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# Conservative subgroup separability for surfaces with boundary

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If $F$ is a compact surface with boundary, then a finitely generated subgroup without peripheral elements of $G=\pi_1(F)$ can be separated from finitely many other elements of $G$ by a finite index subgroup of $G$ corresponding to a finite cover $\widetilde{F}$ with the same number of boundary components as $F$.

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Suppose $F$ is a compact surface with nonempty boundary. A nontrivial element of $\pi_1(F)$ is *peripheral* if it is represented by a loop freely homotopic into $\partial F$. A covering space $p\colon \widetilde{F}\to F$ is called *conservative* if $F$ and $\widetilde{F}$ have the same number of boundary components: $|\partial F|=|\partial \widetilde{F}|$.

**Theorem 0.1** (Main Theorem) *Let $F$ be a compact, connected surface with $\partial F\neq \phi$ and $H\subset \pi_1(F)$ a finitely generated subgroup. Assume that no element of $H$ is peripheral. Given a (possibly empty) finite subset $B\subset \pi_1(F)\setminus H$, there exists a finite-sheeted cover $p\colon \widetilde{F}\to F$ such that:*

(i) *There is a compact, connected, $\pi_1$–injective subsurface $S\subset \widetilde{F}$ such that $p_*(\pi_1(S))=H$.*

(ii) *$p_*(\pi_1(\widetilde{F}))$ contains no element of $B$.*

(iii) *$\widetilde{F}\setminus S$ is connected and $incl_*\colon H_1(S)\to H_1(\widetilde{F})$ is injective.*

(iv) *The covering is conservative.*

This theorem, for $F$ orientable and without (iii), is due to Masters and Zhang [5] and is a key ingredient in their proof that cusped hyperbolic 3–manifolds contain quasi-Fuchsian surface groups [4; 5]. Without (iii) and (iv) the theorem is a special case of well-known theorems on subgroup separability of free groups (see Hall, Jr [1]) and surface groups (see Scott [6; 7]). For a discussion of subgroup separability and 3–manifolds, see Long and Reid [3].

The proof in [5] uses the folded graph techniques due to Stallings; see Kapovich and Myasnikov [2]. The shorter proof below uses cut and cross-join of surfaces. A cover is

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called *good* if properties (i)–(ii) hold and *very good* if (i)–(iii) hold. The idea is to start with a good cover and then pass to a second cover which is very good. Then *cross-join operations* (defined below) are used to reduce the number of boundary components of a very good cover until it is conservative.

## 1 Constructing a very good cover

We first explain a geometric condition on a cover of $F$ which ensures it is good, and then use Theorem 1.3 to construct a very good cover.

Choose a basepoint $x$ in the interior of $F$ and suppose $p\colon F_H \to F$ is the cover corresponding to $H$. There is a compact, connected, incompressible subsurface $S$ in the interior of $F_H$ which is a retract of $F_H$ and which contains a lift $\tilde{x}$ of $x$. Each element $g \in \pi_1(F, x)$ determines a unique lift $\tilde{x}(g) \in F_H$ of the basepoint $x$. The surface $S$ can be chosen large enough to contain $\{\tilde{x}(b) : b \in B\}$. Then $p|_S\colon S \to F$ is a local homeomorphism.

If $\pi\colon F' \to F$ is any cover and there is a lift of $p|_S$ to $\theta\colon S \to F'$ (thus $\pi \circ \theta = p|_S$) which is injective, we say $S$ *lifts to an embedding in the cover* $F'$. The work of M Hall [1] and P Scott [6] shows there is a finite cover $F' \to F$ such that $S$ lifts to an embedding in $F'$.

