.$$

The points of ∂X are defined to be equivalence classes of sequences in X as follows.

Definition 4.1: A sequence $\{a_i\}$ of points in X is said to *converge to infinity* if

$$\lim_{i,j\to\infty} (a_i.a_j) = \infty.$$

Note:

(1) This definition is independent of the choice of basepoint since $|(x.y)_w - (x.y)_{w'}| \leq d(w,w')$.
(2) If $\{a_i\}$ converges to infinity then
$$\lim_{i\to\infty} d(a_i,w) = \infty$$
since $(x.y) \leq \min\{d(x,w), d(y,w)\}$.

Let $S_\infty(X)$ denote the set of all sequences convergent to infinity, and define the relation

$$\{a_i\}R\{b_i\} \text{ iff } \lim_{i\to\infty} (a_i.b_i) = \infty.$$

Note that a sequence which converges to infinity is related to all of its subsequences.

The relation is symmetric and reflexive and independent of the choice of basepoint. For a general metric space R is not transitive as the following example shows.

Example. Let X be the Cayley graph of the group

$$\mathbf{Z} \times \mathbf{Z} = < x, y \mid xy = yx >$$

(with respect to the generators x and y) and let w be the vertex corresponding to the identity element, $e = x^0y^0$. Define $a_n = x^n$, $b_n = y^n$ and $c_n = x^ny^n$. Then the sequences $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ all converge to infinity and $(a_n.c_n) = (b_n.c_n) = n$ while $(a_n.b_n) = 0$. Thus $\{a_i\}R\{c_i\}$ and $\{b_i\}R\{c_i\}$, but $\{a_i\}$ and $\{b_i\}$ are not related.

Example. With X as in the last example, consider the sequence defined by

$$p_n = \begin{cases} x^m, & \text{if } n = 2m \\ x^my^m, & \text{if } n = 2m+1. \end{cases}$$

Then $\{p_n\} \in S_\infty(X)$ but the two subsequences consisting of odd and even terms are tending to infinity in different directions.

Recall that we say X is δ-hyperbolic if for all $x, y, z \in X$

$$(x.y) \geq \min\{(x.z), (y.z)\} - \delta.$$

Lemma 4.2. *When X is δ-hyperbolic the relation R is transitive.*

Proof. If $\{a_i\}, \{b_i\}, \{c_i\} \in S_\infty(X)$ with $\{a_i\}R\{b_i\}$ and $\{b_i\}R\{c_i\}$ then $\{a_i\}R\{c_i\}$ since $(a_i.c_i) \geq min\{(a_i.b_i), (b_i.c_i)\} - \delta$.

□

From now on we assume that X is a δ-hyperbolic metric space.

Observe that for $\{a_i\}, \{b_i\} \in S_\infty(X)$ we have

$$\{a_i\}R\{b_i\} \text{ iff } \lim_{i,j\to\infty}(a_i.b_j) = \infty.$$

Definition 4.3 The *boundary of* X is $\partial X = S_\infty(X)/R$. We say that $\{a_i\} \in S_\infty(X)$ *converges to* $x \in \partial X$ if $x = [\{a_i\}]$ and we write $a_i \to x$.

Note. If $x \in X$ then '$x_i \to x$' has the usual meaning of convergence in the metric sense.

Example. Take $X = \mathbf{R}$ with the usual metric and 0 as the basepoint. If $\{a_i\} \in S_\infty(X)$ then either $a_i > 0$ for almost all values of i or $a_i < 0$ for almost all values of i. This defines two distinct equivalence classes of sequences which we call $+\infty$ and $-\infty$ respectively and $\partial X = \{-\infty, +\infty\}$ as expected. Similarily for $\mathbf{R}$-trees the boundary is the set of ends.

In order to put a topology on the set $\hat{X} = X \cup \partial X$ we first extend the inner product to the boundary.

Definition 4.4: For $x, y \in \hat{X}$,

$$(x.y)_S = \inf\{\liminf_i (x_i.y_i)\},$$

where the infimum is taken over all pairs of sequences $x_i \to x$ and $y_i \to y$.

Note. This definition may seem unduly complicated. The next example illustrates the need for this complexity even in a very simple metric space.