**Proposition 1.1** (Good cover) *Under the hypotheses of the main theorem, if* $\pi\colon F' \to F$ *is any cover and* $S$ *lifts to an embedding in* $F'$*, then the cover is good.*

**Proof** With the notation above, a based loop representing an element $b \in B$ lifts to a path in $F'$ that starts at the basepoint $\tilde{x} \in L = \theta(S)$ but ends at some other point $\tilde{x}(b) \neq \tilde{x}$ in $L$. □

**Addendum 1.2** (Very good cover) *There is a very good cover* $\tilde{F} \to F$ *of finite degree with* $|\partial \tilde{F}|$ *even.*

**Proof** We start with a good cover $F'$ of $F$ with finite degree and the subsurface $S \subset F'$ described above and then construct a cover of $F'$ with the required properties. Let $p\colon \tilde{F} \to F'$ be the regular cover given by the kernel of the map of $\pi_1(F')$ onto $H_1(F', S; \mathbb{Z}/2)$. There is a lift $\tilde{S}$ of $S$ to this cover by construction. The conclusions follow from Theorem 1.3 below. □

The following allows us to lift a $\pi_1$–injective subsurface to a regular cover where it is $H_1$–injective and nonseparating.

**Theorem 1.3** *Suppose $F$ is a compact, connected surface, possibly with boundary, which contains a compact, connected subsurface $S \neq F$. Assume that $S \cap \partial F$ is a (possibly empty) union of components of $\partial F$ and no component of $\mathrm{cl}(F \setminus S)$ is a disc or a boundary parallel annulus. Let $p\colon \widetilde{F} \to F$ be the cover corresponding to the kernel of the natural homomorphism of $\pi_1(F)$ onto $G = H_1(F, S; \mathbb{Z}/2)$. If $\widetilde{S}_0$ is a connected component of $p^{-1}(S)$ then $X = \mathrm{cl}(\widetilde{F} \setminus \widetilde{S}_0)$ is connected and the map $i_*\colon H_1(\widetilde{S}_0) \to H_1(\widetilde{F})$ induced by inclusion is injective. Moreover $|\partial \widetilde{F}|$ is even.*

**Proof** The hypotheses imply $G \neq 0$. Let $Y$ be a connected component of $X$. Then $\partial Y = (Y \cap \partial \widetilde{F}) \sqcup (Y \cap \widetilde{S}_0)$. We claim that $p(Y) \supset S$. Otherwise $p|_Y\colon Y \to \mathrm{cl}(F \setminus S)$ is a covering map which is injective since $p|(Y \cap \widetilde{S}_0)$ is injective. Thus $Y$ is a lift of a component $Z$ of $\mathrm{cl}(F \setminus S)$.

If $Z \cap S$ is connected, then since $Z$ is not a disc or boundary parallel annulus, the image of $H_1(Z; \mathbb{Z}/2)$ in $G$ is not trivial. Thus $Z$ does not lift to the $G$–cover, a contradiction.

Hence $Z \cap S$ contains at least two distinct circle components $B_1, B_2$. There is a loop $\alpha = \beta \cdot \gamma \subset F$ which is the union of two arcs connecting $B_1$ and $B_2$: one arc $\beta \subset Z$ and one arc $\gamma \subset S$. Since $\alpha$ has mod 2 algebraic intersection number 1 with the boundary component $B_1$ of $S$ it is a nonzero element of $G$. It follows that the lift $\widetilde{\beta} \subset Y$ of $\beta$ has endpoints in different components of $p^{-1}(S)$, since otherwise $\alpha$ would lift to a loop. But $\partial \widetilde{\beta} \subset \partial Y \subset \partial \widetilde{S}_0$ which is a contradiction. Thus $p(Y) \supset S$.

It follows that $Y$ contains some component $\widetilde{S}_1 \neq \widetilde{S}_0$ of $p^{-1}(S)$ in its interior. However the cover is regular so there is a covering transformation $\tau$ taking $\widetilde{S}_0$ to $\widetilde{S}_1$. Thus if $\widetilde{S}_0$ is not orientable then $Y$ is not orientable and if $\widetilde{S}_0$ contains a component of $\partial \widetilde{F}$ then so does $Y$.