Example. Let X be the Cayley graph (with respect to the generators x and y) of

$$\mathbf{Z} \times \mathbf{Z}_2 = < x, y \mid y^2 = 1,\ xy = yx > .$$

This space, like $\mathbf{R}$ above, has a boundary consisting of two points which we will also call $+\infty$ and $-\infty$. Let

$$a_n = x^n,\ b_n = x^{-n},\ c_n = yx^n,\ d_n = yx^{-n}$$

$$z_m = \begin{cases} x^{2n}, & \text{if } m = 2n \\ yx^{2n+1} & \text{m = 2n+1.} \end{cases}$$

Then a_n, c_n and z_n all tend towards $+\infty$, while b_n and d_n tend towards $-\infty$.

Observe that $(d_n.z_n)$ is 0 if n is even, and 1 if n is odd. Thus we use lim inf rather than lim in the definition. Furthermore $(a_n.b_n) = 0$, while $(c_n.d_n) = 1$. Thus we take the infimum over all sequences in the definition.

Some properties of the inner product on $\hat{X}$ are given in the following lemmas.

Lemma 4.5.

(1) *If* $x \in X$ *and* $y \in \hat{X}$ *then* $(x.y)_S = \inf\{\liminf_i (x.y_i)\}$ *where the infimum is taken over sequences* $y_i \to y$, *i.e. it suffices to consider the constant sequence at* x.

(2) *For* $x, y \in X$, $(x.y)_S = (x.y)$, *i.e.* $(\,.\,)_S$ *restricts to* $(\,.\,)$ *away from the boundary.*

Proof of (1). Let $x_i \to x$ and $y_i \to y$ be any pair of sequences. Since $x, x_i, y_i \in X$

$$\begin{aligned}
|(x_i.y_i) - (x.y_i)| =& \frac{1}{2}|d(x_i, w) + d(y_i, w) - d(x_i, y_i) \\
& - d(x, w) - d(y_i, w) + d(x, y_i)| \\
=& \frac{1}{2}|d(x_i, w) - d(x, w) + d(x, y_i) - d(x, x_i)| \\
\leq& \frac{1}{2}\{|d(x_i, w) - d(x, w)| + |d(x, y_i) - d(x_i, y_i)|\} \\
\leq& \frac{1}{2}\{d(x_i, x) + d(x, x_i)\} \\
=& d(x_i, x)
\end{aligned}$$

$$\Rightarrow \liminf_i (x_i.y_i) = \liminf_i (x.y_i).$$

Proof of (2). This follows from 4.5.1.

□

Lemma 4.6.

(1) *If* $x, y \in \hat{X}$ *then*

$$(x.y)_S = \infty \Leftrightarrow x, y \in \partial X \text{ and } x = y.$$

(2) *If* $x \in \partial X$ *and* $\{x_i\}$ *is any sequence of points in* X *then*

$$(x_i.x)_S \to \infty \Leftrightarrow \{x_i\} \in S_\infty(X) \text{ and } x_i \to x.$$

(3) *If* $x, y \in \hat{X}$ *there are sequences* $\bar{x}_i \to x$, $\bar{y}_i \to y$ *with*

$$\lim_{i\to\infty} (\bar{x}_i.\bar{y}_i) = (x.y)_S$$

and if x *or* y *lies in* X *then the corresponding sequence can be chosen to be the constant sequence.*

(4) *If* $x, y \in \partial X$ *and* $x_i \to x$, $y_i \to y$ *then*

$$(x.y)_S \leq \liminf_i (x_i.y_i) \leq (x.y)_S + 2\delta.$$

(5) *If* $x, y, z \in \hat{X}$ *then*

$$(x.y)_S \geq \min\{(x.z)_S, (z.y)_S\} - \delta.$$

(6) *Let* $x, y \in \hat{X}$ *and* $y_i \to y$; *then*

$$\liminf_i (x.y_i)_S \geq (x.y)_S.$$

Proof of (1).

($\Leftarrow$): This follows from the definition of $(x.y)_S$.

($\Rightarrow$): First we show that $x, y \in \partial X$. Suppose, for example, that $x \in X$; then

$$(x.y)_S = \inf\{\liminf_i (x.y_i)\}$$

where the infimum is taken over sequences $y_i \to y$. But $(x.y_i) \leq d(w, x) < \infty$. Thus $x \in X \Rightarrow (x.y)_S < \infty$.