Choose some Riemannian metric on $F$. This metric pulls back to one on $\widetilde{F}$ which is preserved by covering transformations. If $X$ is not connected, let $Y$ be a component of $X$ with smallest area.

As shown above, $Y$ contains an $\widetilde{S}_1 \neq \widetilde{S}_0$ in its interior. The covering transformation $\tau$ taking $\widetilde{S}_0$ to $\widetilde{S}_1$ takes each component of $\widetilde{F} \setminus \widetilde{S}_0$ to a component of $\widetilde{F} \setminus \widetilde{S}_1$ with the same area. One of the components of $\widetilde{F} \setminus \widetilde{S}_1$ contains $\widetilde{S}_0$, so all the others must be strictly contained in $Y$, which contradicts that $Y$ has minimal area. Hence $X = Y$ is connected.

To show the injectivity of $i_*$, note the long exact homology sequence of the pair $(\widetilde{F}, \widetilde{S}_0)$ yields

$$0 \longrightarrow H_2(\widetilde{F}) \xrightarrow{j_*} H_2(\widetilde{F}, \widetilde{S}_0) \xrightarrow{\delta} H_1(\widetilde{S}_0) \xrightarrow{i_*} H_1(\widetilde{F}),$$

so that we have the following equivalences: $\ker i_* = 0$ if and only if $\mathrm{Image}\,\delta = 0$ if and only if $j_*$ is an isomorphism. By excision $H_2(\widetilde{F}, \widetilde{S}_0) \cong H_2(X, X \cap \widetilde{S}_0) \cong H_2(X, \partial X \cap \partial\widetilde{S}_0)$.

Suppose $\partial F \neq \phi$. Then $X \cap \partial\widetilde{F} \neq \phi$, since otherwise $\partial\widetilde{F} \subset \widetilde{S}_0$, but we have shown $\tau(\widetilde{S}_0) \subset X$, which is a contradiction. Now $X \cap \partial\widetilde{F} \neq \phi$ implies $H_2(X, \partial X \cap \partial\widetilde{S}_0) = 0$, so that $\mathrm{Image}\,\delta = 0$ hence $\ker i_* = 0$.

The remaining case is $\partial F = \phi$. Here $X \cap \widetilde{S}_0 = \partial\widetilde{S}_0 = \partial X$. If $\widetilde{F}$ is orientable, then so is $X$, and it follows that $j_*$ is an isomorphism, hence $\ker i_* = 0$.

If $\widetilde{F}$ is nonorientable, we claim $X$ must also be nonorientable; hence $H_2(X, \partial X) = 0$ so that $0 = \mathrm{Image}\,\delta = \ker i_*$.

Indeed, if $X$ is orientable then $\widetilde{F}$ is orientable. This is because $\tau(\widetilde{S}_0) \subset X$ so $\tau(\widetilde{S}_0)$ orientable. This is a lift of $S$ so $S$ is orientable. Thus the homomorphism $\pi_1(F) \to \mathbb{Z}_2$ that sends a loop to $0$ if and only if it is orientation preserving vanishes on $\pi_1(S)$ and so factors through $G$. It follows that every orientation reversing loop in $F$ has nonzero image in $G$ so $\widetilde{F}$ is orientable.

It remains to show $|\partial\widetilde{F}|$ is even. The action of $G$ on $\widetilde{F}$ is free. Since $\mathbb{Z}_2^2$ does not act freely on $S^1$ it follows that if $|\partial\widetilde{F}|$ is odd then $G \cong \mathbb{Z}_2$ and is generated by some component $C$ of $\partial F$. Let $Z$ be the component of $\mathrm{cl}(F \setminus S)$ that contains $C$. By excision $\mathbb{Z}_2 \cong G \cong H_1(Z, Z \cap S)$. Since $Z$ is not a disc or an annulus with $C$ one of the boundary components the only other possibility is that $Z = \mathrm{cl}(F \setminus S)$ is a pair of pants with only one boundary component in $S$. But then $|\partial F|$ is even hence so is $|\partial\widetilde{F}|$. □

The following is easily deduced from the proof of Theorem 1.3 and will be used in the next two sections of the paper.