Now suppose that $x, y \in \partial X$ with $a_i \to x$ and $b_i \to y$. Then

$$\liminf_i (a_i.b_i) \geq \inf_{x_i \to x,\, y_i \to y} \{\liminf_i (x_i.y_i)\} = \infty.$$

Thus

$$\lim_i (a_i.b_i) = \infty \text{ and } x = y.$$

Proof of (2). ($\Leftarrow$): Let $\{x_i\} \in S_\infty(X)$ with $x_i \to x$ and suppose that

$$\lim_i (x_i.x)_S \neq \infty.$$

Then the sequence of real numbers, $\{(x_i.x)_S\}$, has a bounded subsequence. Since $\{x_i\}$ is related to all its subsequences we can assume, by passage to a subsequence, that for some M, $(x_i.x)_S < M$, $\forall i$. Thus

$$\inf\{\liminf_j (x_i.y_j)\} < M,$$

where the infimum is taken over all sequences $y_j \to x$. Thus for each x_i, we can find a sequence $y_j^{(i)} \to x$ as $j \to \infty$ satisfying $(x_i.y_j^{(i)}) < M$, $\forall j$. Since X is δ-hyperbolic we also have

$$(x_i.y_j^{(i)}) \geq \min\{(x_i.x_k), (x_k.y_j^{(i)})\} - \delta.$$

This gives a contradiction because the quantity $(x_i.x_k)$ can be made arbitrarily large since $\{x_i\} \in S_\infty(X)$, and the quantity $(x_k.y_j^{(i)})$ can be made arbitrarily large since $\{x_k\} R \{y_j^{(i)}\}$.

($\Rightarrow$): First we show that $\{x_i\} \in S_\infty(X)$. Suppose that $(x_i.x)_S \to \infty$. Then there is a sequence $\{n_i\}$ of real numbers with

$$\lim_{i \to \infty} n_i = \infty$$

such that, for all i,

$$\inf\{\liminf_j (x_i.y_j)\} > n_i.$$

So, for each i, there is a sequence $y_j^{(i)} \to x$ as $j \to \infty$ satisfying $(y_j^{(i)}.x_i) > n_i$ for each j. Thus

$$\begin{aligned}
(x_i.x_j) &\geq \min\{(x_i.y_k^{(i)}), (y_k^{(i)}.x_j)\} - \delta \\
&\geq \min\{(x_i.y_k^{(i)}), (y_k^{(i)}.y_l^{(j)}), (y_l^{(j)}.x_j)\} - 2\delta \\
&\geq \min\{n_i, (y_k^{(i)}.y_l^{(j)}), n_j\} - 2\delta \\
&\geq \min\{n_i, n_j\} - 2\delta
\end{aligned}$$

for k and l large enough since $\{y_k^{(i)}\}R\{y_l^{(j)}\}$. Thus

$$\lim_{i,j\to\infty}(x_i.x_j)=\infty$$

and $\{x_i\}\in S_\infty(X)$.

Finally we show that $x_i\to x$. Assume $\{x_i\}\in S_\infty(X)$ with $(x_i.x)\to\infty$. Let $y_j^{(i)}\to x$ be as above and fix a specific sequence $z_i\to x$. Then

$$\begin{aligned}(x_i.z_i)&\geq \min\{(x_i.y_j^{(i)}),(y_j^{(i)}.z_i)\}-\delta\\ &\geq \min\{(x_i.y_j^{(i)}),(y_j^{(i)}.z_j),(z_j.z_i)\}-2\delta\\ &\geq \min\{(x_i.y_j^{(i)}),(z_j.z_i)\}-2\delta\end{aligned}$$

for j large enough since $\{y_j^{(i)}\}R\{z_j\}$. So

$$\lim_{i\to\infty}(x_i.z_i)=\infty$$

and $x_i\to x$.