**Remark 1.4** If $F$ is a surface and $S \subset F$ is a subsurface and $X = \mathrm{cl}(F \setminus S)$ is connected and $X \cap \partial F \neq \phi$ then $i_*\colon H_1(S) \to H_1(F)$ is injective.

## 2 Cross-joining covers

Suppose $F$ is a surface and $\alpha_1$ and $\alpha_2$ are disjoint arcs properly embedded in $F$. Let $N(\alpha_i) \equiv \alpha_i \times [-1, 1]$ be disjoint regular neighborhoods of the arcs $\alpha_i$ in $F$ such that $\alpha_i \equiv \alpha_i \times 0$ and $N(\alpha_i) \cap \partial F = (\partial\alpha_i) \times [-1, 1]$. The sets $\alpha_i \times (0, \pm 1] \subset F$ are called the $\pm$ *sides* of $\alpha_i$.

Given a homeomorphism $h\colon N(\alpha_1) \to N(\alpha_2)$ taking the $+$ side of $\alpha_1$ to the $+$ side of $\alpha_2$, the *cross-join* of $F$ along $(\alpha_1, \alpha_2)$ is the surface $K$ defined as follows. The surface $F^- = F \setminus (\alpha_1 \cup \alpha_2)$ contains four subsurfaces $\alpha_i \times (0, \pm 1]$. Let $F^{\text{cut}}$ be the surface obtained by completing these subsurfaces to $\alpha_i \times [0, \pm 1]$. Thus $F^{\text{cut}}$ has two copies $\alpha_i^+, \alpha_i^-$ of $\alpha_i$ in $\partial F^{\text{cut}}$ and identifying these copies suitably produces $F$. The surface $K$ is the quotient of $F^{\text{cut}}$ obtained by using $h$ to identify $\alpha_1^-$ to $\alpha_2^+$ and $\alpha_1^+$ to $\alpha_2^-$. Note that here we do not require $F$ to be connected, so that $\alpha$ and $\beta$ might be in different components of $F$.

There are two special cases of cross-join which will be used to change the number of boundary components of a surface:

**Lemma 2.1** *Suppose the compact surface $F$ contains two disjoint properly embedded arcs $\alpha$ and $\beta$. In addition suppose that*

(1) *either $F$ is connected and the endpoints of $\alpha, \beta$ lie on four distinct components of $\partial F$,*

(2) *or $F$ is the union of two connected components $A$ and $B$ and $\alpha \subset A$ has both endpoints on the same boundary component and $\beta \subset B$ has endpoints on distinct boundary components.*

*Then a surface $K$ obtained by cross-joining along these arcs has $|\partial K| = |\partial F| - 2$. Furthermore $\chi(K) = \chi(F)$ and $K$ is connected.*

**Proof** We verify that $K$ is connected. In the first case this follows since the arcs do not disconnect the boundary components on which they have endpoints; therefore $F \setminus (\alpha \cup \beta)$ is connected. In the second case it follows because $B \setminus \beta$ is connected, and every point in $K$ is connected to a point in this subset by an arc. $\square$

Suppose $p\colon \widetilde{F} \to F$ is a (possibly not connected) covering of surfaces and $\alpha$ is an arc properly embedded in $F$. Suppose $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ are two distinct lifts of $\alpha$ to $\widetilde{F}$; then they are disjoint. The map $p$ provides a homeomorphism between small regular neighborhoods of these two arcs. Using this to cross-join produces a surface $\widetilde{F}'$ and since the identifications are compatible with $p$ there is a covering map $p'\colon \widetilde{F}' \to F$.