Proof of (3). Let $x,y\in\partial X$ and let

$$A=(x.y)_S=\inf\{\liminf_i(x_i.y_i)\}.$$

Then for each $n\in\mathbf{Z}_+$ there are sequences $x_i^{(n)}\to x$ and $y_i^{(n)}\to y$ satisfying

$$A\leq\liminf_i(x_i^{(n)}.y_i^{(n)})\leq A+\frac{1}{n}.$$

By passing to subsequences we can assume that, for each i,

$$A\leq(x_i^{(n)}.y_i^{(n)})\leq A+\frac{1}{n}.$$

Now $x_i^{(n)}\to x$ and $y_i^{(n)}\to y$, so, by 4.6.2,

$$(x_i^{(n)}.x)\to\infty \text{ and } (y_i^{(n)}.y)\to\infty \text{ as } i\to\infty.$$

So, for each n, there is an i_n such that $(x_{i_n}^{(n)}.x)>n$ and $(y_{i_n}^{(n)}.y)>n$. Define $\bar{x}_i=x_{i_n}^{(n)}$ and $\bar{y}_n=y_{i_n}^{(n)}$. By 4.6.2, $\bar{x}_n\to x$ and $\bar{y}_n\to y$ and since $(\bar{x}_n.\bar{y}_n)=(x_{i_n}^{(n)}.y_{i_n}^{(n)})$

$$A\leq(\bar{x}_n.\bar{y}_n)\leq A+\frac{1}{n}.$$

If one of the points, say x, actually lies in X, then the proof is similar but uses 4.5.1 and a constant sequence for x. If both x and y lie in X constant sequences can be used for both.

Proof of (4). It follows from the definition of $(x.y)_S$ that $(x.y)_S \le \liminf(x_i.y_i)$.

To get the other inequality we observe that, by 4.6.3, there are sequences $\bar{x}_i \to x$ and $\bar{y}_i \to y$ satisfying

$$\lim_{i\to\infty}(\bar{x}_i.\bar{y}_i) = (x.y)_S$$

so that

$$(\bar{x}_i.\bar{y}_i) \ge \min\{(\bar{x}_i.x_i), (x_i.y_i), (y_i.\bar{y}_i)\} - 2\delta.$$

Now $(\bar{x}_i.x_i) \to \infty$ and $(y_i.\bar{y}_i) \to \infty$, so, taking lim inf of both sides gives

$$(x.y)_S = \liminf(\bar{x}_i.\bar{y}_i) \ge \liminf(x_i.y_i) - 2\delta.$$

Proof of (5). By 4.6.3 there are sequences $x_i \to x$ and $y_i \to y$ satisfying

$$\lim_{i\to\infty}(x_i.y_i) = (x.y)_S.$$

Now, if $z_i \to z$ is any sequence converging to z,

$$(x_i.y_i) \ge \min\{(x_i.z_i), (z_i.y_i)\} - \delta.$$

Taking the lim inf of both sides gives

$$\begin{aligned}(x.y)_S &\ge \min\{\liminf(x_i.z_i), \liminf(z_i.y_i)\} - \delta \\ &\ge \min\{(x.z)_S, (z.y)_S\} - \delta\end{aligned}$$

by 4.6.4.

Proof of (6). Suppose that $y_i \to y$ satisfies

$$\liminf_i(x.y_i)_S < (x.y)_S - \epsilon < (x.y)_S.$$

By passing to a subsequence we can assume that $(x.y_i)_S < (x.y)_S - \epsilon$. Then for each i, there is a sequence $x_j^{(i)} \to x$ with

$$\liminf_j(x_j^{(i)}.y_i) < (x.y)_S - \epsilon$$

Again passing to subsequences, we can assume that

$$(x_j^{(i)}.y_i) < (x.y)_S - \epsilon.$$

Since $x_j^{(i)} \to x$ we can choose, for each i, a j_i satisfying $(x_{j_i}^{(i)}.x) > i$. This allows us to define a sequence $\bar{x}_i = x_{j_i}^{(i)}$. Then $\bar{x}_i \to x$ by 4.6.2 and

$$\liminf_i(\bar{x}_i.y_i) \le (x.y)_S - \epsilon < (x.y)_S$$

contradicting 4.6.4.

□

Note: As the extended inner product $(\ ,\ )_S$, has all these properties, we shall henceforth drop the suffix, S.

We now propose a basis of open sets for a topology on $\hat{X}$.

Definition 4.7: Let $\mathcal{B}$ be the collection of subsets of $\hat{X}$ consisting of

(1) the usual basis for the metric topology on X, i.e. open neighbourhoods $B_r(x) = \{y \in X, d(x,y) < r\}$, for each $x \in X$ and $r > 0$, and
(2) all sets of the form $N_{x,k} = \{y \in \hat{X} \mid (x.y) > k\}$, for each $x \in \partial X$ and $k > 0$.