An important special case is when $\widetilde{F}$ is a $(d+1)$–fold cover which is the disjoint union of a 1–fold cover $F_1 \to F$ and some connected $d$–fold cover $F_d \to F$. Then cross-joining an arc in $F_1$ with one in $F_d$ produces a connected cover of degree $d+1$.

To produce a new cover $F'$ of $F$ by a cross-join along two arcs in some cover $\widetilde{F}$ requires the arcs to be disjoint from each other. If $S$ is embedded in $\widetilde{F}$ and these arcs are also disjoint from $S$, then $S$ lifts to an embedding in $F'$, so the cover $F'$ is good. We call the combination of these two properties the *disjointness condition*.

There is a metric condition, involving some arbitrary choice of Riemannian metric on $F$, that ensures the disjointness condition is satisfied and therefore that the new cover is good. The next lemma provides a uniform upper bound on the lengths of the arcs we will use to cross-join *in any cover* of $F$.

**Lemma 2.2** (Short arcs) *Suppose $F$ is a compact, connected surface with a Riemannian metric such that the diameter of $F$ is $\ell$. If $\widetilde{F}$ is a finite connected cover of $F$ then:*

(1) *If $A$ and $B$ are distinct components of $\partial F$ then there is an arc $\alpha$ in $F$ connecting them and length$(\alpha) \leq \ell$.*

(2) *If some component $A$ of $\partial F$ has (at least) two preimages in $\partial \widetilde{F}$ then there is an embedded arc $\alpha$ in $F$ of length at most $2\ell$ which lifts to an arc with endpoints on distinct preimages of $A$.*

**Proof** The first claim is obvious. For the second claim, since every point in $\widetilde{F}$ is within a distance at most $\ell$ of some point in $p^{-1}(A)$ and $\widetilde{F}$ is connected, some point in $\widetilde{F}$ is within a distance at most $\ell$ of points in two distinct components of $p^{-1}(A)$. This gives an arc $\beta$ in $\widetilde{F}$ of length at most $2\ell$ which connects two distinct components of $p^{-1}(A)$.

Let $\gamma\colon [0, 2R] \to \widetilde{F}$ be a shortest arc connecting two distinct components of $p^{-1}(A)$ and parametrized by arc length. Then $R \leq \ell$. To complete the proof we show that $\gamma$ projects to an embedded arc in $F$. Observe that

$$d_{\widetilde{F}}(\gamma(t), p^{-1}(A)) = \min(t, 2R - t)$$

otherwise there is a shorter arc connecting two distinct components of $p^{-1}(A)$. It follows that

$$d_F(p(\gamma(t)), A) = \min(t, 2R - t)$$

This means that the distance in $F$ of a point on $p \circ \gamma$ from $A$ is given by arc length along $p \circ \gamma$. It follows that $\alpha = p \circ \gamma$ is the required embedded arc. □

An arc of length at most $2\ell$ is called *short*. The next lemma provides a conservative cyclic cover with large diameter of a surface $F$. If a short arc in $F$ connects two distinct boundary components, then so does every covering translate of it. If $S$ lifts to the cover then there are many different translates of the short arc that are far from each other and far from the lift of $S$. In particular the disjointness condition is satisfied by suitable translates of a lifted short arc in this cover.

**Lemma 2.3** (Big covers) *Suppose $F$ is a compact connected surface with $k \geq 2$ boundary components and which contains a compact, connected, incompressible subsurface $S \subset \mathrm{interior}(F)$ with $F \setminus S$ connected. Given $n > 0$ there is a conservative finite cyclic cover $\widetilde{F} \to F$ of degree bigger than $n$ and a lift, $\widetilde{S}$, of $S$ to $\widetilde{F}$. Furthermore $\widetilde{F} \setminus \widetilde{S}$ is connected and the map $i_*\colon H_1(\widetilde{S}) \to H_1(\widetilde{F})$ induced by inclusion is injective.*