Proposition 4.8. *The set $\mathcal{B}$ is a basis for a topology.*

Proof. We need to show that $\mathcal{B}$ satisfies the two requirements for a basis, i.e. that the elements of $\mathcal{B}$ form a cover of $\hat{X}$ and that if $B_1, B_2 \in \mathcal{B}$ with $y \in B_1 \cap B_2$ then there is a $B_3 \in \mathcal{B}$ with $y \in B_3 \subset B_1 \cap B_2$.

$\mathcal{B}$ has been chosen so that the first requirement is automatically satisfied. The proof that $\mathcal{B}$ satisfies the second condition breaks up into cases according to the types of the two basis elements.

Case 1: If B_1 and B_2 are both of type (1) the proof is the usual metric space proof.

Case 2: If one of the neighbourhoods is of type (1) and one is of type (2) then the neighbourhood B_3 will be of type (1). We need to show that if $y \in B(x,\epsilon) \cap N_{z,k}$ then there is an ϵ' satisfying $B(y,\epsilon') \subset B(x,\epsilon) \cap N_{z,k}$.

Since $y \in B(x,\epsilon)$ there is an ϵ_1 satisfying $B(y,\epsilon_1) \subset B(x,\epsilon)$. On the other hand, $y \in N_{z,k}$ means that $(z.y) > k$, so there is an ϵ_2 with $(z.y) > k + \epsilon_2 > k$. Set $\epsilon' = \min\{\epsilon_1, \epsilon_2\}$.

It follows that $B(y,\epsilon') \subset B(x,\epsilon)$. To see that $B(y,\epsilon') \subset N_{z,k}$ let $p \in B(y,\epsilon')$ and apply 4.6.3 to give a sequence $z_i \to z$ satisfying

$$\lim_{i\to\infty}(z_i.p) = (z.p).$$

As in the proof of 4.5.1,

$$|(z_i.p) - (z_i.y)| \leq d(p,y) < \epsilon' \leq \epsilon_2,$$

so that

$$-\epsilon_2 < (z_i.p) - (z_i.y) < \epsilon_2.$$

Applying lim inf to both sides gives

$$-\epsilon_2 \leq \liminf_i (z_i.p) - \liminf_i (z_i.y) \leq \epsilon_2$$

$$\Rightarrow -\epsilon_2 \leq (z.p) - \liminf_i (z_i.y) \leq \epsilon_2.$$

So by 4.5.1,

$$-\epsilon_2 \leq (z.p) - (z.y).$$

$$\begin{aligned} \Rightarrow (z.p) &= (z.p) - (z.y) + (z.y) \\ &> (z.p) - (z.y) + k + \epsilon_2 \\ &\geq -\epsilon_2 + k + \epsilon_2 \\ &= k. \end{aligned}$$

Case 3: If both neighbourhoods are of type (2), we need to show that if

$$y \in N_{x,k} \cap N_{x',k'}$$

then there is a $B_3 \in \mathcal{B}$ with

$$y \in B_3 \subset N_{x,k} \cap N_{x',k'}.$$

If $y \in X$ then this B_3 will be of type (1). By Case 2, there is an ϵ_1 with $B(y, \epsilon_1) \subset N_{x,k}$ and an ϵ_2 with $B(y, \epsilon_2) \subset N_{x',k'}$. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ we get

$$B(y, \epsilon) \subset N_{x,k} \cap N_{x',k'}.$$

If $y \in \partial X$ then the neighbourhood B_3 will be of type (2). Since $N_{y,r_1} \subset N_{y,r_2}$ for $r_1 > r_2$ it suffices to show that, if $x \in \partial X$ and $y \in \partial X \cap N_{x,k}$, with $y \neq x$, then there is an m satisfying $N_{y,m} \subset N_{x,k}$.

Suppose that this is not the case. Then for each m there is a

$$y_m \in (N_{y,m}) \cap (\hat{X} - N_{x,k}),$$

i.e. $(y.y_m) > m$ while $(x.y_m) \leq k$. We consider two subcases:

3a) A subsequence of the y_m's actually lies in X. By passing to this subsequence we get $(y_m.y) > m$, $(y_m.x) \leq k$ and $y_m \in X$. Thus by 4.6.2, $y_m \to y$, giving a contradiction since

$$k < (x.y) \leq \liminf_m (x.y_m) \leq k,$$

where the second inequality follows from 4.6.6.