**Proof** Let $Y$ be the surface obtained from $F \setminus \mathrm{interior}(S)$ by gluing a disc onto each component of $\partial S$. Then $Y$ is a connected surface with $k$ boundary components and there is a natural isomorphism of $H_1(F)/H_1(S)$ onto $H_1(Y)$. Choose a prime $p > \max(k, n)$. Because $Y$ is connected, there is an epimorphism from $H_1(Y)$ onto $\mathbb{Z}/p$ which sends one component of $\partial Y$ to $k-1$ and all the other $(k-1)$ components of $\partial Y$ to $-1$. Now $(k-1)$ is coprime to $p$ because $2 \leq k < p$. Therefore this defines a *conservative* cyclic $p$–fold cover $\widetilde{Y}$ of $Y$. It also determines a conservative cyclic $p$–fold cover $\widetilde{F}$ of $F$ such that $S$ lifts to $\widetilde{S}$. Since $\widetilde{Y}$ is connected, it follows that $X = \widetilde{F} \setminus \widetilde{S}$ is connected. Also $X \cap \partial\widetilde{F} = \partial\widetilde{F}$ is not empty. Hence $i_*$ is injective by Remark 1.4. □

## 3 Proof of main theorem

In this section all covers are of finite degree. Given a cover $p\colon \widetilde{F} \to F$, the *excess* number of boundary components $E(p)$ for this cover is defined as $E(p) = |\partial\widetilde{F}| - |\partial F|$. By Addendum 1.2 there is a very good cover $p\colon \widetilde{F} \to F$ with $|\partial\widetilde{F}|$ even. If $E(p) = 0$ the theorem is proved; otherwise we construct another very good cover with smaller excess, and repeating this process a finite number of times yields a very good cover with zero excess.

These constructions use various very good covers of $F$. We will choose a lift of $S$ to each cover and *identify* this lift with $S$ and refer to the lift as $S$. This should not cause confusion.

We will also use the big cover Lemma 2.3 to replace a very good cover $\widetilde{F} \to F$ by another very good cover $F' \to F$ with the same excess and very large diameter. This process is called *taking a big cover*. We will rename $F'$ as $\widetilde{F}$.

Given $\widetilde{F}$ very good, we will use (except in one case) one of the two cross-joins described in Lemma 2.1 to produce a new *connected* very good cover $F'$ of $F$ with smaller excess. We first take a big cover so that there are lifts of a short arc that are far apart and disjoint from $S$ in $\widetilde{F}$. Then we change $\widetilde{F}$ with a cross-join to produce a good connected cover $F'$, which has smaller excess by Lemma 2.1.

To verify $F'$ is very good we check below that $F' \setminus S$ is *connected*, then by Remark 1.4 this implies $incl_* f\colon H_1(S) \to H_1(F')$ is injective, so $F'$ is very good.

First observe that the cyclic cover produced by Lemma 2.3 leaves $\widetilde{F} \setminus S$ connected. Since the cross-join arcs are disjoint from $S$ they also determine a cross-join of $\widetilde{F} \setminus S$. This cross-joined subsurface is $F' \setminus S$ which is connected by Lemma 2.1 as required.

**Case when** $|\partial F| = 1$ By Lemma 2.2 there is a properly embedded, short arc, $\alpha$, in $F$ which is covered by an arc $\beta$ with endpoints on two distinct boundary circles of $\partial \widetilde{F}$. After taking a big cover we may assume the diameter of $\widetilde{F}$ is much larger than the length of $\alpha$ and the diameter of $S$; thus $\beta$ can be chosen to be disjoint from $S$ in $\widetilde{F}$.

Cross-join $(\widetilde{F}, \beta)$ with $(F, \alpha)$ to obtain a cover $F'$ with one fewer boundary circle than $\widetilde{F}$. There is a lift of $S$ to $F'$ and $F' \setminus S$ is connected. Repeat the process until we obtain a cover with only one boundary component. This completes the proof when $|\partial F| = 1$.

**Case when** $|\partial F| \geq 2$ First we show how to make $E(p)$ even by performing a cross-join if needed. This first step will increase the number of boundary components.