3b) Suppose only a finite number of the y_m's lie in X. By passing to a subsequence we can assume all the y_m's lie on ∂X.

There is a k' such that

$$k < k' < (x.y) < \infty, \text{ while } (x.y_m) \leq k < k'$$

and $(y_m.y) > m$. By 4.6.3 we can choose, for each m, a pair of sequences $x_i^{(m)} \to x$ and $y_i^{(m)} \to y_m$ satisfying

$$\lim_i (x_i^{(m)}.y_i^{(m)}) = (x.y_m) \leq k < k'.$$

By excluding a finite number of terms, we can further assume that, for each m and for each i, $(x_i^{(m)}.y_i^{(m)}) < k'$. Since $x_i^{(m)} \to x$ we can require that

$$(*) \qquad (x_i^{(m)}.x) > m,$$

for each i and m.

Similarily, since, by 4.6.6,

$$\liminf_i (y_i^{(m)}.y) \geq (y_m.y) > m$$

we can require that

$$(**) \qquad (y_i^{(m)}.y) > m,$$

for each i and m. Now we define $\bar{x}_i = x_i^{(i)}$ and $\bar{y}_i = y_i^{(i)}$. Equations $(*)$ and $(**)$ together with 4.6.2 give $\bar{x}_i \to x$ and $\bar{y}_i \to y$. This gives a contradiction since

$$\begin{aligned} k' < (x.y) &= \inf\{\liminf_i (x_i.y_i)\} \\ &\le \liminf_i (\bar{x}_i.\bar{y}_i) \\ &\le k'. \end{aligned}$$

□

<u>Note</u>: With this topology on $\hat{X}$, the inner product

$$(\ .\ ) : \hat{X} \times \hat{X} \to \mathbf{R}$$

is not continuous. For suppose that $x, y \in \partial X$ and we have a pair of sequences $x_i \to x$ and $y_i \to y$. Then, by 4.6.2, $(x_i.x) \to \infty$ so that

$$\{x_i\} \cap N_{x,k} \neq \emptyset \text{ for all } k.$$

Thus $\{x_i\}$ converges to x in the topological sense. Similarily, $\{y_i\}$ converges to y in the topological sense. If $(\ .\ )$ were continuous then we would have

$$(x_i.y_i) \to (x.y).$$

However, as we have seen in the examples above, $\{(x_i.y_i)\}_i$ need not converge and even $\liminf(x_i.y_i)$ may not be $(x.y)$.

The Rips complex

In this section we describe Rips' construction of a complex on which a word hyperbolic group acts in an specially nice way, allowing one to deduce properties of the group. More details on the construction, and some extensions of the methods to automatic groups are given in J. Alonso's preprint [A] (see also [CDP, chap.5]).

Theorem 4.11. (Rips)

A word hyperbolic group G acts simplicially on a simplicial complex P satisfying:

(1) *P is contractible, locally finite and finite dimensional;*
(2) *on the vertices of P, G acts freely and transitively;*
(3) *the quotient complex P/G is compact.*

Before giving the construction required to prove the theorem, we state several corollaries:

Corollary 4.12. *Let G be a word hyperbolic group.*

(1) *G is finitely presented and is of type FP_∞ (i.e. $\mathbf{Z}$ has a free $\mathbf{Z}G$ resolution of finite type).*
(2) *The fact the action is free on the set of vertices means that the stabilizer of each simplex is finite.*
(3) *If G is torsion–free, then P/G is a finite $K(G,1)$, and G is of type FL, and has finite cohomological dimension (written $cd(G) < \infty$).*
(4) *If G is virtually torsion–free, then the virtual cohomological dimension is finite ($vcd(G) < \infty$).*
(5) *$H_\star(G;\mathbf{Q})$ and $H^\star(G;\mathbf{Q})$ are finite dimensional.*

Proof of Corollary. (1)

To prove this, we quote the following theorem

Theorem. (K. Brown, [Br2])

Let X be a contractible G-complex such that the stabilizer of each cell is finitely presented and of type FP_∞. Suppose that X has a filtration by G-equivariant subcomplexes $\{X_j\}_{j\geq 1}$ such that each X_j is finite mod G. If in addition, for each j there is a j' such that the inclusion maps induce trivial maps $\pi_1(X_j) \to \pi_1(X_{j'})$ and $\widetilde{H}_i(X_j) \to \widetilde{H}_i(X_{j'})$, then G is finitely presented and of type FP_∞.

The filtration used in this context is the sequence of complexes P_d defined below, and a variation of the slight variation of the argument given in the proof of 4.14 shows that every finite subcomplex of P_d collapses in some $P_{d'}$. (The details are given in [A].)

(2) Let S be the stabilizer of the simplex $\{g_1, \ldots, g_n\}$. Define a map from S to the set of permutations of n letters by simply recording the action of each $s \in S$ on the g_i. By (2) in Rips' Theorem, this map is injective so that S is finite.

(3) and (4) This follows from (1). Serre has shown that if G is virtually torsion–free, then $vcd(G) < \infty$ if and only if G acts on some contractible, finite dimensional CW complex with finite stabilizers.

(5) A spectral sequence argument shows that $H_\star(G;\mathbf{Q}) \equiv H_\star(P/G;\mathbf{Q})$ and $H^\star(G;\mathbf{Q}) \equiv H^\star(P/G;\mathbf{Q})$.

□

We now build the complex P_d. The standard resolution of a group G is obtained by taking Y to be the simplex spanned by G. This has a vertex for each element of G, and a simplex for each finite subset of G.

Definition 4.13 Let X be a metric space, and let d be a positive real. We define the simplicial complex $P_d(X)$ to have a vertex for each point in X, and a simplex for each finite subset of X which has diameter at most d.

Proposition 4.14.

Let G be a finitely generated group and let Γ be the Cayley gaph of G with respect to some finite set of generators, regarded as a metric space in the usual way. If Γ is a δ-hyperbolic metric space, then $P_d(\Gamma)$ is contractible for $d \geq 4\delta + 1$.

Proof. As $P_d(\Gamma)$ is simplicial, it is contractible if and only if $\pi_i(P_d(\Gamma)) = 0$ for all $i \geq 0$. It is sufficient to prove that every finite subcomplex K of $P_d(\Gamma)$ is contractible. Choose the identity element x_0 as base point for G.

Case (i) $\max\limits_{y \in K^0} d(x_0, y) \leq d/2$.

Then K lies inside a simplex of $P_d(\Gamma)$, and so is contractible.

Case (ii) $\max\limits_{y \in K^0} d(x_0, y) > d/2$.

Let y_0 be the point in K^0 furthest from x_0. Let y_0' be the point on a geodesic from x_0 to y_0 such that $d(y_0', x_0) = d(y_0, x_0) - [d/2]$, where $[r]$ denotes the integral part of r. We now define a function

$$f : K^0 \to P_d(\Gamma) \text{ by } f(y_0) = y_0',\ f(y) = y,\ y \in K^0 - \{y_0\}.$$

Claim: f can be extended to a simplicial map $K \to P_d(\Gamma)$.

We must show that whenever σ is a simplex in K, $f(\sigma)$ is a simplex in $P_d(\Gamma)$. But simplices of $P_d(\Gamma)$ consist of sets of elements of G of diameter at most d, so we must show that

$$(\star) \qquad \forall y \in K^0,\ d(y, y_0) \leq d \Rightarrow d(y, y_0') \leq d$$

Recall that (see Corollary 2.4) Γ is δ-hyperbolic if and only if for all $x, y, z, t \in \Gamma$,

$$d(x, y) + d(z, t) \leq \max\{d(x, z) + d(y, t), d(x, t) + d(y, z)\} + 2\delta.$$

Replacing (x, y, z, t) by (y, y_0', y_0, x_0), we get

$$d(y, y_0') + d(x_0, y_0) \leq \max\{d(y, y_0) + d(x_0, y_0'), d(y_0, y_0') + d(x_0, y)\} + 2\delta$$

$$\Leftrightarrow d(y, y_0') \leq$$
$$\max\{d(y, y_0) + d(x_0, y_0') - d(x_0, y_0), d(y_0', y_0) + d(y, x_0) - d(x_0, y_0)\} + 2\delta$$
$$\leq \max\{d - [d/2], d/2\} + 2\delta \leq d - [d/2] + 2\delta.$$

This is $\leq d$ when $d \geq 4\delta + 1$, as required.

It remains to show that f is homotopic to the inclusion map; but this follows immediately by noticing that $f(\sigma) \cup \sigma$ is contained in a simplex of $P_d(\Gamma)$, as $d(y_0, y_0') \leq d$.

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