Suppose $E(p)$ is odd. By Addendum 1.2 $|\partial \widetilde{F}|$ is even, so $|\partial F|$ is odd. We can make $E(p)$ even by cross-joining $(\widetilde{F}, \widetilde{\alpha})$ and $(F, \alpha)$ to obtain a cover $p'\colon F' \to F$. To perform the cross-join, choose a short embedded arc $\alpha \subset F$ with endpoints on two distinct circles $C$, $C'$ of $\partial F$ and a lift $\widetilde{\alpha} \subset \widetilde{F}$, with endpoints on two preimages $\widetilde{C}$, $\widetilde{C}'$. After taking a big cover we can assume that $\widetilde{\alpha}$ is disjoint from $S$ in $\widetilde{F}$. Then $F'$ is the cross-join of $(F, \alpha)$ and $(\widetilde{F}, \widetilde{\alpha})$. The surface $S$ lifts to $F'$ and $F' \setminus S$ is connected by Lemma 2.1.

Here is the outline of the rest of the proof. If $E(p) \neq 0$ then it is even. We proceed as follows using suitable cross-joins to construct new coverings. If there are two different components $C, C' \subset \partial F$ which both have more than one preimage in $\partial \widetilde{F}$ then we find a short arc $\alpha$ in $F$ connecting $C$ and $C'$ and cross-join $\widetilde{F}$ to itself along two suitable lifts of $\alpha$ in $\widetilde{F}$. This reduces the excess by $2$. After finitely many steps we obtain a cover so that at most one component $C \subset \partial F$ has more than one preimage. A single *cyclic cross-join* (defined below) is done simultaneously to reduce the excess to zero. Here are the details.

Suppose $A$ and $B$ are distinct circles in $\partial F$ which both have (at least) two distinct preimages $\widetilde{A}_i, \widetilde{B}_i$ for $i = 1, 2$ in $\partial \widetilde{F}$. Choose a short arc $\gamma$ in $F$ with endpoints on $A$ and $B$. Let $\alpha_i$ be a lift of $\gamma$ with one endpoint on $\widetilde{A}_i$ and $\beta_i$ a lift with an endpoint on $\widetilde{B}_i$. Inductively we assume that $\widetilde{F} \setminus S$ is connected. After taking a big cover we may assume that these arcs are all far apart and far from $S$. Thus there is a cover, obtained by cross-joining along any pair of distinct arcs chosen from this set of four, and $S$ lifts to this cover.

We claim that there is a pair of these arcs which have endpoints on four distinct boundary components of $\widetilde{F}$. It follows from Lemma 2.1 that cross-joining along this pair reduces the excess by 2 and $S$ lifts to the cover $F'$ so produced. Furthermore, since $\widetilde{F}\setminus S$ is connected it follows that $F'\setminus S$ is connected by Lemma 2.1.

If $\alpha_1$ and $\alpha_2$ do not both have endpoints on the same lift $\widetilde{B}$ of $B$ the pair $(\alpha_1,\alpha_2)$ works. Similarly if $\beta_1$ and $\beta_2$ do not both have endpoints on the same lift $\widetilde{A}$ of $A$ the pair $(\beta_1,\beta_2)$ works. The remaining case is (after relabeling) $\alpha_1$ and $\alpha_2$ both have endpoints on a component $\widetilde{B}\neq\widetilde{B}_2$ which covers $B$ and $\beta_1,\beta_2$ both have endpoints on some component $\widetilde{A}\neq\widetilde{A}_2$ which covers $A$. Then $\alpha_2$ connects $A_2$ to $\widetilde{B}\neq\widetilde{B}_2$ and $\beta_2$ connects $B_2$ to $\widetilde{A}\neq\widetilde{A}_2$. Thus the pair $(\alpha_2,\beta_2)$ works.

Repeating this process a finite number of times reduces the excess by an even number until either $|\partial\widetilde{F}|=|\partial F|$ or else there is a unique component $C$ of $\partial F$ with more than one preimage. In the latter case the excess is even so there is an odd number of preimages $p^{-1}(C)=\{C_0,\dots,C_{2k}\}$.

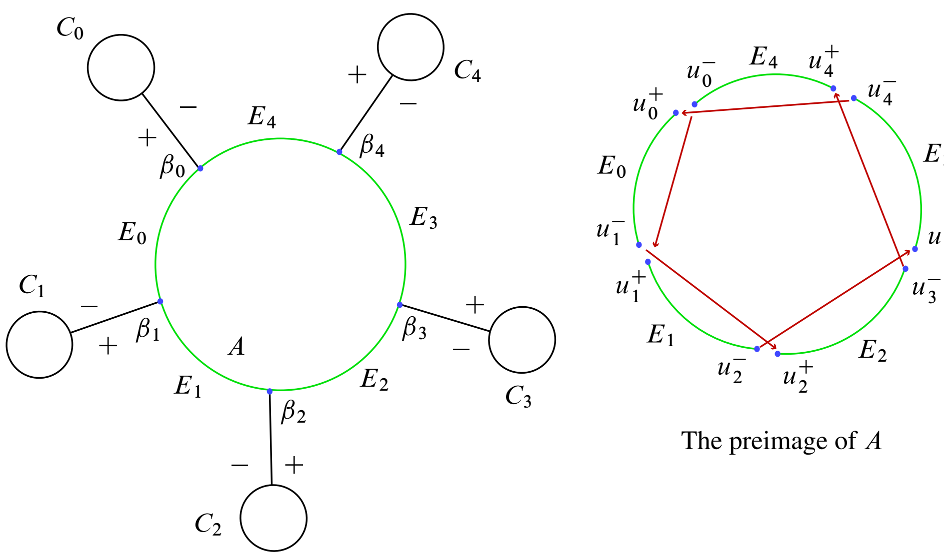


Figure 1: Cyclic cross-joining, $2k+1=5$ illustrated

Refer to Figure 1. Choose a component $A$ of $\partial\widetilde{F}$ that does not cover $C$. This is possible because $|\partial F|\geq 2$. Let $\beta$ be a short arc in $F$ with endpoints on $p(A)$ and $C$. For each $i$ there is a lift $\beta_i$ of $\beta$ with one endpoint on $C_i$ and the other on $A$. After taking a big cover we may assume all these lifts are far apart and far from $S$. Orient each arc $\beta_i$ so it points from $A$ to $C_i$ and call the left side $+$ and the right side $-$. Now cross-join cyclically as follows. Cut $\widetilde{F}$ along the union of these arcs and join the $-$ side of $\beta_i$ to the $+$ side of $\beta_{i+1}$, with all integer subscripts taken mod $2k+1$.

The resulting cover has a single preimage of $C$. Indeed, each $C_i$ has been cut at one point to give an interval $D_i = [t_i^+, t_i^-]$ where the label $i$ denotes an endpoint of $\beta_i$ and $t_i^{\pm}$ is on the $\pm$ side of $\beta_i$. These intervals are then glued by identifying $t_i^-$ in $D_i$ to $t_{i+1}^+$ in $D_{i+1}$. The result is obviously connected; it is a single circle.

To analyse the preimage of $p(A)$ the circle $A$ was cut at $2k+1$ points to produce $2k+1$ subarcs $E_i = [u_i^+, u_{i+1}^-]$ where $u_i^{\pm}$ is on the $\pm$ side of $\beta_i$. Then $E_i$ is glued to $E_{i+2}$ by identifying $u_{i+1}^-$ with $u_{i+2}^+$ (see figure 1). Since there are $2k+1$ intervals and the $i$–th one is glued to the $(i+2)$–th one the result is connected because 2 is coprime to $2k+1$. This gives the required conservative cover completing the proof of the main theorem. □

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